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n-CONVEXITY AND MAJORIZATION

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ABSTRACT. The fact that the n^{th} order divided difference of an $(n+2)$-convex function is a symmetric, convex function of its arguments, and is therefore Schur convex, allows us to apply the theory of Majorization in order to derive inequalities for such functions. Several consequences of this result are presented. In a separate section the theory of majorization is used to compute bounds on the derivatives of polynomials.

1. n-Convexity and Schur convexity. The first two definitions are given in [**2**].

DEFINITION 1. Let $x, y \in \mathbf{R}^{n+1}$ be given. We say that y is *majorized* by $x(y \prec x)$ if and only if $\sum_{i=0}^{n} x_i = \sum_{i=0}^{n} y_i$ and

$$\sum_{i=0}^{k} x_{[i]} \geq \sum_{i=0}^{k} y_{[i]}, \quad k = 0, \ldots, n-1, \tag{1}$$

where $x_{[0]} \geq \cdots \geq x_{[n]}$ denotes a decreasing rearrangement of $x_0, \ldots, x_n$.

Numerous example of majorization are given in [**2**].

DEFINITION 2. Let $x, y \in \mathbf{R}^{n+1}$ be given. A function $\varphi : \mathbf{R}^{n+1} \to \mathbf{R}$ is called *Schur convex* if and only if $x \prec y \Rightarrow \varphi(x) \leq \varphi(y)$.

The next definition can be found in [**4**].

DEFINITION 3. A function f is $(n+2)$-convex on (a, b) if and only if, for all $a < x_0 < \cdots < x_{n+2} < b$, the divided differences $[x_0, \ldots, x_{n+2}]f$

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are nonnegative. In particular, a 2-convex function is convex in the classical sense.

We note that f is $(n+2)$-convex on (a,b) if and only if $f^{(n)}$ is continuous and convex there.

Let f be $(n+2)+$ convex on (a,b), and, for $a <_0 < \cdots < x_n < b$, define

$$\varphi(x) := [x_0, \ldots, x_n]f, \quad x = (x_0, \ldots, x_n). \tag{2}$$

Since the divided difference is symmetric, i.e., independent of the order of its arguments, and $f^{(n)}$ is continuous in (a,b), φ may be extended to all of $(a,b)^{n+1}$.

LEMMA 1. *Let f be $(n+2)+$ convex on (a,b) and let φ be defined as in* (2). *Then φ is Schur convex on $(a,b)^{n+1}$.*

PROOF. The proof in [**5**] shows that φ is a convex function of x for $x \in (a,b)^{n+1}$. Since it also symmetric, Proposition C.2 in Chapter 3 of [**2**] implies that it is Schur convex.

A more direct proof can be constructed with the aid of *Schur's condition* [**2**, p. 57]:

$$\left(\frac{\partial\varphi}{\partial x_i} - \frac{\partial\varphi}{\partial x_j}\right) \cdot (x_i - x_j) \geq 0.$$

Since, as is well known,

$$\frac{\partial\varphi}{\partial x_i} = [x_0, \ldots, x_{i-1}, x_i, x_o, x_{i+1}, \ldots, x_n]f,$$

and, for an $(n+2)$-convex function$f, [z_0, \ldots, z_{n+1}]f$ is an increasing function of its arguments, φ satisfies Schur's condition, and is therefore Schur convex.

We thus get

THEOREM 1. *Let f be $(n+2)$-convex on (a,b). If $x, y \in (a,b)^{n+1}$ and $x \prec y$, then $[x_0, \ldots, x_n]f \leq [y_0, \ldots, y_n]f$.*

A sufficient condition on x and y for (1) to be satisfied is given in Lemma 2.

LEMMA 2. *Let $x, y \in \mathbf{R}^{n+1}$ be given. Then $y \prec x$ if the following conditions hold:*

$$\sum_{i=0}^{n} x_i = \sum_{i=0}^{n} y_i \tag{3}$$

$$\sum_{i=0}^{k} x_i \leq \sum_{i=0}^{k} y_i, \quad k = 0, \cdots, n-1 \tag{4}$$

$$x_{i-1} \leq y_i, \quad i = 1, \cdots, n. \tag{5}$$

Lemma 2 (and the variations discussed below) may be demonstrated by applying the results of [**3**] and certain characterizations of majorization presented in [**2**]. We give here a more direct proof.

PROOF OF LEMMA 2. Note first that (3) and (4) are equivalent to (3) and

$$\sum_{i=k+1}^{n} x_i \geq \sum_{i=k+1}^{n} y_i, \quad k = 0, \ldots, n-1. \tag{6}$$

Thus, with (5), we get

$$\sum_{i=k+1}^{n} x_i \geq \sum_{i=k+1}^{n} y_i \geq \sum_{i=k+1}^{n} x_{i-1} = \sum_{i=k}^{n-1} x_i,$$

and hence $x_n \geq x_k, \quad k = 0, \ldots, n-1$. From (6) we also get $x_n \geq y_n$ and

$$x_n - y_k \geq \sum_{i=k+1}^{n} (y_i - x_{i-1}) \geq 0, \quad k = 1, \ldots, n-1, \tag{7}$$

which holds for $k = 0$ as well. In particular, $x_n = x_{[0]} \geq y_{[0]}$. In order to show that the remainder of the conditions (1) are satisfied, fix $1 \leq k \leq n-1$ and let $x_{[0]}, \ldots, x_{[k]} = x_n, x_{j_1}, \ldots, x_{j_k}$ and $y_{[0]}, \ldots, y_{[k]} = y_{l_0}, \ldots, y_{l_k}$. Further, set $l := \min_{0 \leq i \leq k}\{l_i\}$. We then have

$$\begin{aligned}
\sum_{i=0}^{k}(x_{[i]} - y_{[i]}) = (x_n - y_1) &= \sum_{i=1}^{k} x_{[i]} - \sum_{\substack{i=0 \\ l_i \neq l}}^{k} y_{l_i} \\
&= (x_n - y_l) + \Big(\sum_{i=1}^{k} x_{[i]} - \sum_{\substack{i=0 \\ l_i \neq}}^{k} x_{l_i - 1}\Big) \\
&\quad + \Big(\sum_{\substack{i=0 \\ l_i \neq l}}^{k} x_{l_i - 1} - \sum_{\substack{i=0 \\ l_i \neq l}}^{k} y_{l_i}\Big).
\end{aligned}$$

From the definition of $x_{[i]}$, it follows that

$$\sum_{i=1}^{k} x_{[i]} - \sum_{\substack{i=0 \\ l_i \neq l}}^{k} x_{l_i - 1} \geq 0,$$

and (7) yields

$$x_n - y_l \geq \sum_{i=l+1}^{n} (y_i - x_{i-1}),$$

hence it only remains to show that

$$\sum_{i=l+1}^{n} (y_i - x_{i-1}) \geq \sum_{\substack{i=0 \\ l_i \neq l}}^{k} (y_{l_i} - x_{li-1})$$

But this is true since, for each i in the sum, $l_i \geq l + l$ and hence the right-hand side is a partial sum of the left-hand side, all of whose terms are nonnegative.

REMARK 1. If the reverse inequalities hold in (4) and (5) then the conclusion of Lemma 2 is still valid. If one of (4) and (5) is replaced by the reverse inequality, then $x \prec y$. The proofs of these assertions are similar to that of Lemma 2.

We note that Theorem (2.2) of [**1**] is a consequence of Lemma 2 and the Schur convexity of $[x_0, \ldots, x_n]f$ when f is an $(n+2)$-convex function.

2. Weak majorization. A useful form of majorization is given in

DEFINITION 4.

a) y is *weakly majorized from above* by $x(y \prec^{w} x)$ if

$$\sum_{i=0}^{k} x_{(i)} \leq \sum_{i=0}^{k} y_{(i)}, \quad k = 0, \ldots, n,$$

where $x_{(0)} \leq \cdots \leq x_{(n)}$ is an increasing rearrangement of $x_0, \ldots, x_n$.

b) y is *weakly majorized from below* by $x(y \prec_{w} x)$, if

$$\sum_{i=0}^{k} y_{[i]} \leq \sum_{i=0}^{k} x_{[i]}, \quad k = 0, \ldots, n,$$

with $x_0, \ldots, x_n$ a decreasing rearrangement of $x_0, \ldots, x_n$.

LEMMA 3. *If* $\sum_{i=0}^{k} x_i \leq \sum_{i=0}^{k} y_i$, $k = 0, \ldots, n$, *and either* $x_{i-1} \leq y_i$, $i = 1, \ldots, n$, *or* $y_{i-1} \leq x_i$, $i = 1, \ldots, n$, *then* $y \prec^{\omega} x$. *If the reverse inequalities hold then* $y \prec_{\omega} x$.

PROOF. The proof of these assertions may be carried out in much the same manner as in the proof of Lemma 2.

The importance of weak majorization is given in

PROPOSITION 3. **[2]**. *$y \prec_\omega x \Rightarrow \varphi(y) \le \varphi(x)$ for all increasing, Schur convex functions, and $y \prec^\omega x \Rightarrow \varphi(y) \le \varphi(x)$ for all decreasing, Schur convex functions.*

The following theorem generalizes Theorem (2.2) of **[1]**.

THEOREM 2. *Let f be $(n+2)$-convex in (a,b) and let $x, y \in (a,b)^{n+1}$ be given. Then*

$$[x_0, \ldots, x_n]f \le [y_0, \ldots, y_n]f$$

if any of the following conditions are satisfied:

a) $x \prec y$;

b) $x \prec_\omega y$ *and* f *is* $(n+1)$*-convex;*

c) $x \prec^\omega y$ *and* $-f$ *is* $(n+1)$*-convex.*

PROOF. As noted above, the function $\varphi(x) := [x_0, \ldots, x_n]f$ is Schur convex. If f is $(n+1)$-convex then φ is increasing, hence if $-f$ is $(n+1)$-convex then φ is decreasing. The theorem now follows from the definition of Schur convexity and from Proposition 3.

REMARK 2. We note that the results of Theorem 2 are valid for the closed interval $[a,b]$, provided $f^{(n)}$ is continuous on $[a,b]$.

3. Some applications. The results of §1 can be used to demonstrate that certain functions are Schur convex. We give a few examples:

EXAMPLE 1. The complete symmetric functions

$$C_k(x) := \sum_{0 \le i_1 \le \cdots \le i_k \le n} x_{i_1} \ldots x_{i_k}, \quad k = 1, \ldots, n, \quad C_0(x) \equiv 1,$$

can be generated by taking divided differences of monomials:

$$C_k(x) = [x_0, \ldots, x_n]t^{n+k}, \quad k \ge 0.$$

Since t^{n+k} is $(n+2)$-convex for $t \ge 0$ and $k \ge 0$, the functions C_k are Schur convex, provided that $x_i \ge 0, \quad i = 0, \ldots, n$.

EXAMPLE 2. Let

$$f(t) := \frac{1}{1-at}.$$

Then

$$\varphi(x) := [x_0, \dots, x_n]f = \frac{a^n}{\prod_{i=0}^n (1-ax_i)}.$$

Since

$$f^{(n+2)}(t) = \frac{(n+2)!a^{n+2}}{(1-at)^{n+3}},$$

it follows that φ is Schur convex, provided n is even and $ax_i < 1$ $(i = 0, \dots, n)$, or n is odd and either $ax_i > 1$ $(i = 0, \dots, n)$, or $ax_i < 1$ $(i = 0, \dots, n)$ with $a > 0$.

Theorem 2 can be used to derive divided difference inequalities for $(n+2)$-convex functions (see [**1**]).

EXAMPLE 3. For a given $x \in (a,b)^{n+1}$, denote by ζ the average $\frac{1}{n+1}\sum_{i=0}^n x_i$, and let $z := (\zeta, \dots, \zeta)$. Then $z \prec x$, hence $\varphi(z) \le \varphi(x)$ for all Schur convex functions φ. In particular, for $\varphi(x) := [x_0, \dots, x_n]f$, with f an $(n+2)$-convex function, we get

$$\frac{f^{(n)}(\zeta)}{n!} \le [x_0, \dots, x_n]f.$$

In [**1**] several applications of this inequality are given.

4. Derivatives of polynomials. In this section we apply the theory of Majorization to the problem of computing bounds on the derivatives of polynomials. These results are independent of the results of the previous sections.

We recall that the *elementary symmetric functions* are defined as

$$S_k(x) := \sum_{0 \le i_1 < \cdots < i_k \le n} x_{i_1}, \dots, x_{i_k}, \quad S_0(x) \equiv 1.$$

As easily follows from Schur's condition, the functions $S_k(x)$ are increasing and Schur concave for $x_i \ge 0$. Since $-S_k$ is thus decreasing and Schur convex it follows that, for $x, y \in \mathbf{R}^{n+1}$,

$$x \prec^\omega y \Rightarrow -S_k(x) \le -S_k(y),$$

i.e.,

$$S_k(y) \le S_k(x).$$

Let $p(t) := \prod_{i=0}^{n}(t-\xi_i)$, where $\xi_i \le \eta_i$, $i = 0,\dots,n$. We have $p'(t) = \sum_{i=0}^{n}\prod_{j\ne i}(t-\xi_j)$, hence $p(b) = S_n(x)$ and $p'(b) = S_{n-1}(x)$, where $x_i := b - \xi_i$, $i = 0,\dots,n$. In general,

$$\frac{p^{(j)}(b)}{j!} = S_{n-j}(x), \quad j = 0,\dots,n.$$

Now suppose that $q(t) := \prod_{i=0}^{n}(t-\eta_i), \eta_i \le b \;\; i = 0,\dots,n$, and define

$$y_i := b - \eta_i, \quad i = 0,\dots,n.$$

As observed above, if $x \prec^{\omega} y$ then $S_k(y) \le S_k(x)$, that is,

$$q^{(k)}(b) \le p^{(k)}(b), \quad k = 0,\dots,n.$$

A simple calculation shows that $x \prec^{\omega} y$ if

$$\sum_{i=0}^{k}\xi_i \le \sum_{i=0}^{k}\eta_i, \quad k = 0,\dots,n, \tag{8}$$

provided that $b \ge \xi_0 \ge \cdots \ge \xi_n$ and $b \ge \eta_0 \ge \cdots \ge \eta_n$, or, by Lemma 3, if (8) holds and either $\xi \le \eta_{i-1}$, $i = 1,\dots,n$, or $\eta_i \le \xi_{i-1}$, $i = 1,\dots,n$.

We now give an application of this result.

EXAMPLE 4. Let $p(t) := \prod_{i=1}^{n}(t-\eta_i)$ with $1 \ge \eta_1 \ge \cdots \ge \eta_n \ge -1$ and let $T_n(t)$ be the monic Chebyshev polynomial of degree n on $[-1,1]$:

$$T_n(t) := \prod_{i=1}^{n}(t-\xi_i),$$

with $\xi_i = \cos\frac{2i-1}{2n}\pi$, $i = 1,\dots,n$, the zeros of T_n arranged in decreasing order in $(-1,1)$.

Now

$$\sum_{i=1}^{k}\xi_i = \frac{\sin\frac{k\pi}{n}}{2\sin\frac{\pi}{2n}}, \quad k = 1,\dots,n,$$

as readily follows from the identity

$$2\sin\frac{\pi}{2n}\cos\frac{2i-1}{2n}\pi = \sin\frac{i\pi}{n} - \sin\frac{i-1}{n}\pi.$$

Thus we get the following result:

If

$$\sum_{i=1}^{k}\eta_i \geq \frac{\sin\frac{k\pi}{n}}{2\sin\frac{\pi}{2n}}, \quad k=1,\ldots,n,$$

then

$$p(1) \leq T_n(1) = \frac{1}{2^{n-1}}$$

and

$$p^{(k)}(1) \leq T_n^{(k)}(1) = \frac{n^2(n^2-1)\ldots(n^2-(k-1)^2)}{2^{n-1}\cdot 1\cdot 3\ldots(2k-1)}, \quad k=1,\ldots,n.$$

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