ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 19, Number 1, Winter 1989

CHARACTERIZATION OF MEASURES ASSOCIATED WITH ORTHOGONAL POLYNOMIALS ON THE UNIT CIRCLE

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ABSTRACT. Measures with almost everywhere positive absolutely continuous components are described in terms of certain integrals involving the corresponding orthogonal polynomials on the unit circle.

1. The results. Let $d\mu$ be a finite positive Borel measure on the interval $[0, 2\pi)$ such that its support is an infinite set. Let $\{\varphi_n(d\mu)\}$ $(n \in N)$ be the unique system of orthonormal polynomials associated with $d\mu$, that is, the polynomials $\varphi_n(d\mu, z) = \kappa_n z^n + \cdots$ $(\kappa_n = \kappa_n(d\mu) > 0)$ satisfy

$$(2\pi)^{-1} \int_0^{2\pi} \varphi_m(d\mu, e^{i\theta}) \overline{\varphi_n(d\mu, e^{i\theta})} d\mu(\theta) = \delta_{mn} \quad (m \in N, n \in N).$$

In what follows $\phi_n(d\mu)$ denotes the monic orthogonal polynomial, that is, $\phi_n(d\mu) = \kappa_n^{-1}\varphi_n(d\mu)$. The reverse polynomial Π^* of a polynomial Π of degree n is given by $\Pi^*(z) = z^n\overline{\Pi}(z^{-1})$.

The recurrence formula

$$\phi_n(d\mu, z) = z\phi_{n-1}(d\mu, z) + \phi_n(d\mu, 0)\phi_{n-1}^*(d\mu, z), \tag{1.1}$$

$|\phi_n(d\mu, 0)| < 1$, $n = 1, 2, \ldots$, (cf. [**19**, formula (11.4.7), p. 293]) plays an essential role in the theory of orthogonal polynomials and its applications (cf. [**13, 15**]). This recurrence formula fully characterizes the orthogonal polynomials and the associated measures which is the

Recieved by the editor on September 3, 1986.
1980 *Subject Classification*: 42C05.
Key Words and Phrases: Orthogonal Polynomials, Recurrence Formulas.
The material is based upon work supported by the National Science Foundation under Grant No. DMS 84-19525, by the United States Information Agency under Senior Research Fulbright Grant No. 85-41612, by the Catholic University of Louvain in Louvain-La-Neuve, Belgium, by the University of Alberta in Edmonton, Canada, and by NATO.

content of the analogue of J. Favard's theorem for the unit circle according to which any system of polynomials $\{\phi_n\}$ satisfying

$$\phi_n(z) = z\phi_{n-1}(z) + \phi_n(0)\phi^*_{n-1}(z), \quad |\phi_n(0)| < 1,$$

$n = 1, 2, \ldots$, is orthogonal with respect to a measure $d\mu$ which is uniquely determined up to a positive constant factor (cf. [**2**, Theorem 8.1, p. 156]).

It was shown in [**9**] that E.A. Rachmanov's theorem

$$\mu' > 0 \text{ a.e. in } [0, 2\pi) \Rightarrow \lim_{n\to\infty} \phi_n(d\mu, 0) = 0 \tag{1.2}$$

on the reflection coefficients $\phi_n(d\mu, 0)$ (cf. [**17**; 1°, p. 207] and [**18**, Theorem, p. 106]) can be proved in two steps by first proving the existence of an absolute constant K such that

$$|\phi_{n+1}(d\mu, 0)| \le K \int_0^{2\pi} ||\varphi_n(d\mu, e^{i\theta})|^2 |\varphi_{n+1}(d\mu, e^{i\theta})|^{-2} - 1| d\theta \tag{1.3}$$

holds for every measure $d\mu$ and $n \in N$ [**9**; Theorem 2, p. 64], and then by establishing

$$\begin{aligned} &\mu' > 0 \text{ a.e. in } [0, 2\pi) \\ &\Rightarrow \lim_{n\to\infty} \sup_{l\in N} \int_0^{2\pi} ||\varphi_n(d\mu, e^{i\theta})|^2 |\varphi_{n+l}(d\mu, e^{i\theta})|^2 - 1| d\theta = 0 \end{aligned} \tag{1.4}$$

[**9**; Theorem 3, p. 64]. This approach to Rachmanov's theorem has recently proved to be applicable to a variety of situations. For instance, G. López [**5**] used it to solve some problems raised by A.A. Goncar on convergence of multipoint Padé approximations. Another application of this method is given in [**4**] which was subsequently used in [**6**] to solve Freud's conjecture (cf. [**15**]). Rachmanov's theorem (1.2) is the corner stone in generalizing Szegö's theory of orthogonal polynomials (cf. [**8, 10-12**], [**13-16**]). In addition, it is also of great significance in applications of orthogonal polynomials and continued fractions (cf. [**7**]).

On the basis of (1.1) it is easy to show that

$$\begin{aligned} &\lim_{n\to\infty} \phi_n(d\mu, 0) = 0 \Leftrightarrow \lim_{n\to\infty} z\varphi_n(d\mu, z)\varphi_{n+1}(d\mu, z)^{-1} = 1 \\ &\qquad \text{uniformly for } |z| \ge 1 \end{aligned} \tag{1.5}$$

(cf. [**3**, p. 81]). Thus, by (1.3) and (1.5), we have

$$\begin{aligned}&\lim_{n\to\infty}\phi_n(d\mu,0)=0\\ &\iff \lim_{n\to\infty}\int_0^{2\pi}||\varphi_n(d\mu,e^{i\theta})|^2|\varphi_{n+1}(d\mu,e^{i\theta})|^{-2}-1|d\theta=0.\end{aligned}\tag{1.6}$$

One of the main results of the present paper is the following theorem which yields a description of measures analogous to (1.6).

THEOREM 1.1. *For all measures $d\mu$*

$$\begin{aligned}&\mu'>0 \text{ a.e. in } [0,2\pi) \iff\\ &\lim_{n\to\infty}\sup_{l\in N}\int_0^{2\pi}||\varphi_n(d\mu,e^{i\theta})|^2|\varphi_{n+l}(d\mu,e^{i\theta})|^{-2}-1|d\theta=0.\end{aligned}\tag{1.7}$$

In view of (1.4), the implication " $\Leftarrow$ " needs to be proved only. This is accomplished via Corollary 1.5 which follows from our second main result given as

THEOREM 1.2. *The inequality*

$$\begin{aligned}&\int_0^{2\pi}||\varphi_n(d\mu,e^{i\theta})|^2\mu'(\theta)-1|d\theta\\ &\le \sup_{l\in N}\int_0^{2\pi}||\varphi_n(d\mu,e^{i\theta}|^2|\varphi_{n+l}(d\mu,e^{i\theta})|^{-2}-1|d\theta,\end{aligned}\tag{1.8}$$

$n\in N$, holds for every measure $d\mu$.

As to the sharpness of (1.8), we can prove the following

THEOREM 1.3. *Given $d\mu$, the inequality*

$$\begin{aligned}&\sup_{l\in N}\int_0^{2\pi}||\varphi_n(d\mu,e^{i\theta})|^2|\varphi_{n+l}(d\mu,e^{i\theta})|^{-2}-1|d\theta\\ &\le\left\{24\pi\int_0^{2\pi}||\varphi_n(d\mu,e^{i\theta})|^2\mu'(\theta)-1|d\theta\right\}^{1/2}\end{aligned}\tag{1.9}$$

holds for every $n \in N$.

One of the consequences of Theorems 1.2 and 1.3 is that they imply the equivalence of Theorem 3 in [**9**, p. 64] and Corollary 2.2 (formula (2.2)) in [**10**]. Another result which follows from inequality (1.3) and Theorem 1.3 is

COROLLARY 1.4. *There exists a constant* K^* *such that, for every measure* $d\mu$, *the inequality*

$$|\phi_{n+1}(d\mu,0)|^2 \leq K^* \int_0^{2\pi} ||\varphi_n(d\mu,e^{i\theta})|^2\mu'(\theta) - 1|d\theta, \tag{1.10}$$

$n \in N$ *holds.*

COROLLARY 1.5. *For every measure* $d\mu$ *we have the inequality*

$$\begin{aligned} &\operatorname{meas}\{\theta : \theta \in [0,2\pi), \mu'(\theta) = 0\} \\ &\leq \inf_{n\in N} \sup_{l\in N} \int_0^{2\pi} ||\varphi_n(d\mu,e^{i\theta})|^2|\varphi_{n+l}(d\mu,e^{i\theta})|^{-2} - 1|d\theta \end{aligned} \tag{1.11}$$

where $\operatorname{meas}\{A\}$ *denotes the Lebesque measure of the set* A.

We define the singular component $d\mu_s$ of the measure $d\mu$ as the difference between the measure and its absolutely continuous component $(d\mu_s(\theta) = d\mu(\theta) - \mu'(\theta)d\theta)$. It is well known that

$$\begin{aligned} &\mu' > 0 \text{ a.e. in } [0,2\pi) \Rightarrow \\ &\lim_{n\to\infty} \int_0^{2\pi} |\varphi_n(d\mu,e^{i\theta})|^2 d\mu_s(\theta) = 0 \end{aligned} \tag{1.12}$$

(cf. [**10**, Corollary 5.1]). Theorems 1.1 and 1.2 explain the reasons that lie behind (1.12) via the following

COROLLARY 1.6. *For every measure* $d\mu$

$$\begin{aligned} &\int_0^{2\pi} |\varphi_n(d\mu,e^{i\theta})|^2 d\mu_s(\theta) \\ &\leq \sup_{l\in N} \int_0^{2\pi} ||\varphi_n(d\mu,e^{i\theta}|^2|\varphi_{n+l}(d\mu,e^{i\theta})|^{-2} - 1|d\theta \end{aligned} \tag{1.13}$$

holds for $n \in N$.

It is not clear how the results of this paper can be carried over to orthogonal polynomials on the real line. For instance, we do not know whether inequality (1.3) has an appropriate analogue for the recursion coefficients associated with real orthogonal polynomials. It is probable that if such analogues exist then they will involve Turán type determinants (cf. [**10**], [**15**]).

2. The proofs.

PROOF OF THEOREM 1.2. Let $m \in N$, and let H_m be a trigonometric polynomial of degree at most m. Since the first n moments of $|\varphi_n(d\mu)|^{-2}d\theta$ equal those of $d\mu$ (cf. [**1**; Theorem 5.2.2, p. 198]), we have

$$\begin{aligned}
&\int_0^{2\pi} [|\varphi_n(d\mu, e^{i\theta})|^2\mu'(\theta) - 1]H_m(\theta)d\theta \\
&= -\int_0^{2\pi} |\varphi_n(d\mu, e^{i\theta})|^2 H_m(\theta)d\mu_s(\theta) \\
&\quad + \int_0^{2\pi} [|\varphi_n(d\mu, e^{i\theta})|^2|\varphi_{n+l}(d\mu, e^{i\theta})|^{-2} - 1]H_m(\theta)d\theta
\end{aligned}$$

for all $l \geq m$. Therefore we obtain

$$\begin{aligned}
&\left|\int_0^{2\pi} [|\varphi_n(d\mu, e^{i\theta})|^2\mu'(\theta) - 1]H_m(\theta)d\theta\right| \\
&\leq \left|\int_0^{2\pi} |\varphi_n(d\mu, e^{i\theta})|^2 H_m(\theta)d\mu_s(\theta)\right| \\
&\quad + ||H_m||_\infty \sup_{l \in N}\int_0^{2\pi} ||\varphi_n(d\mu, e^{i\theta})|^2|\varphi_{n+l}(d\mu, e^{i\theta})|^{-2} - 1|d\theta,
\end{aligned}$$

and since the set of all trigonometrical polynomials is dense in $C_{2\pi}$, we also have

$$
\begin{aligned}
&\left|\int_0^{2\pi} [|\varphi_n(d\mu, e^{i\theta})|^2\mu'(\theta) - 1]f(\theta)d\theta\right| \\
(2.1)\qquad &\leq \left|\int_0^{2\pi} |\varphi_n(d\mu, e^{i\theta})|^2 f(\theta)d\mu_s(\theta)\right| \\
&\quad + ||f||_\infty \sup_{l\in N}\int_0^{2\pi} ||\varphi_n(d\mu, e^{i\theta})|^2|\varphi_{n+l}(d\mu, e^{i\theta})|^{-2} - 1|d\theta
\end{aligned}
$$

for every $f \in C_{2\pi}$. According to [**9**; Lemma 1, p. 65], given the singular component $d\mu_s$ of the measure $d\mu$, there exists a sequence of 2π-periodic continuous functions $\{h_k\}$ such that $0 < h_k(\theta) \leq 1$ ($k \in N$, $\theta \in [0, 2\pi]$), $\lim h_k(\theta) = 1 (k \to \infty)$ almost everywhere in $[0, 2\pi)$, and

$$\lim_{k\to\infty}\int_0^{2\pi} |\varphi_n(d\mu, e^{i\theta})|^2 h_k(\theta)d\mu_s(\theta) = 0.$$

Applying (2.1) with (fh_k) instead of f, and letting $k \to \infty$, we obtain

$$
\begin{aligned}
&\left|\int_0^{2\pi} [|\varphi_n(d\mu, e^{i\theta})|^2\mu'(\theta) - 1]f(\theta)d\theta\right| \\
(2.2)\qquad &\leq ||f||_\infty \sup_{l\in N}\int_0^{2\pi} ||\varphi_n(d\mu, e^{i\theta})|^2|\varphi_{n+l}(d\mu, e^{i\theta})|^{-2} - 1|d\theta,
\end{aligned}
$$

initially for all 2π-periodic continuous functions f, but then, by approximation arguments, (2.2) can be extended to every $f \in L^\infty[0, 2\pi]$ as well. Theorem 1.2 clearly follows from inequality (2.2) applied with the special choice of $f = \operatorname{sign}[|\varphi_n(d\mu)|^2\mu' - 1]$. □

PROOF OF THEOREM 1.3. Let $l \in N$ and $n \in N$. Then

$$\int_0^{2\pi} |\varphi_n(d\mu, e^{i\theta})|^2|\varphi_{n+l}(d\mu, e^{i\theta})|^{-2}d\theta = 2\pi$$

(cf. [**1**; Theorem 5.2.2, p. 198]), and consequently, by Schwarz's inequality, we obtain

$$
\begin{aligned}
&\int_0^{2\pi} ||\varphi_n(d\mu,e^{i\theta})|^2|\varphi_{n+l}(d\mu,e^{i\theta})|^{-2}-1|d\theta \\
&\le \left\{\int_0^{2\pi}[|\varphi_n(d\mu,e^{i\theta})||\varphi_{n+l}(d\mu,e^{i\theta})|^{-1}-1]^2 d\theta\right\}^{1/2} \\
(2.3)\qquad &\times\left\{\int_0^{2\pi}[|\varphi_n(d\mu,e^{i\theta})||\varphi_{n+l}(d\mu,e^{i\theta})|^{-1}+1]^2 d\theta\right\}^{1/2} \\
&= 2\left\{4\pi^2-\left[\int_0^{2\pi}|\varphi_n(d\mu,e^{i\theta})||\varphi_{n+l}(d\mu,e^{i\theta})|^{-1}d\theta\right]^2\right\}^{1/2}.
\end{aligned}
$$

The next step consists of estimating the expression in the brackets on the right-hand side of (2.3) from below. We accomplish this by applying Hölder's inequality as outlined in [**9**; Formula (11), p. 67]. We have

$$
\begin{aligned}
&\int_0^{2\pi}[|\varphi_n(d\mu,e^{i\theta})|^2\mu'(\theta)]^{1/3}d\theta \\
&= \int_0^{2\pi}[|\varphi_n(d\mu,e^{i\theta})||\varphi_{n+l}(d\mu,e^{i\theta})|^{-1}]^{1/2} \\
&\qquad \times_1 |\varphi_{n+l}(d\mu,e^{i\theta})|^2\mu'(\theta)]^{1/4}\times[|\varphi_n(d\mu,e^{i\theta})|^2\mu'(\theta)]^{1/12}d\theta,
\end{aligned}
$$

and thus Hölder's inequality with indices (1/2, 1/4, 1/4) yields

$$
\begin{aligned}
&\int_0^{2\pi}[|\varphi_n(d\mu,e^{i\theta})|^2\mu'(\theta)]^{1/3}d\theta \\
&\le\left\{\int_0^{2\pi}|\varphi_n(d\mu,e^{i\theta})||\varphi_{n+l}(d\mu,e^{i\theta})|^{-1}d\theta\right\}^{1/2} \\
(2.4)\qquad &\times\left\{\int_0^{2\pi}|\varphi_{n+l}(d\mu,e^{i\theta})|^2\mu'(\theta)d\theta\right\}^{1/4} \\
&\times\left\{\int_0^{2\pi}[|\varphi_n(d\mu,e^{i\theta})|^2\mu'(\theta)]^{1/3}d\theta\right\}^{1/4}.
\end{aligned}
$$

Since the polynomials are orthonormal, the second integral on the righthand side of (2.4) is at most 2π. Therefore, assuming that μ' does not vanish almost everywhere in $[0,2\pi)$, we obtain

$$
\begin{aligned}
(2.5)\qquad &(2\pi)^{-1}\left[\int_0^{2\pi}[|\varphi_n(d\mu,e^{i\theta})|^2\mu'(\theta)]^{1/3}d\theta\right]^3 \\
&\le\left[\int_0^{2\pi}|\varphi_n(d\mu,e^{i\theta})||\varphi_{n+l}(d\mu,e^{i\theta})|^{-1}d\theta\right]^2.
\end{aligned}
$$

If $\mu' = 0$ almost everywhere in $[0, 2\pi)$, then (2.5) is obviously satisfied as well. Applying inequality (2.5) to estimate the expression in the brackets on the right-hand side of (2.3) we get

$$\begin{aligned}
&\int_0^{2\pi} ||\varphi_n(d\mu, e^{i\theta})|^2 |\varphi_{n+l}(d\mu, e^{i\theta})|^{-2} - 1| d\theta \\
&\le 4\pi \left\{1 - \left[(2\pi)^{-1} \int_0^{2\pi} [|\varphi_n(d\mu, e^{i\theta})|^2 \mu'(\theta)]^{1/3} d\theta\right]^3\right\}^{1/2},
\end{aligned} \tag{2.6}$$

and factoring the expression $1 - [\ \cdot\ ]^3$ in the braces on the right-hand side of (2.6) yields

$$\begin{aligned}
&\int_0^{2\pi} ||\varphi_n(d\mu, e^{i\theta})|^2 |\varphi_{n+l}(d\mu, e^{i\theta})|^{-2} - 1| d\theta \\
&\le 4\pi \left\{1 - (2\pi)^{-1} \int_0^{2\pi} [|\varphi_n(d\mu, e^{i\theta})|^2 \mu'(\theta)]^{1/3} d\theta\right\}^{1/2} \\
&\qquad \times \left\{1 + \left[(2\pi)^{-1} \int_0^{2\pi} [|\varphi_n(d\mu, e^{i\theta})|^2 \mu'(\theta)]^{1/3} d\theta\right]\right. \\
&\qquad\qquad \left. + \left[(2\pi)^{-1} \int_0^{2\pi} [|\varphi_n(d\mu, e^{i\theta})|^2 \mu'(\theta)]^{1/3} d\theta\right]^2\right\}^{1/2}.
\end{aligned} \tag{2.7}$$

Since the $\varphi_n(d\mu)$ are orthonormal, Hölder's inequality shows that the expression in the last two brackets in the right-hand side of (2.7) is at most 1. Thus

$$\begin{aligned}
&\int_0^{2\pi} ||\varphi_n(d\mu, e^{i\theta})|^2 |\varphi_{n+l}(d\mu, e^{i\theta})|^{-2} - 1| d\theta \\
&\le 4\pi 3^{1/2} \left\{1 - (2\pi)^{-1} \int_0^{2\pi} [|\varphi_n(d\mu, e^{i\theta})|^2 \mu'(\theta)]^{1/3} d\theta\right\}^{1/2} \\
&= 4\pi 3^{1/2} \left\{(2\pi)^{-1} \int_0^{2\pi} \left[1 - [|\varphi_n(d\mu, e^{i\theta})|^2 \mu'(\theta)]^{1/3}\right] d\theta\right\}^{1/2} \\
&\le 4\pi 3^{1/2} \left\{(2\pi)^{-1} \int_0^{2\pi} |1 - [|\varphi_n(d\mu, e^{i\theta})|^2 \mu'(\theta)]^{1/3}| d\theta\right\}^{1/2}.
\end{aligned} \tag{2.8}$$

Finally, since $|1 - [|\varphi_n(d\mu)|^2 \mu']^{1/3}| \le |1 - [|\varphi_n(d\mu)|^2 \mu']|$, formula (1.9) follows immediately from inequality (2.8). □

PROOF OF COROLLARY 1.5. We have

$$\operatorname{meas}\{\theta : \theta \in [0,2\pi), \mu'(\theta) = 0\} = \int_{\mu'(\theta)=0\ \theta\in[0,2\pi)} [1-|\varphi_n(d\mu, e^{i\theta})|^2\mu'(\theta)]d\theta,$$

and therefore (1.11) follows from (1.8). □

PROOF OF COROLLARY 1.6. In view of the identity

$$\int_0^{2\pi} |\varphi_n(d\mu, e^{i\theta})|^2 d\mu_s(\theta) = \int_0^{2\pi} [1 - |\varphi_n(d\mu, e^{i\theta})|^2\mu'(\theta)]d\theta,$$

inequality (1.13) is an immediate consequence of (1.8).

Acknowledgement. The author expresses his sincere gratitude to Doron S. Lubinsky who made numerous suggestions regarding the presentation.

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