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APPROXIMATION BY CHENEY-SHARMA-KANTOROVIČ POLYNOMIALS IN THE L_p-METRIC

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1. Properties of CSB-polynomials. Based on the identity
(1.1)

$$\sum_{k=0}^{n} p_{nk}(x;\beta) := (1+n\beta)^{-n} \sum_{k=0}^{n} \binom{n}{k} x(x+k\beta)^{k-1}[1-x+(n-k)\beta]^{n-k} = 1,$$

$x \in I := [0,1], \beta \in \mathbf{R}, n \in \mathbf{N}$, a partition of unity originating from a more general identity of Jensen [**6**], Cheney and Sharma [**1**] associated with a bounded function $f : I \to \mathbf{R}$ the polynomial

$$(1.2) \qquad (P_{n,\beta} f)(x) := \sum_{k=0}^{n} p_{nk}(x;\beta) f\Big(\frac{k}{n}\Big)$$

of degree n, depending on a parameter β and reducing to the n-th Bernstein polynomial for $\beta = 0$. We shall refer to it as the n-th Cheney-Sharma-Bernstein polynomial (briefly: CSB-polynomial). The CSB-operators $P_{n,\beta}$ defined by (1.2) are positive, linear, polynomial and preserve, due to (1.1), constant functions. In [**1**] it is proved that the sequence $(P_{n,\beta})_{n \in \mathbf{N}}$ gives a positive polynomial approximation method on the space $C(I), || \cdot ||_\infty$ (i.e. $\lim_{n\to\infty} ||f - P_{n,\beta} f||_\infty = 0$ for all $f \in C(I)$) if the parameters β are chosen to be nonnegative and are coupled with n (i.e. $\beta = \beta_n$) in such a way that

$$(1.3) \qquad n\beta_n \to 0 \text{ for } n \to \infty.$$

Using estimates in [**1**] it can easily be shown that

$$(1.4) \qquad (P_{n,\beta_n} t)(x) = x + o\Big(\frac{1}{n}\Big),$$

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$$(1.5)\qquad (P_{n,\beta_n}t^2)(x) = x^2 + \frac{x(1-x)}{n} + o\Big(\frac{1}{n}\Big)$$

pointwise for $x \in I$ and $n \to \infty$ if (1.3) is satisfied. By an argument similar to that given in the proof of Theorem 5 it can be shown moreover that, pointwise for $x \in I$,

$$(1.6)\qquad n(P_{n,\beta_n}(t-x)^4)(x) \to 0 \text{ for } n \to \infty$$

if (1.3) is replaced by the stronger coupling

$$(1.7)\qquad n^2\beta_n \to c \quad (c > 0) \text{ for } n \to \infty.$$

Utilizing (1.4), (1.5) and (1.6) we have, by Mamedov's theorem [**9**], the following Voronovskaja-theorem for CSB-polynomials: If f is bounded on I and possesses a second derivative at a point x and if (1.7) is satisfied, then

$$(1.8)\qquad P_{n,\beta_n}f(x) - f(x) = \frac{x(1-x)}{2n}f''(x) + o\Big(\frac{1}{n}\Big)(n \to \infty).$$

This formula is the same as for Bernstein polynomials and corrects a result contained in [**1**].

2. $\mathbf{L_p}$-approximation by CSK-polynomials. CSB-polynomials are not suitable for the approximation of functions $f \in L_p(I)$, $1 \le p \le \infty$, in the L_p-metric. According to an idea of Kantorovič the point evaluations of f in (1.2) are replaced by integral means over suitable small and disjoint intervals around the knots leading to the polynomial of degree n

$$(2.1)\qquad (A_{n,\beta}f)(x) := (n+1)\sum_{k=0}^{n}\Big(\int_{I_k} f(t)dt\Big)p_{nk}(x;\beta),$$

where $I_k := \Big[\frac{k}{n+1}, \frac{k+1}{n+1}\Big]$. Since this polynomial reduces to the n-th Kantorovič polynomial for $\beta = 0$ we shall refer to it as the n-th Cheney-Sharma-Kantorovič polynomial (briefly: CSK-polynomial). These polynomials have been introduced by Habib and Umar as generalized Bernstein polynomials and studied in two subsequent papers [**2**], [**3**]. However their statements are mostly incorrect and fragmentary [**13**].

This motivates a new and systematic treatment.

The CSK-operators $A_{n,\beta}$ defined by (2.1) for $f \in L_p(I)$, $1 \le p \le \infty$, are positive, linear, polynomial and preserve, due to (1.1), constant functions. We write $A_{n,\beta}f$ as a singular integral of the type

$$A_{n,\beta}f(x) = \int_0^1 H_{n,\beta}(x,t)f(t)dt$$

with the positive kernel

$$H_{n,\beta}(x,t) = (n+1)\sum_{k=0}^{n} p_{nk}(x;\beta)1_{I_k}(t),$$

where $1_{I_k}(t)$ denotes the characteristic function of the interval I_k with respect to I. Utilizing the estimate

$$\int_0^1 p_{nk}(x;\beta)dx \le (1+n\beta)\binom{n}{k}\int_0^1 z^k(1-z)^{n-k}dz = \frac{1+n\beta}{n+1}$$

we have, for all n and x or t respectively,

$$\int_0^1 H_{n,\beta}(x,t)\,dt = \sum_{k=0}^{n} p_{nk}(x;\beta) = 1,$$
$$\int_0^1 H_{n,\beta}(x,t)\,dx \le (n+1)\sum_{k=0}^{n}\frac{1+n\beta}{n+1}1_{I_k}(1) = 1+n\beta$$

and thus by a theorem of Orlicz [**10**] the operator norm $||A_{n,\beta}||_p$ is bounded by $1+n\beta$. If $(A_{n,\beta_n})_{n\in\mathbf{N}}$ is a sequence of CSK-operators with nonnegative parameters satisfying (1.3) then the corresponding sequence of operator norms is hence bounded by some constant $C > 1$. For $f \in C(I)$ and arbitrary $x \in I$ we easily obtain

$$|A_{n,\beta_n}f(x) - P_{n,\beta_n}f(x)| \le \omega_1\Big(f;\frac{1}{n+1}\Big)_\infty,$$

where $\omega_1(f;\cdot)_\infty$ denotes the ordinary modulus of continuity of f with respect to the sup-norm, and consequently

$$
\begin{aligned}
&||A_{n,\beta_n}f - P_{n,\beta_n}f||_p \leq ||A_{n,\beta_n}f - P_{n,\beta_n}f||_\infty \\
&\leq \omega_1\Big(f;\frac{1}{n+1}\Big)_\infty = o(1) \quad (n\to\infty).
\end{aligned} \tag{2.2}
$$

Now

$$
\begin{aligned}
&||f - A_{n,\beta_n}f||_p \leq ||A_{n,\beta_n}f - P_{n,\beta_n}f||_p + ||P_{n,\beta_n}f - f||_p \\
&\leq \omega_1\Big(f;\frac{1}{n+1}\Big)_\infty + ||P_{n,\beta_n}f - f||_\infty = o(1)\ (n\to\infty)
\end{aligned}
$$

holds on account of (2.2) and the fact that $(P_{n,\beta_n})_{n\in\mathbf{N}}$ is a linear approximation method on the space $(C(I), ||\cdot||_\infty)$. Since this space is dense in $L_p(I)$ with respect to the L_p-norm and $||A_{n,\beta_n}||_p \leq C$ for $n \in \mathbf{N}$ we have proved the following

THEOREM 1. *If* $n\beta_n \to 0$ *for* $n\to\infty$, *then*

$$\lim_{n\to\infty} ||f - A_{n,\beta_n}f||_p = 0 \tag{2.3}$$

for all $f \in L_p(I)$, $1 \leq p \leq \infty$.

As an application of this theorem we obtain the following criterion of compactness for a bounded subset

$$K := \{f \in L_p(I) |\ ||f||_p \leq M,\ M \text{ a positive constant}\}$$

of $L_p(I)$: K is compact with respect to the L_p-norm if and only if (2.3) holds uniformly for all $f \in K$.

The proof of this criterion proceeds just along the lines of an argument given by G.G. Lorentz [**5**, p. 33] for Kantorovič polynomials.

3. Degree of $\mathbf{L_p}$-approximation by CSK-polynomials. Long and tedious calculations (see [**13**]) using estimates in [**1**] show that

$$(A_{n,\beta_n}t)(x) = x + \frac{1-2x}{2n} + o\Big(\frac{1}{n}\Big), \tag{3.1}$$

$$(3.2) \qquad (A_{n,\beta_n}t^2)(x) = x^2 + \frac{x(2-3x)}{n} + o\Big(\frac{1}{n}\Big)$$

pointwise for $x \in I$ and $n \to \infty$ if (1.3) is satisfied.

We start with the approximation of functions belonging to the Sobolev spaces $L_p^r(I) := \{f^{(r-1)} \in AC(I) | f^{(r)} \in L_p(I)\}$, $r = 1, 2$, $1 \le p \le \infty$, which are smooth subspaces of $L_p(I)$.

THEOREM 2. *If $n\beta_n \to 0$ for $n \to \infty$, then*

$$||f - A_{n,\beta_n}f||_p \le \frac{C}{\sqrt{n}}||f'||_p, \quad n \in \mathbf{N}, \quad n \ge n_0,$$

for all $f \in L_p^1(I)$, $1 \le p \le \infty$, where C is some positive constant.

PROOF. We apply the following very remarkable quantitative result of V.A. Popov [**12**] on positive linear operators mapping the space $M(I)$ of bounded and measurable functions on I into itself and preserving constant functions: If

$$(Lt)(x) = x + \alpha(x), \ (Lt^2)(x) = x^2 + \beta(x)$$

and

$$M := \sup_{x \in I} |\beta(x) - 2x\alpha(x)| \le 1,$$

then

$$(3.3) \qquad ||g - Lg||_p \le B\tau_1(g; \sqrt{M})_p, \quad g \in M(I), \quad 1 \le p \le \infty.$$

Here B is some positive constant and $\tau_1(g;\delta)_p$ denotes the first order τ-modulus of g with step size δ in the L_p-metric given by

$$\tau_1(g;\delta)_p := ||\omega_1(g, \cdot; \delta)||_p,$$

where

$$\omega_1(g, x; \delta) := \sup\{|g(t+h) - g(t)| : t, t+h \in [x - \delta/2, \ x + \delta/2] \cap I\}.$$

The following two properties of this modulus of smoothness will be needed ([12]):

$$\tau_1(g;\lambda\delta)_p \le (2]\lambda[+2)^2\tau_1(g;\delta)_p, \quad \lambda \in \mathbf{R}^+, \tag{3.4}$$

$$\tau_1(g;\delta)_p \le \delta\|g'\|_p, \quad g \in L_p^1(I). \tag{3.5}$$

For $L = A_{n,\beta_n}$, we derive immediately from (3.1), (3.2) that

$$M = \frac{1}{n}\max_{x\in I} x(1-x) + o\left(\frac{1}{n}\right) \le \frac{A}{n}, \quad n \in \mathbf{N}$$

(A a suitable positive real constant) if (1.3) is satisfied. In view of (3.3) and (3.5) we have therefore, for all $f \in L_p^1(I)$ and almost all $n \in \mathbf{N}$ (say $n \ge n_0$),

$$\begin{aligned}\|f - A_{n,\beta_n}f\|_p &\le B\tau_1\left(f;\sqrt{\frac{A}{n}}\right)_p \\ &\le (2]\sqrt{A}[+2)^2 B\tau_1\left(f;\frac{1}{\sqrt{n}}\right)_p \le \frac{C}{\sqrt{n}}\|f'\|_p\end{aligned}$$

(C a positive real constant), which completes the proof. □

A quite different measure for the smoothness of functions is the first order K-functional of J. Peetre [**11**] which is, for $g \in L_p(I)$, $1 \le p \le \infty$ (with $g \in C(I)$ for $p = \infty$), defined by

$$K_{1,p}(t;g) := \inf_{h\in L_p^1(I)}(\|g-h\|_p + t\|h'\|_p) \quad (t > 0) \tag{3.6}$$

and which is equivalent to the usual first order ω-modulus of g in the L_p-metric, i.e., there are constants $c_1 > 0$ and $c_2 > 0$ independent of g and p such that

$$c_1\omega_1(g;t)_p \le K_{1,p}(t;g) \le c_2\omega_1(g;t)_p \quad (t > 0). \tag{3.7}$$

Combining (3.6), (3.7) and Theorem 2 by a smoothing argument in a similar way to what we have done in [**8**, p. 246] for Kantorovič polynomials we obtain the following upper bound for the degree of L_p-approximation of nonsmooth functions by our method.

THEOREM 3. *If $n\beta_n \to 0$ for $n \to \infty$, then*

$$\|f - A_{n,\beta_n} f\|_p \le M\omega_1\Big(f; \frac{1}{\sqrt{n}}\Big)_p, \quad n \in \mathbf{N}, \tag{3.8}$$

for all $f \in L_p(I)$, $1 \le p \le \infty$, where M is some positive constant. Especially if $f \in \mathrm{Lip}(\alpha, L_p)$, $0 < \alpha \le 1$, then

$$\|f - A_{n,\beta_n} f\|_p = o(n^{-\alpha/2}) \quad (n \to \infty).$$

REMARK. Since the second order ω-modulus has the property $w_2(g;\delta)_p \le \delta^2\|g''\|_p$ for $g \in L_p^2(I)$, $1 \le p \le \infty$ (with $g \in C^2(I)$ for $p = \infty$) and since $(A_{n,\beta_n} t)(x) \ne x$ for all $n \in \mathbf{N}$, $x \in I$, an estimate of the type (3.8) with $\omega_1(f;\cdot)_p$ replaced by $\omega_2(f;\cdot)_p$ cannot exist.

The following theorem shows that the degree of approximation can be $o(n^{-1})$ for suitable subspaces of $L_p(I)$.

THEOREM 4. *If $n\beta_n \to 0$ for $n \to \infty$, then*

$$\|f - A_{n,\beta_n} f\|_p \le \frac{C_p}{n}[\|f'\|_p + \|f''\|_p], \quad n \in \mathbf{N}, \tag{3.9}$$

for all $f \in L_p^2(I)$, $p > 1$, where C_p is a positive real constant depending only on p.

PROOF. Fix $x \in I$ and $n \in \mathbf{N}$. Then

$$E(x) := A_{n,\beta_n} f(x) - f(x) = \int_0^1 H_{n,\beta_n}(x,t)[f(t) - f(x)]\, dt.$$

From

$$f(t) - f(x) = (t-x)f'(x) + (t-x)\int_x^\xi f''(u)\, du$$

for arbitrary $t \in I$ and $\xi = \xi(t)$ between x and t we obtain

$$E(x) := f'(x)A_{n,\beta_n}(t-x)(x) + \int_0^1 H_{n,\beta_n}(x,t)(t-x)\Big\{\int_x^\xi f''(u)\,du\Big\}\,dt.$$

Because of

$$|A_{n,\beta_n}(t-x)^i(x)| \le \frac{A_i}{n}, \quad x \in I, \quad n \in \mathbf{N}, \quad i \in \{1,2\}$$

(A_i positive real constants independent of n and x) being an immediate consequence of (3.1) and (3.2), there follows

$$\begin{aligned}|E(x)| &\le \frac{A_1}{n}|f'(x)| + \int_0^1 H_{n,\beta_n}(x,t)(t-x)^2 \left|\sup_{\substack{t\in I\\ t\neq x}} \frac{1}{t-x}\int_x^t |f''(u)|\,du\right| \\ &= \frac{A_1}{n}|f'(x)| + \theta_{f''}(x)A_{n,\beta_n}(t-x)^2(x) \le \frac{C}{n}(|f'(x)| + \theta_{f''}(x)),\end{aligned}$$

where $C := \max(A_1, A_2)$ and

$$\theta_{f''}(x) := \sup_{\substack{t\in I\\ t\neq x}} \frac{1}{t-x}\int_x^t |f''(u)|\,du, \quad x \in I,$$

is the Hardy-Littlewood majorant of f'' on I. For $p > 1$ it is known that $f'' \in L_p(I)$ implies $\theta_{f''} \in L_p(I)$ and

$$\text{(3.10)} \qquad \int_0^1 \theta^p_{f''}(x)\,dx \le 2\Big(\frac{p}{p-1}\Big)^p \int_0^1 |f''(x)|^p\,dx$$

(cf. [**14**, Theorem 13.15]). Applying Minkowski's inequality to the last inequality for $|E(x)|$ and taking into account (3.10), we obtain

$$\begin{aligned}||f - A_{n,\beta_n}f||_p &\le \frac{C}{n}\Big[||f'||_p + \frac{p\sqrt{2}}{p-1}||f''||_p\Big] \\ &\le C\sqrt{2}\frac{p}{p-1}\,\frac{1}{n}[||f'||_p + ||f''||_p] =: \frac{C_p}{n}[||f'||_p + ||f''||_p],\end{aligned}$$

which completes the proof. □

REMARK. For $p = 1$ the above proof breaks down. This case is left as an open problem.

4. The Voronovskaja theorem for CSK-polynomials. When determining the asymptotic form of CSB-approximation Cheney and Sharma [**1**] had to replace the coupling (1.3) by the stronger condition (1.7). As the proof of the following theorem shows (1.7) is indispensable in the case of CSK-approximation, too, a fact which has not been taken into account by Habib and Umar [**3**, Theorem 1.2].

THEOREM 5. *If $n^2\beta_n \to c(c > 0)$ for $n \to \infty$ and if $f \in L_1(I)$ possesses a second derivative at a point x, then*

$$A_{n.\beta_n}f(x) - f(x) = \frac{(1-2x)f'(x) + x(1-x)f''}{x/(2n)} + o\Big(\frac{1}{n}\Big)\ (n \to \infty).$$

REMARK. This formula is the same as for Kantorovič polynomials (cf. [**7**]).

PROOF. In order to be able to apply Mamedov's theorem we show that $I := n(A_{n.\beta_n}(t-x)^4)(x) \to o(n \to \infty)$. Now

$$\begin{aligned} I &= n(n+1)\sum_{k=0}^{n}\Big(\int_{I_k}(t-x)^4dt\Big)p_{nk}(x;\beta_n) \\ &= n(n+1)\Big[\sum_{|\frac{k}{n}-x|<n^{-\alpha}} + \sum_{|\frac{k}{n}-x|\geq n^{-\alpha}}\Big] =: I_1 + I_2, \end{aligned}$$

where $\frac{1}{4} < \alpha < \frac{1}{2}$.

For $k \in \mathbf{N}$ with $|\frac{k}{n} - x| < n^{-\alpha}$ there holds $|\frac{k}{n+1} - x| < 2n^{-\alpha}$ and from this we easily deduce

$$|I_1| = o(n^{1-4\alpha})\sum_{k=0}^{n} p_{nk}(x;\beta_n) = o(n^{1-4\alpha}),$$

$$\begin{aligned}|I_2| &\le n \sum_{|\frac{k}{n-x}|\ge n^{-\alpha}} p_{nk}(x;\beta_n)\\ &\le n(1+n\beta_n)^{-n} \sum_{|\frac{k}{n}-x|\ge n^{-\alpha}} \binom{n}{k}(x+n\beta_n)^k[1-x+n\beta_n]^{n-k}\\ &= \Big(\frac{1+2n\beta_n}{1+n\beta_n}\Big)^n \sum_{|\frac{k}{n}-x|\ge n^{-\alpha}} \binom{n}{k}\Big(\frac{x+n\beta_n}{1+2n\beta_n}\Big)^k\Big(1-\frac{x+n\beta_n}{1+2n\beta_n}\Big)^{n-k}.\end{aligned}$$

Substituting $y_n := \frac{x+n\beta_n}{1+2n\beta_n}$ we observe that $0 \le y_n \le 1$ and $y_n = x + 0(n^{-1})$ since $n^2\beta_n \to c$. Thus $|\frac{k}{n} - y_n| \ge \frac{1}{2}n^{-\alpha}$ for sufficiently large n, say for $n \ge n_0$. (1.7) implies further on that $(\frac{1+2n\beta_n}{1+n\beta_n})^n$ is bounded by e^c. Then, for $n \ge n_0$,

$$|I_2| \le e^c \sum_{|\frac{k}{n}-y_n|\ge\frac{1}{2}n^{-\alpha}} \binom{n}{k} y_n^k(1-y_n)^{n-k}$$

which is of order $o(n^{-s})$ for each $s > 0$, by inequality 1.5 (8) in [**5**]. Mamedov's theorem together with (3.1), (3.2) completes the proof. □

REMARK. The estimate of I_2 in the above proof uses certain ideas from [**4**].

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