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ON THE RIEMANN CONVERGENCE OF POSITIVE LINEAR OPERATORS

H. MEVISSEN, R.J. NESSEL AND E. VAN WICKEREN

ABSTRACT. Inspired by work of Pólya (1933) on the convergence of quadrature formulas, we previously introduced a concept of sequential convergence in the space of Riemann integrable functions under which this classical space is the completion of continuous functions. It is the purpose of this paper to continue the discussion of its consequences to approximation theory by extending the (qualitative as well as quantitative) Bohman-Korovkin theorem from continuous to Riemann integrable functions.

1. Riemann convergence. Let $B = B[a,b]$ be the space of functions, everywhere defined and bounded on the compact interval $[a,b]$, and consider the subspaces $C = C[a,b]$ and $R = R[a,b]$ of functions, continuous and Riemann integrable, respectively. Obviously, $C \subset R \subset B$, and under the norm

$$||f|| = ||f||_B := \sup\{|f(u)| : u \in [a,b]\}$$

each of these spaces, thus in particular R, is a Banach space, in other words, each of them is complete under uniform convergence. But this norm is not appropriate for approximation in R since, e.g., polynomials are only dense in C. This disadvantage is avoided by the following concept of sequential convergence, in fact even well-defined in B.

DEFINITION. *A sequence $\{f_n\} \subset B$ is called (Riemann) R-convergent to $f \in B$ (in notation: $\mathbf{R} - \lim f_n = f$) if, for $n \to \infty$,*

$$(1) \qquad ||f_n||_B = O(1),$$

$$(2) \qquad \overline{\int} \sup_{k \geq n} |f_k - f| = o(1),$$

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where $\overline{\int} f = \overline{\int}_{[a,b]} f(u)\,du$ *denotes the upper Riemann integral of* f *over* $[a, b]$.

Obviously, uniform convergence implies R-convergence. On the other hand, to compare R-convergence with the usual ones of Lebesgue theory, first of all note that the integrand in (2) is monotone in n - this is characteristic: It compensates the missing countable additivity of the algebra of Riemann measurable sets. Moreover, in many proofs the equiboundedness (1) serves as a substitute for arguments based upon dominated convergence in Lebesgue theory. Obviously, R-convergence as induced on R in particular implies $\int |f_n - f| = o(1)$, thus convergence in L^1-norm. In fact, (1), (2) strengthen L^1-convergence on R to ensure that limits continue to stay in R (cf. Theorem 1 (a)). Moreover, R-convergence implies pointwise convergence Lebesgue almost everywhere and is related to convergence in measure inasmuch as (1), (2) are equivalent to (1) together with the fact that, for each $\varepsilon > 0$,

$$\overline{\mu}(\{u \in [a, b] : \sup_{k \geq n} |f_k(u) - f(u)| \geq \varepsilon\}) = o(1), \tag{2*}$$

$\overline{\mu}$ denoting the outer Jordan content. Let us, however, emphasize that embedding into Lebesgue integration does not seem to be suitable since, in view of the applications (e.g., to quadrature rules), one is interested to have point evaluation functionals well-defined everywhere. In fact, they are not bounded on L^1, even when restricted to continuous functions.

Obviously, each R-convergent sequence is an R-Cauchy sequence, that is, satisfies (1) and

$$\lim_{n \to \infty} \overline{\int} \sup_{j,k \geq n} |f_j - f_k| = 0.$$

In fact, the converse is valid as well. In other words, B is (sequentially) R-complete. As already mentioned, it is useful to have Riemann convergence well-defined on the whole set B, but it turns out to be particularly significant when considered on the subset R. It is therefore essential that R also is R-complete. From the point of view of approximation the most important feature then is that the R-closure of standard classes of smooth functions yields R (and not B). Indeed, THEOREM 1.

(a) $R[a,b]$ *is (sequentially) R-complete.*

(b) $C[a,b]$ *is R-dense in* $R[a,b]$.

For proofs see [**2**]. Summarizing, Theorem 1 finally justifies the terminology: It is R which not only is R-complete, but, e.g., polynomials are (B-dense in C and therefore) R-dense in R. Of course one may now ask for further topological properties. For example, the structure of the two conditions (1), (2) is strongly related to two-norm convergence or Saks spaces. In fact, the following question is of particular interest: Is there even a metrical structure which generates R-convergence? In the meantime this question is answered in the negative.

The R-density of polynomials may now be considered as the starting point for a discussion of approximation in R. In [**2**] we already derived Banach-Steinhaus theorems which in particular cover the original Pólya result [**4**], and extended the classical approximate identity argument to the present setting. In the following we continue this program by examining qualitative as well as quantitative aspects of the Bohman-Korovkin theorem.

2. Convergence of positive linear operators. The classical Bohman-Korovkin theorem states that a sequence $\{T_n\}$ of positive linear operators of C into itself constitutes an approximation process on C, i.e., $\lim_{n\to\infty} \|T_n f - f\| = 0$ for each $f \in C$, if and only if uniform convergence can be established for the three test functions $p_i(u) := u^i$, $i = 0, 1, 2$. The analogue for R now reads

THEOREM 2. *Let* $\{T_n\}_{n=1}^{\infty}$ *be a sequence of positive linear operators, mapping* $R[a,b]$ *into itself. The following assertions are equivalent (with* $\varphi_x(u) := (u-x)^2$*):*

(i) $R - \lim T_n f = f \quad (f \in R[a,b])$,

(ii) $R - \lim T_n p_i = p_i \quad (i = 0, 1, 2)$,

(iii) $R - \lim T_n p_0 = p_0, \quad R - \lim (T_n \varphi_x)(x) = 0$.

PROOF. As usual the implications (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) are trivial. To prove (iii) $\Rightarrow$ (i), first note that (1) is obvious since $|T_n f| \leq ||f|| T_n p_0$. To establish (2) we follow the procedure of Fejér [**3**] and Weyl [**6**], concerned with the extension of results from C to R via step functions.

(a). Let $f \in C$. Thus, for $\varepsilon > 0$, there exists $\delta > 0$ such that, for $u, x \in [a, b]$,

$$|f(u) - f(x)| \leq \begin{cases} \varepsilon, & |u - x| < \delta \\ 2||f|| \leq 2||f|| \varphi_x(u)\delta^{-2}, & |u - x| \geq \delta. \end{cases}$$

As usual for C, this implies the inequality

$$|(T_k f)(x) - f(x)| \leq (\varepsilon + ||f||)|(T_k p_0)(x) - p_0(x)| + \varepsilon + 2||f||\delta^{-2}(T_k \varphi_x)(x)$$

so that (iii) yields

$$\limsup_{n \to \infty} \overline{\int} \sup_{k \geq n} |T_k f - f| \leq \varepsilon.$$

Since ε is arbitrary, this proves (i) for continuous functions.

(b). Let $f = \chi_I$ be the characteristic function of a subinterval $I \subset [a, b]$ with endpoints c, d. For $\varepsilon > 0$ consider the continuous functions

$$g(u) := \begin{cases} 1, & u \in [c + \varepsilon, d - \varepsilon] \\ 0, & u \notin [c, d], \\ \text{linear}, & \text{otherwise}, \end{cases}$$

$$h(u) := \begin{cases} 1, & u \in [c, d] \\ 0, & u \notin [c - \varepsilon, d + \varepsilon], \\ \text{linear}, & \text{otherwise}, \end{cases}$$

restricted to $[a, b]$. Then

$$g \leq f \leq h, \quad \int [h - g] \leq 2\varepsilon,$$

and therefore

$$T_k g - g - [h - g] \leq T_k f - f \leq T_k h - h + [h - g],$$

$$\overline{\int} \sup_{k \geq n} |T_k f - f| \leq \overline{\int} \sup_{k \geq n} |T_k h - h| + \overline{\int} \sup_{k \geq n} |T_k g - g| + 2\varepsilon.$$

Since $g, h \in C$, it follows from (a) that, for each $\varepsilon > 0$,

$$\limsup_{n\to\infty} \overline{\int} \sup_{k\geq n} |T_k f - f| \leq 2\varepsilon,$$

which establishes (i) for step functions.

(c). Let $f \in R$ be arbitrary. By Riemann's integrability criterion for $\varepsilon > 0$ there exist step functions m, M with

$$m := \sum_{i=1}^{j} \alpha_i \chi_{I_i}, \quad M := \sum_{i=1}^{j} \beta_i \chi_{I_i},$$

$$m \leq f \leq M, \quad \int [M - m] \leq \varepsilon.$$

Analogously to (b) it follows that

$$\overline{\int} \sup_{k\geq n} |T_k f - f| \leq \overline{\int} \sup_{k\geq n} |T_k M - M| + \overline{\int} \sup_{k\geq n} |T_k m - m| + \varepsilon,$$

and since m, M are step functions, (b) implies the convergence of the integrals on the right-hand side. □

Let us emphasize that the finite test condition (ii) still involves only the continuous functions p_i so that (ii) is in fact known, even uniformly, for the standard processes. In these cases convergence on R follows without any additional assumptions or verifications.

3. A quantitative Bohman-Korovkin theorem. Concerning quantitative aspects of approximation in R it is essential that an appropriate measure of smoothness is already available from the literature, namely the τ-modul ($f \in B, \delta > 0$)

$$\tau(f, \delta) := \overline{\int}_{[a,b]} \omega(f, x, \delta) dx,$$

$$\omega(f, x, \delta) := \sup\{|f(u) - f(v)| : u, v \in U_\delta(x) := [x - \delta, x + \delta] \cap [a, b]\},$$

employed by the Bulgarian school of approximation during the last decade (for measurable f and then with $\overline{\int}$ replaced by the Lebesgue integral, see [**5**]). In fact, one has (cf. [**5**, p. 25])

$$f \in R \Longleftrightarrow \lim_{\delta \to 0+} \tau(f, \delta) = 0, \tag{3}$$

parallel to the characterization

$$f \in C \Longleftrightarrow \lim_{\delta \to 0+} ||\omega(f, x, \delta)||_B = 0$$

of continuous functions via the classical modulus of continuity. In the following use is made of the observation that for $f \in R$ the τ-modul is indeed given by a Riemann integral.

LEMMA 1. *If $f \in R[a,b]$, then also $\omega(f, x, \delta) \in R[a,b]$ for each $\delta > 0$.*

PROOF. By Lebesgue's theorem $f \in R$ if and only if the set D of discontinuities of f has Lebesgue measure zero. Consider

$$\begin{aligned} E := &\{x \in [a+\delta, b-\delta] : x - \delta \in D \text{ or } x + \delta \in D\} \\ &\cup \{x \in [a, a+\delta] : x + \delta \in D\} \cup \{x \in [b - \delta, b] : x - \delta \in D\}. \end{aligned}$$

Obviously, E has Lebesgue measure zero, too, and therefore it is enough to show that $\omega(f, x, \delta)$ is continuous for $x \notin E$. To this end, let us confine to the case $x \in [a + \delta, b - \delta] \backslash E$ (the other two cases are analogous). Then f is continuous at $x - \delta, x + \delta$, i.e., for $\varepsilon > 0$ there exists $\eta > 0$ such that

$$\begin{aligned} &|f(u_1) - f(v_1)| < \varepsilon, \qquad |f(u_2) - f(v_2)| < \varepsilon \\ &(u_1, v_1 \in U_\eta(x - \delta), \quad u_2, v_2 \in U_\eta(x + \delta)). \end{aligned}$$

For $y \in U_\eta(x)$ one has

$$\begin{aligned} \omega(f, y, \delta) &\leq \sup\{|f(u) - f(v)| : u, v \in U_\delta(x) \cup U_\eta(x - \delta) \cup U_\eta(x + \delta)\} \\ &\leq \Big\{ \sup_{u,v \in U_\delta(x)} + \sup_{u,v \in U_\eta(x-\delta)} + \sup_{u,v \in U_\eta(x+\delta)} \Big\} |f(u) - f(v)| \\ &\leq \omega(f, x, \delta) + 2\varepsilon, \end{aligned}$$

since $U_\eta(x\underline{+}\delta) \cap U_\delta(x) \neq \emptyset$. The argument being symmetric in x and y, the proof is complete. $\square$

Towards a quantitative extension of Theorem 2 let us first establish the following pointwise inequality.

LEMMA 2. *For $f \in R[a,b]$ and $u, x \in [a,b], \delta > 0$,*

$$\begin{aligned} &|f(u) - f(x)| \\ &\leq \omega(f,x,\delta) + \delta^{-1}(u-x)^2 \int_{|t-x|\geq\delta/2} \omega(f,t,\delta)(t-x)^{-2}dt. \end{aligned} \tag{4}$$

PROOF. For $u, x \in [a,b]$ there exists an integer $n \geq 0$ with

$$n\delta \leq |u-x| < (n+1)\delta. \tag{5}$$

Setting $x_k := x + (u-x)k/(n+1)$ for $k = 0, \ldots, n+1$, one has

$$|x_{k+1} - x_k| = |u-x|/(n+1) \leq \delta,$$
$$|f(x_1) - f(x_0)| = |f(x_1) - f(x)| \leq \omega(f,x,\delta).$$

Moreover, for $u \neq x$ and $k = 1, \ldots, n$,

$$\left| \int_{x_k}^{x_{k+1}} (t-x)^{-2}dt \right| = (n+1)/|u-x|k(k+1) \geq 1/n|u-x| \geq \delta(u-x)^{-2},$$

and therefore (cf. Lemma 1)

$$|f(x_{k+1}) - f(x_k)| \leq \delta^{-1}(u-x)^2 \left| \int_{x_k}^{x_{k+1}} \omega(f,t,\delta)(t-x)^{-2}dt \right|.$$

Altogether this yields

$$\begin{aligned} |f(u) - f(x)| &\leq \sum_{k=0}^{n} |f(x_{k+1}) - f(x_k)| \\ &\leq \omega(f,x,\delta) + \delta^{-1}(u-x)^2 \sum_{k=1}^{n} \left| \int_{x_k}^{x_{k+1}} \omega(f,t,\delta)(t-x)^{-2}dt \right| \\ &\leq \omega(f,x,\delta) + \delta^{-1}(u-x)^2 \int_{|t-x|\geq\delta/2} \omega(f,t,\delta)(t-x)^{-2}dt, \end{aligned}$$

since, for t between x_1 and $x_{n+1} = u$, one has (cf. (5))

$$|t-x| \geq |x_1 - x| = |u-x|/(n+1) \geq n\delta/(n+1) \geq \delta/2.$$

$\square$

Parallel to the classical assertions in C (cf. [**1**, p. 28]) one may now formulate the following quantitative Bohman-Korovkin theorem for R.

THEOREM 3. *Let $\{T_n\}_{n=1}^{\infty}$ be a sequence of positive linear operators, mapping $R[a,b]$ into itself such that*

$$T_n p_0 = p_0. \tag{6}$$

Setting $\mu_n := \sup_{k\geq n} ||(T_k\varphi_x)(x)||_B$ there holds true the estimate

$$\overline{\int} \sup_{k\geq n} |T_k f - f| \leq 5\tau(f, \mu_n^{1/2}) \quad (f \in R[a,b]). \tag{7}$$

PROOF. In view of (4), (6), for any $\delta > 0$,

$$\begin{aligned}
I_n :&= \overline{\int} \sup_{k\geq n} |(T_k f)(x) - f(x)| dx \leq \overline{\int} \sup_{k\geq n} T_k(|f - f(x)p_0|)(x)\, dx \\
&\leq \int \omega(f, x\ \delta) dx \\
&\quad + \delta^{-1} \overline{\int} \sup_{k\geq n} (T_k\varphi_x)(x) \int_{|t-x|\geq\delta/2} \omega(f,t,\delta)(t-x)^{-2}\, dt\, dx \\
&\leq \tau(f,\delta) + \mu_n\delta^{-1} \int\int_{|t-x|\geq\delta/2} (t-x)^{-2} dx\, \omega(f,t,\delta) dt \\
&\leq \tau(f,\delta)(1 + 4\mu_n\delta^{-2})
\end{aligned}$$

upon using Fubini's theorem for Riemann integrable functions (cf. Lemma 1). If $\mu_n = 0$, then $I_n \leq \limsup_{\delta\to 0^+} \tau(f,\delta) = 0$ in view of (3), otherwise (7) follows with $\delta = \mu_n^{1/2}$.

Along the same lines one may also establish a quantitative Bohman-Korovkin theorem with regard to the L^1-norm which then regains the result

$$\int |T_n f - f| \leq 748\tau(f, ||(T_n\varphi_x)(x)||_B^{1/2}),$$

given in [**5**, p. 113] via more intricate arguments.

4. Applications. Both the qualitative as well as the quantitative Bohman-Korovkin theorem for R only involve continuous test functions so that for many processes all conditions are already given by results, known from uniform convergence. Let us discuss the matter in connection with Bernstein polynomials and spline approximation.

The Bernstein polynomials are defined for $f \in R[0,1]$ by

$$(B_n f)(x) := \sum_{k=0}^{n} f\left(\frac{k}{n}\right)\binom{n}{k} x^k (1-x)^{n-k}.$$

The fact that Bernstein polynomials form an approximation process on $C[0,1]$ with regard to uniform convergence then, in particular, implies $R - \lim B_n p_i = p_i$ for $i = 0, 1, 2$, and thus by Theorem 2

$$R - \lim B_n f = f \text{ for every } f \in R[0,1].$$

This delivers a constructive proof for the density of the set of polynomials in $R[a,b]$. Further, $B_n p_0 = p_0$ and $(B_n \varphi_x)(x) = x(1-x)/n$, thus $\mu_n = 1/4n$. An application of Theorem 3 therefore yields

COROLLARY 2. *For any* $f \in R[0,1]$,

$$\overline{\int} \sup_{k \geq n} |B_k f - f| \leq 5\tau(f, 1/2n^{1/2}).$$

Finally consider interpolating splines on the equidistant nodes $x_{kn} := k/n$ in the interval [0,1]. With the continuous functions

$$g_{kn}(x) := \begin{cases} 1, & x = x_{kn}, \\ 0, & x \notin (x_{k-1,n}, x_{k+1,n}), \\ \text{linear}, & \text{otherwise}, \end{cases}$$

the spline operator S_n is defined for $f \in R[0,1]$ by

$$(S_n f)(x) := \sum_{k=0}^{n} f(x_{kn}) g_{kn}(x) \quad (x \in [0,1]).$$

Obviously, $S_n p_0 = p_0$ and

$$(S_n \varphi_x)(x) := \sum_{k=0}^{n} (x_{kn} - x)^2 g_{kn}(x) = \sum_{g_{kn}(x) \neq 0} (x_{kn} - x)^2 g_{kn}(x) \leq 1/n^2,$$

since $|x_{kn} - x| \leq 1/n$ if $g_{kn}(x) \neq 0$. Theorem 3 then delivers the quantitative result

COROLLARY 3. *For any* $f \in R[0,1]$,

$$\overline{\int} \sup_{k \geq n} |S_k f - f| \leq 5\tau(f, 1/n).$$

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LEHRSTUHL A FÜR MATHEMATIK, RWTH AACHEN, HOSCHSCHULE, TEMPLERGRABEN 55, 5100 AACHEN, FEDERAL REPUBLIC OF GERMANY