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MONOTONE POLYNOMIAL APPROXIMATION IN L^p

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ABSTRACT. Jackson type estimates on the rate of approximation of monotone functions in $L^p[-1,1]$ by means of monotone polynomials are obtained. The estimates involve an L^p-modulus of continuity or equivalently a Peetre functional that weighs differently the behavior of the function in the middle of the interval and near the end points.

1. Introduction. The first significant Jackson type estimates for monotone polynomial approximation of a monotone function were obtained by Lorentz and Zeller [**5**] for approximation in the sup norm. Shvedov [**6**] extended these results to the case of approximating a monotone $f \in L^p[-1,1]$, $1 \leq p < \infty$. He showed that the order of approximation in the L^p-norm of such a function by means of monotone polynomials of degree $\leq n$ can be estimated by $\omega(f, {}^1/_n)_p$ where $\omega(f, \cdot)_p$ denotes the modulus of continuity of f in the L^p-norm. Shvedov [**7**] went on to improve the above estimates by replacing $\omega(f, {}^1/_n)_p$ by $\omega_2(f, {}^1/_n)_p$. Moreover he showed that we cannot expect an estimate involving higher order moduli of smoothness with constants independent of f and n. Ivanov [**2,3**] introduced certain τ-moduli of smoothness in order to prove an L^p-estimate analogous to the Timan pointwise estimates for nonconstrained polynomial approximation. Recently [**4**] we used the τ-modulus to obtain L^p-estimates for monotone polynomial approximation.

In this note we will give a somewhat different estimate involving a recently defined new modulus of continuity due to Ditzian and Totik [**1**]. This modulus of continuity is in our opinion much more convenient to work with than the τ-moduli. We are indebted to Z.Ditzian for discussing with us some of the results in that yet unpublished paper [**1**].

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Let $\varphi(x) = \sqrt{1 - x^2}$, and let

$$\Delta_{h\varphi} f(x) = f\left(x - \frac{h}{2}\varphi(x)\right) - f\left(x + \frac{h}{2}(x)\right) \text{ if } x \pm \frac{h}{2}\varphi(x) \in [-1, 1]$$
$$= 0 \text{ otherwise.}$$

Define

$$\omega^{\varphi}(f, t)_p = \sup_{0 \leq h \leq t} ||\Delta_{h\varphi} f||_p.$$

Then Ditzian and Totik [1] proved that $\omega^{\varphi}(f, \cdot)_p$ is equivalent to the Peetre functional

$$(1) \qquad K_{\varphi}(f, t) = \inf\{||f - g||_p + t||\varphi g'||_p\},$$

where the infimum is taken over all $g \in L^p[-1, 1]$ which are locally absolutely continuous and such that $\varphi g' \in L^p[-1, 1]$. That is, for some constants $C_1 > 0$ and C_2, we have

$$(2) \qquad C_1 K_{\varphi}(f, t) \leq \omega^{\varphi}(f, t)_p \leq C_2 K_{\varphi}(f, t)$$

for all $0 < t < 1$.

2. Estimates of monotone approximation. We shall prove

THEOREM 1. *Let $f \in L^p[-1, 1], 1 \leq p < \infty$, be monotone. Then, for each $n \geq 1$, there exists a monotone polynomial p_n of degree $\leq n$ such that*

$$(3) \qquad ||f - p_n||_p \leq C\omega^{\varphi}\left(f, \frac{1}{n}\right)_p,$$

where C is an absolute constant.

In the sequel C will always denote an absolute constant not necessarily the same in different occurrences even in the same line.

REMARKS. (i) Evidently $\omega^{\varphi}(f, \frac{1}{n})_p \leq \omega(f, \frac{1}{n})_p$ and in some cases, e.g., when f is increasing too fast near the end points, a lot smaller.

(ii) The proof of Theorem 1 also provides the same estimate for the case of nonconstrained L^p polynomial approximation. The result in this case is a special case of a theorem due to Ditzian and Totik [**1**], our proof however is different.

(iii) Ditzian and Totik [**1**] also prove some inverse theorems relating the rate of L^p approximation to the modulus $\omega^\varphi(f,\cdot)_p$ which enable us to characterize monotone functions in certain smoothness classes by means of their rate of monotone L^p approximation. If $E_n(f)_p$ denotes the rate of best L^p approximation by polynomials of degree $\le n$, then Ditzian and Totik [**1**] proved the inequality

$$\omega^\varphi\left(f,\frac{1}{n}\right) \le Cn^{-1}\sum_{k=1}^{n} E_k(f)_p.$$

Thus as a consequence of Theorem 1 we have

COROLLARY 2. *A monotone $f \in L^p[-1,1]$ is approximable by monotone polynomials of degree $\le n$, at the rate of $n^{-\alpha}, 0 < \alpha < 1$, if and only if $\omega^\varphi(f,t)_p = O(t^\alpha)$.*

3. Proof of Theorem 1. Let $\xi_i = \cos\frac{n-i}{n}\pi$, $i = 0,\dots,2n$. With a function $f \in L^p[-1,1]$ associate the step function

$$\begin{aligned} F(x) &= \frac{1}{\xi_{i+1}-\xi_i}\int_{\xi_i}^{\xi_{i+1}} f(t)dt,\ \xi_i \le x < \xi_{i+1}, 0 \le i \le n-1, \\ F(\xi_n) &= F(\xi_{n-1}). \end{aligned} \tag{4}$$

Evidently F is monotone in $[-1,1]$ if f is.

For an integrable function $h(t)$ on $[-\pi,\pi]$ let $J_n(h,u)$ be the Jackson integral of h, namely

$$J_n(h,u) = \int_{-\pi}^{\pi} K_n(u-t)h(t)dt,$$

where

$$K_n(t) = \lambda_n\left(\frac{\sin nt/2}{\sin t/2}\right)^4$$

and

$$\int_{-\pi}^{\pi} K_n(t)dt = 1.$$

Denote $T_n f(x) = J_n(F(\cos t), \cos^{-1} x)$ where F is defined by (4). We note that if F is monotone in $[-1, 1]$, then by virtue of Lorentz and Zeller [**5**] so is $T_n f$. Thus $T_n f$ is monotone in $[-1, 1]$ whenever f is.

We will first prove our theorem under the assumption that f is locally differentiable. Note that we do not assume that f is monotone.

PROPOSITION 3. *Let $f \in L^p[-1, 1], 1 \le p < \infty$, be locally differentiable and such that $\varphi f' \in L^p[-1, 1]$. Then, for each $n \ge 1$,*

$$||f - T_n f||_p \le \frac{C}{n} ||\varphi f'||_p. \tag{5}$$

PROOF. For $\xi_i \le x < \xi_{i+1}, 0 \le i \le n-1$,
(6)

$$\begin{aligned} f(x) - F(x) &= \frac{1}{\xi_{i+1} - \xi_i} \int_{\xi_i}^{\xi_{i+1}} [f(x) - f(t)]dt \\ &= \frac{1}{\xi_{i+1} - \xi_i} \left(\int_{\xi_i}^{x} (t - \xi_i) f'(t)dt - \int_{x}^{\xi_{i+1}} (\xi_{i+1} - t) f'(t)dt \right). \end{aligned}$$

It is readily seen that if $\xi_i \le x \le \xi_{i+1}, 1 \le i \le n-2$, then

$$\xi_{i+1} - \xi_i \le C \frac{\sqrt{1 - x^2}}{n} \tag{7}$$

and

$$\xi_{i+2} - \xi_i \le C \frac{\sqrt{1 - x^2}}{n}. \tag{8}$$

Also if $-1 \le x \le \xi_1$, then

$$x - \xi_0 = x + 1 \le C \frac{\sqrt{1 - x^2}}{n} \tag{9}$$

and if $\xi_{n-1} \le x \le 1$, then

$$\xi_n - x = 1 - x \le C \frac{\sqrt{1 - x^2}}{n}.$$

Hence if $\xi_i \le x < \xi_{i+1}$, $1 \le i \le n-2$, we get, from (6) and (7),

$$
\begin{aligned}
|f(x) - F(x)| &\le \frac{C}{\xi_{i+1} - \xi_i} \int_{\xi_i}^{\xi_{i+1}} \frac{\sqrt{1-t^2}}{n} |f'(t)| dt \\
&\le \frac{C}{n} M_g(x),
\end{aligned} \tag{10}
$$

where M_g denotes the Hardy Littlewood maximal function of g and $g = \varphi f'$ (see e.g., [**8** p. 53],). We do not necessarily take a symmetric interval about x for M_g. If $-1 \le x < \xi_1$, then using (9) we can estimate the first term in (6) exactly as in (10). Similarly we have the proper estimate for the second term in (6) when $\xi_{n-1} \le x \le 1$.

This leaves us with the need to estimate the second term in (6) for $-1 \le x < \xi_1$ and (which is similar) to estimate the first term in (6) for $\xi_{n-1} \le x \le 1$. Toward this end denote

$$\psi_1(x) = \frac{1}{1+\xi_1} \int_x^{\xi_1} (\xi_1 - t) f'(t) dt.$$

We estimate the L^p-norm of $\psi_1(x)$ over the interval $[-1, \xi_1]$.

$$
\begin{aligned}
||\psi_1||_{L^p[-1,\xi_1]} &= \frac{1}{1+\xi_1} \left(\int_{-1}^{\xi_1} \left| \int_x^{\xi_1} (\xi_1 - t) f'(t) dt \right|^p dx \right)^{1/p} \\
&\le \left(\int_{-1}^{\xi_1} \left| \int_{-1}^{\xi_1} \chi_x(t) f'(t) dt \right|^p dx \right)^{1/p},
\end{aligned}
$$

where $\chi_x(t)$ is the characteristic function of $[x, \xi_1]$ and we estimated $\xi_1 - t$ by $1 + \xi_1$. Now the Minkowsky inequality yields

$$
\begin{aligned}
||\psi_1||_{L^p[-1,\xi_1]} &\le \int_{-1}^{\xi_1} \left(\int_{-1}^{\xi_1} |\chi_x(t) f'(t)|^p dx \right)^{1/p} dt \\
&= \int_{-1}^{\xi_1} |f'(t)| \left(\int_{-1}^{t} dx \right)^{1/p} dt \\
&= \int_{-1}^{\xi_1} |f'(t)| (1+t)^{1/p} dt
\end{aligned}
$$

which, by (9),

$$\leq Cn^{-1/p}\int_{-1}^{\xi_1}|f'(t)|\left(\sqrt{1-t^2}\right)^{1/p}dt.$$

Hölder's inequality now gives

$$(11)\qquad \begin{aligned} ||\psi_1||_{L^p[-1,\xi_1]} &\leq Cn^{-1/p}\left(\int_{-1}^{\xi_1}\frac{dt}{\sqrt{1-t^2}}\right)^{1/q} \\ &\quad \left(\int_{-1}^{\xi_1}|f'(t)|^p(\sqrt{1-t^2})^p dt\right)^{1/p} \\ &= C\pi^{1/q}n^{-1}\left(\int_{-1}^{\xi_1}|f'(t)|^p(\varphi(t))^p dt\right)^{1/p}. \end{aligned}$$

Similarly, if

$$\psi_2(x) = \frac{1}{1-\xi_{n-1}}\int_{\xi_{n-1}}^{x}(t-\xi_{n-1})f'(t)dt,$$

then

$$(12)\qquad ||\psi_1||_{L^p[\xi_{n-1},1]} \leq Cn^{-1}\left(\int_{\xi_{n-1}}^{1}|f'(t)|^p(\varphi(t))^p dt\right)^{1/p}.$$

Combining (10) through (12) it follows that

$$||f-F||_p \leq Cn^{-1}(||g||_p + ||M_g||_p),$$

whence by [**8**, Theorem 3.7, p. 58] we have, for $1 < p < \infty$,

$$(13)\qquad ||f-F||_p \leq Cn^{-1}||f'\varphi||_p.$$

For $p = 1$, (13) follows immediately from the first inequality in (10) together with (11) and (12). To establish (5) we need to show that

$$(14)\qquad ||F - T_n f||_p \leq Cn^{-1}||f'\varphi||_p.$$

For $\xi_i \le x < \xi_{i+1}, 0 \le i \le n-1$, let $x = \cos y$. Then

$$F(x) - T_n f(x) = \int_{-\pi}^{\pi} [F(\xi_i) - F(\cos(y-t))] K_n(t)dt$$
$$= \sum_{j=1}^{n} \int_{\frac{j-1}{n}\pi}^{\frac{j}{n}\pi} [F(\xi_i) - F(\cos(y-t))] K_n(t)dt.$$

When $\frac{j-1}{n}\pi < t < \frac{j}{n}\pi$ and $\frac{n-i-1}{n}\pi < y < \frac{n-1}{n}\pi$, we have

$$\frac{n-(i+j)-1}{n}\pi < y - t < \frac{n-(i+j)+1}{n}\pi$$

so $F(\cos(y-t)) =$ either $F(\xi_{i+j-1})$ or $F(\xi_{i+j})$. Hence

$$|F(\xi_i) - F(\cos(y-t))| \le \sum_{k=1}^{j} |F(\xi_{i+k-1}) - F(\xi_{i+k})|.$$

At the same time

$$\int_{\frac{j-1}{n}\pi}^{\frac{j}{n}\pi} K_n(t)dt \le \frac{C}{j^4}, \quad \text{for } j \ge 1.$$

Together they imply

$$|F(x) - T_n f(x)| \le C \sum_{j=1}^{n} \frac{1}{j^4} \sum_{k=1}^{j} |F(\xi_{i+k-1}) - F(\xi_{i+k})|$$
$$\le C \sum_{k=1}^{n} \frac{1}{k^3} |F(\xi_{i+k-1}) - F(\xi_{i+k})|.$$

Therefore

(15)

$$||F - T_n f||_p^p = \int_{-1}^{1} |F(x) - T_n f(x)|^p dx$$
$$= \sum_{i=0}^{n-1} \int_{\xi_i}^{\xi_{i+1}} |F(x) - T_n f(x)|^p dx$$
$$\le C \sum_{i=0}^{n-1} \left(\sum_{k=1}^{n} \frac{1}{k^3} |F(\xi_{i+k-1}) - F(\xi_{i+k})| \right)^p (\xi_{i+1} - \xi_i)$$
$$\le C \sum_{i=0}^{n-1} \sum_{k=1}^{n} \frac{1}{k^3} |F(\xi_{i+k-1}) - F(\xi_{i+k})|^p (\xi_{i+1} - \xi_i)$$

where we have used Hölder's inequality for the inner sum and incorporated a bound for $(\sum_{k=1}^{n}\frac{1}{k^3})^{p/q}(\frac{1}{p}+\frac{1}{q}=1)$ into C. Thus, in order to complete the proof of (14), we need an estimate of $|F(\xi_{i+1})-F(\xi_i)|$. We will show that it can be estimated as in (10) and (11).

For $\xi_i \le x < \xi_{i+1}, 1 \le i \le n-3$, we have, by (10),

$$\begin{aligned} |F(\xi_{i+1})-F(\xi_i)| &\le |F(\xi_{i+1})-f(x)|+|f(x)-F(x)| \\ &\le |F(\xi_{i+1})-f(x)|+\frac{C}{n}M_g(x). \end{aligned} \tag{16}$$

Now

$$\begin{aligned} F(\xi_{i+1})-f(x) &= \frac{1}{\xi_{i+2}-\xi_{i+1}}\int_{\xi_{i+1}}^{\xi_{i+2}}[f(t)-f(x)]dt \\ &= \frac{1}{\xi_{i+2}-\xi_{i+1}} \\ &\quad \left(\int_{\xi_i}^{\xi_{i+2}}[f(t)-f(x)]dt-\int_{\xi_i}^{\xi_{i+1}}[f(t)-f(x)]dt\right). \end{aligned} \tag{17}$$

So, by virtue of (10) and (8),

$$\begin{aligned} |F(\xi_{i+1})-f(x)| &\le \frac{1}{\xi_{i+2}-\xi_{i+1}}\frac{C}{n}[(\xi_{i+2}-\xi_i)+(\xi_{i+1}-\xi_i)]M_g(x) \\ &\le \frac{C}{n}M_g(x) \end{aligned}$$

where we used the inequality

$$\xi_{i+2}-\xi_{i+1} \le C(\xi_{i+1}-\xi_i).$$

Hence, by (16),

$$|F(\xi_{i+1})-F(\xi_i)| \le \frac{C}{n}M_g(x), \quad \xi_i \le x < \xi_{i+1}, \quad 1 \le i \le n-3. \tag{18}$$

Obviously (18) is valid for $i=n-1$, the lefthand side being zero, so this leaves us with the case $i=0$ and $i=n-2$. For $-1 \le x < \xi_1$, we have, by (17),

$$\begin{aligned}|F(\xi_1) - F(\xi_0)| &\le |F(\xi_1) - f(x)| + |f(x) - F(x)| \\ &\le \frac{1}{\xi_2 - \xi_1}\left|\int_{-1}^{\xi_2}[f(t) - f(x)]dt\right| \\ &\quad + \frac{1}{\xi_2 - \xi_1}\left|\int_{-1}^{\xi_1}[f(t) - f(x)]dt\right| + |f(x) - F(x)| \\ &= \alpha_1(x) + \alpha_2(x) + \alpha_3(x),\end{aligned}$$

say. Now, as in proving (1) and (11), in order to estimate the integrals in (6), for $i = 0$, we get

$$\begin{aligned}||\alpha_j||_{L^p[-1,\xi_2]} &\le Cn^{-1}[||g||_{L^p[-1,\xi_2]} + ||M_g||_{L^p[-1,\xi_2]}] \\ &\le Cn^{-1}||g||_{L^p[-1,\xi_2]}, \quad i = 1, 2, 3.\end{aligned}$$

Hence

$$\begin{aligned}|F(\xi_1) - F(\xi_0)|^p(\xi_1 - \xi_0) &\le C\int_{-1}^{\xi_2}\sum_{j=1}^{3}[\alpha_j(x)]^p dx \\ &\le Cn^{-p}||g||^p_{L^p[-1,\xi_2]}.\end{aligned} \tag{19}$$

For $i = n - 2$, the second integral in (17) is estimated as in (18) while the first one we expand into two integrals as in (6). Then the second integral is estimated as in (18) while the first is estimated as in (19) (see (11)). Finally it is easy to see [**3**, Lemma 2] that, for $0 \le i \le n-1$, and $k \ge 1$

$$\xi_{i+1} - \xi_i \le \frac{3\pi k}{2}|\xi_{i+k} - \xi_{i+k-1}|. \tag{20}$$

We are ready to go back to (15). By virtue of (20),

$$||F - T_n f||_p^p \le C\sum_{i=0}^{n-1}\sum_{k=1}^{n}\frac{1}{k^2}|F(\xi_{i+k+1}) - F(\xi_{i+k})|^p|\xi_{i+k} - \xi_{i+k-1}|,$$

so that, for $1 < p < \infty$, we have, by (18) and (19),

$$\begin{aligned}&\le Cn^{-p}\left[\sum_{i=0}^{n-1}\sum_{k=1}^{n}\frac{1}{k^2}\left|\int_{\xi_{i+k-1}}^{\xi_{i+k}}(M_g(x))^p dx\right| + ||g||_p^p\right] \\ &= Cn^{-p}\left[\left(\sum_{k=1}^{n}\frac{1}{k^2}\right)\int_{-1}^{1}(M_g(x))^p dx + ||g||_p^p\right] \\ &\le Cn^{-p}||f'\varphi||_p^p.\end{aligned}$$

This proves (14) for $1 < p < \infty$. Again (14) is valid for $p = 1$ and the proof is easier using the first inequality in (10) instead of the maximal function M_g. This concludes the proof of (5). □

PROOF OF THEOREM 1. By (2) there exists $g \in L^p[-1,1]$ which is locally absolutely continuous and such that $\varphi g' \in L^p[-1,1]$ which satisfies

$$\|f-g\|_p \le 2\omega^\varphi\Big(f,\frac{1}{n}\Big)_p \tag{21}$$

and

$$\|\varphi g'\|_p \le 2n\omega^\varphi\Big(f,\frac{1}{n}\Big)_p. \tag{22}$$

If F and G are the corresponding step functions, then, for $\xi_i \le x < \xi_{i+1}$,

$$|F(x)-G(x)| \le \frac{1}{\xi_{i+1}-\xi_i}\int_{\xi_i}^{\xi_{i+1}} |f(t)-g(t)|dt$$

(by Hölder's inequality)

$$\le \frac{1}{(\xi_{i+1}-\xi_i)^{1/p}}\left(\int_{\xi_i}^{\xi_{i+1}} |f(t)-g(t)|^p dt\right)^{1/p}.$$

Thus

$$\int_{-1}^{1}|F(x)-G(x)|^p dx \le \sum_{i=0}^{n-1}\frac{1}{\xi_{i+1}-\xi_i}\int_{\xi_i}^{\xi_{i+1}}|f(t)-g(t)|^p dt(\xi_{i+1}-\xi_i)$$
$$= \int_{-1}^{1}|f(t)-g(t)|^p dt.$$

So

$$\|F-G\|_p \le \|f-g\|_p$$

which implies

$$\|T_n f - T_n g\|_p \le C\|f-g\|_g.$$

Now, by (21) and Proposition 1,

$$\|f-T_n f\|_p \le \|f-g\|_p + \|g-T_n g\|_p + \|T_n f - T_n g\|$$
$$\le C\omega^\varphi\Big(f,\frac{1}{n}\Big)_p + Cn^{-1}\|\varphi g'\|_p,$$

which, by (22),

$$\leq C\omega^{\varphi}\left(f, \frac{1}{n}\right)_p.$$

Since $T_n f$ is monotone when f is and is a polynomial of degree $\leq 2n-1$ our proof is complete. □

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