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CONVEXITY PRESERVING CONVOLUTION OPERATORS

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Dedicated to Professor A. Sharma

1. Introduction. Let K be a piecewise smooth, real 2π-periodic non negative function and $\gamma(t) = (f_1(t), f_2(t))$, $t \in [0, 2\pi]$, where $f_i(t)$ $(i = 1, 2)$ are piecewise smooth 2π-periodic functions, be a closed curve in $\mathbf{R}^2$. Then the function

$$\Gamma(x) := \int_0^{2\pi} K(x-t)\gamma(t)dt, \quad x \in [0, 2\pi], \tag{1.1}$$

is also a closed curve in $\mathbf{R}^2$. The following theorem attributed to Loewner was proved in [**3**].

THEOREM (LOEWNER). *A necessary condition for the convolution transform* (1.1) *to map a positively convex curve onto a positively locally convex curve is that the closed curve* $(K'(x), K(x))$, $0 \leq x \leq 2\pi$, *is positively convex.*

The objectives of this paper are to show that the condition that the curve (K', K) be positively convex is also sufficient for the transformation (1.1) to map positively convex curves onto positively locally convex curves, and to give other sufficient conditions for the transformation to map convex curves onto convex curves. In §3, we mention an application to convolution operators with spline kernels from which the present work originates.

2. Convexity preserving kernels. As in §1, γ denotes a closed curve in $\mathbf{R}^2$. We shall assume that all curves are piecewise smooth. By *convexity* of γ we mean that it does not intersect any straight line more than twice. The kernel K in (1.1) is said to be *convex preserving*

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if the curve Γ is convex whenever γ is. We also need the concept of local convexity. The curve $\Gamma(x) = (g_1(x), g_2(x))$, $x \in [0, 2\pi]$ is said to be *locally convex* if the Wronskian

$$W(g_1', g_2') := \left| \begin{pmatrix} g_1'(x) & g_2'(x) \\ g_1''(x) & g_2''(x) \end{pmatrix} \right| \geq 0, \quad x \in [0, 2\pi]. \tag{2.1}$$

The condition (2.1) says that the tangent to the curve turns in the same direction as the point moves along the curve in the positive sense.

We shall prove the following theorems which give sufficient conditions for the kernel K to be "convex preserving".

THEOREM 2.1. *If the closed curve $(K'(x), K(x)), 0 \leq x \leq 2\pi$, is positively convex, then the convolution transform* (1.1) *maps a positively convex curve onto a positively locally convex curve.*

THEOREM 2.2. *A sufficient condition for the kernel K to be convex preserving is that, for any $\alpha \in [0, 2\pi]$, the closed curve $(K'(x), K(x) - K(x+\alpha))$, $0 \leq x \leq 2\pi$, is positively convex.*

REMARK. Theorem 2.1 is the converse of Loewner's Theorem. In [**3**], Loewner's Theorem is stated for convex preserving kernels, but the proof requires only that the transform (1.1) maps convex curves onto locally convex curves.

The proofs of Theorems 2.1 and 2.2 require the following lemmas.

LEMMA 1. *The closed curve $\gamma(t) = (f_1(t), f_2(t))$, $0 \leq t \leq 2\pi$, is positively convex if and only if, for $t_1 < t_2 < t_3 < t_1 + 2\pi$, the determinant*

$$\begin{vmatrix} 1 & f_1(t_1) & f_2(t_1) \\ 1 & f_1(t_2) & f_2(t_2) \\ 1 & f_1(t_3) & f_2(t_3) \end{vmatrix} \geq 0. \tag{2.2}$$

LEMMA 2. *A necessary and sufficient condition for the curve γ to be positively convex is that, for any numbers $0 \leq t_0, t \leq 2\pi$, the* *determinant*

$$\left|\begin{matrix} f_1(t) - f_1(t_0) & f_2(t) - f_2(t_0) \\ f_1'(t) & f_2'(t) \end{matrix}\right| \geq 0. \tag{2.3}$$

Lemma 1 is well-known (see [**3**]). Geometrically, condition (2.3) says that, for any two points $\gamma(t)$ and $\gamma(t_0)$ on the curve γ, the angle from the vector $\gamma(t) - \gamma(t_0)$ to the tangent vector $\gamma'(t)$, in the positive direction, does not exceed 180°. The proof of Lemma 2 is straightforward and we shall omit it.

PROOF OF THEOREM 2.1. Let $\gamma(t) = (f_1(t), f_2(t))$, $0 \leq t \leq 2\pi$, be a positively convex closed curve, and

$$\Gamma(x) = (g_1(x), g_2(x)) := \int_0^{2\pi} K(x-t)(f_1(t), f_2(t))dt.$$

Then we can write

$$(g_1'(x), g_2'(x)) = \int_0^{2\pi} K(x-t)(f_1'(t), f_2'(t))dt \tag{2.4}$$

and

$$(g''(x), g_2''(x)) = \int_0^{2\pi} K'(x-t)(f_1'(t), f_2'(t))dt. \tag{2.5}$$

Because of periodicity,

$$(0,0) = \int_0^{2\pi} (f_1'(t), f_2'(t))dt. \tag{2.6}$$

Set

$$s = \int_0^{2\pi} |\gamma'(t)|dt \tag{2.7}$$

and

$$\alpha(x) = \int_0^{2\pi} K(x-t)|\gamma'(t)|dt. \tag{2.8}$$

From (2.4)-(2.8) and the composition formula (see [**2**]), we have

$$
\begin{vmatrix} s & 0 & 0 \\ \alpha(x) & g_1'(x) & g_2'(x) \\ \alpha'(x) & g_1''(x) & g_2''(x) \end{vmatrix}
$$

$$
\tag{2.9} = \iiint_{t_1<t_2<t_3} \begin{vmatrix} 1 & 1 & 1 \\ K(x-t_1) & K(x-t_2) & K(x-t_3) \\ K'(x-t_1) & K'(x-t_2) & K'(x-t_3) \end{vmatrix} \times \begin{vmatrix} |\gamma'(t_1)| & f_1'(t_1) & f_2'(t_1) \\ |\gamma'(t_2)| & f_1'(t_2) & f_2'(t_2) \\ |\gamma'(t_3)| & f_1'(t_3) & f_2'(t_3) \end{vmatrix} dt_1 dt_2 dt_3.
$$

Because of the convexity of (K', K),

$$
\tag{2.10} \begin{vmatrix} 1 & 1 & 1 \\ K(x-t_1) & K(x-t_2) & K(x-t_3) \\ K'(x-t_1) & K'(x-t_2) & K'(x-t_3) \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 \\ K'(x-t_3) & K'(x-t_2) & K'(x-t_1) \\ K(x-t_3) & K(x-t_2) & K(x-t_1) \end{vmatrix} \geq 0,
$$

for all $t_1 < t_2 < t_3 < t_1 + 2\pi$ for which

$$
\tag{2.11} \begin{vmatrix} |\gamma'(t_1)| & f_1'(t_1) & f_2'(t_1) \\ |\gamma'(t_2)| & f_1'(t_2) & f_2'(t_2) \\ |\gamma'(t_3)| & f_1'(t_3) & f_2'(t_3) \end{vmatrix} = \prod_{i=1}^{3} |\gamma'(t_i) \begin{vmatrix} 1 & f_1'(t_1)/|\gamma'(t_1)| & f_2'(t_1)/|\gamma'(t_1)| \\ 1 & f_1'(t_2)/|\gamma'(t_2)| & f_2'(t_2)/|\gamma'(t_2)| \\ 1 & f_1'(t_3)/|\gamma'(t_3)| & f_2'(t_3)/|\gamma'(t_3)| \end{vmatrix}
$$

is also non negative because of the convexity of γ. It follows from (2.9), (2.10) and (2.11), that

$$
\tag{2.12} \begin{vmatrix} g_1'(x) & g_2'(x) \\ g_1''(x) & g_2''(x) \end{vmatrix} \geq 0 \quad \forall x \in [0, 2\pi],
$$

which means that the curve Γ is locally positively convex. □

PROOF OF THEOREM 2.2. Let f_i and $g_i, i = 1, 2$, be as in the proof of Theorem 2.1, and $x_0 \in [0, 2\pi]$ be a fixed number. Then we can write

$$g_i(x) - g_i(x_0) = \int_0^{2\pi} \{K(x-t) - K(x_0 - t)\} f_i(t)dt \tag{2.13}$$

and

$$g_i'(x) = \int_0^{2\pi} K'(x-t) f_i(t)dt, \quad i = 1, 2. \tag{2.14}$$

Also

$$\begin{cases} 0 = \int_0^{2\pi} \{K(x-t) - K(x_0 - t)\}dt \\ 0 = \int_0^{2\pi} K'(x-t)dt, \end{cases} \tag{2.15}$$

and let

$$a_i = \int_0^{2\pi} f_i(t)dt, \quad i = 1, 2. \tag{2.16}$$

Applying the composition formula using (2.13)-(2.16), we have

$$\begin{vmatrix} 2\pi & a_1 & a_2 \\ 0 & g_1(x) - g_1(x_0) & g_2(x) - g_2(x_0) \\ 0 & g_1'(x) & g_2'(x) \end{vmatrix} \tag{2.17}$$

$$= \iiint_{0 \le t_1 < t_2 < t_3 \le \pi}$$

$$\begin{vmatrix} 1 & 1 & 1 \\ K(x-t_1) - K(x_0 - t_1) & K(x-t_2) - K(x_0 - t_2) & K(x-t_3) - K(x_0 - t_3) \\ K'(x-t_1) & K'(x-t_2) & K'(x-t_3) \end{vmatrix}$$

$$\times \begin{vmatrix} 1 & f_1(t_1) & f_2(t_1) \\ 1 & f_1(t_2) & f_2(t_2) \\ 1 & f_1(t_3) & f_2(t_3) \end{vmatrix} dt_1dt_2dt_3.$$

The first determinant in the integrand is non negative for any x,

$x_0 \in [0, 2\pi]$, because of the convexity of $(K'(x), K(x) - K(x+\alpha))$. Hence, if $\gamma = (f_1, f_2)$ is convex, then, by (2.17),

$$\begin{vmatrix} g_1(x) - g_1(x_0) & g_2(x) - g_2(x_0) \\ g_1'(x) & g_2'(x) \end{vmatrix} \geq 0$$

for $0 \leq x,\ x_0 \leq 2\pi$, which means that the curve $\Gamma = (g_1, g_2)$ is positively convex by Lemma 2. □

The following theorem gives another sufficient condition for a kernel to be convex preserving. The proof is similar to those given above, employing the convexity criterion of Lemma 2.

THEOREM 2.3. *If, for $t_1 < t_2 < t_3 < t_1 + 2\pi$ and $\alpha \in [0, 2\pi]$, the determinant*

$$\begin{vmatrix} K(t_1+\alpha) & K(t_2+\alpha) & K(t_3+\alpha) \\ K'(t_1) & K'(t_2) & K'(t_3) \\ K(t_1) & K(t_2) & K(t_3) \end{vmatrix}$$

is non negative, then the kernel K is convex preserving.

3. Shape preserving trigonometric B-spline kernel. Let k be a positive integer, $h = 2\pi/k$ and $n \leq k-1$ be a non negative integer. For $\nu \in Z$, define

$$t_\nu := \begin{cases} \frac{2^n}{k} \prod_{j=0}^{n\,\prime} \frac{\sin(\nu-j)h/2}{(\nu-j)}, & 0 \leq \nu \leq n, \\ \frac{2^n}{\pi} \prod_{j=0}^{n} \frac{\sin(\nu-j)h/2}{(\nu-j)}, & \text{otherwise}, \end{cases} \tag{3.1}$$

where the product $\prod'$ means that the factor corresponding to $j = \nu$ is 1. The function

$$T_n(x) := \sum_\nu t_\nu e^{i(\nu - n/2)(x-(n+1)h/2)}, \quad x \in [0, 2\pi), \tag{3.2}$$

is called the *trigonometric B-spline* of degree n with uniform knots at νh, $\nu = 0, 1, \ldots, n-1$ (see [**1**]). We extend $T(x)$ to $x \in \mathbf{R}$ in such a way that it is 2π-periodic. If $\gamma(t)$, $t \in [0, 2\pi]$ is a closed curve, we define

$$\Gamma_n(x) = \int_0^{2\pi} T_n(x-t)\gamma(t)dt, \quad x \in [0, 2\pi]. \tag{3.3}$$

Then we have

THEOREM 3.1. *For $n = 1, 2, \ldots$, the curve $z_n(x) = (T_n'(x), T_n(x))$, $x \in [0, 2\pi]$ is positively convex.*

Since z_n is positively convex, in view of Theorem 2.1, we have the

COROLLARY. *For $n = 1, 2, \ldots$, the convolution transform* (3.3) *maps positively convex curves onto positively locally convex curves.*

The proof of Theorem 3.1 is long and complicated (see [**1**]) and will not be given here because of lack of space.

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