ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 19, Number 1, Winter 1989

ATOMIC CHARACTERIZATIONS OF MODULATION SPACES THROUGH GABOR-TYPE REPRESENTATIONS

HANS G. FEICHTINGER

ABSTRACT. Given $s \in \mathbf{R}$ and $1 \leq p, q \leq \infty$ the modulation space $M^s_{p,q}(\mathbf{R}^m)$ can be described as follows, using the Gauss-function $g_0, g_0(x) := \exp(-x^2)$

$$M^s_{p,q}(\mathbf{R}^m) := \Big\{\sigma |\ \sigma \in \mathcal{S}',\ g_0 * \sigma \in L^p(\mathbf{R}^m) \text{ and}$$
$$\|\sigma | M^s_{p,q}\| := \Big[\int_{\mathbf{R}^m} \|M_t g_0 * \sigma\|_p^q (1+|t|)^{sq}\Big]^{1/q} < \infty\Big\}$$

(Writing M_t, $M_t f(x) := \exp(ix \cdot t) f(x), t, x \in \mathbf{R}^m$) for the modulation operator. Among these spaces one has the classical potential spaces $\mathcal{L}^2_s(\mathbf{R}^m) = M^s_{2,2}(\mathbf{R}^m)$ and the remarkable Segal algebra $S_0(\mathbf{R}^m) = M^0_{1,1}(\mathbf{R}^m)$. It is the aim of this paper to show that for these spaces an atomic characterization similar to known characterization of Besov spaces can be given (with dilation being replaced by modulation). Our main theorem is the following: Given $s \in \mathbf{R}$ and some $g_0 \neq 0$, $g_0 \in M^{|s|}_{1,1}(\mathbf{R}^m)$ (e.g., $g \in \mathcal{S}(\mathbf{R}^m)$ or $g \in L^1$ with compactly supported Fourier transform) one has:

THEOREM . *There exist $\alpha_0 > 0$ and $\beta_0 > 0$ such that, for $\alpha \leq \alpha_0$ and $\beta \leq \beta_0$, there exists $C = C(\alpha, \beta) > 0$ with the following property: $f \in M^s_{p,q}(\mathbf{R}^m)$ if and only if $f = \sum_{n,k} a_{n,k} M_{\beta n} L_{\alpha k} g_0$, for some double sequence of coefficients satisfying*

$$\Big[\sum_n \Big(\sum_k |a_{n,k}|^p\Big)^{q/p} (1+|n|)^{sq}\Big]^{1/q} \leq C \|f | M^s_{p,q}(\mathbf{R}^m)\|.$$

The convergence is in the sense of tempered distributions, and in the norm sense for $p, q < \infty$.

Received by the editors on October 28, 1986.

1. Introduction. The investigation of modulation spaces has been suggested by the author, starting at first from the simple idea of replacing the dyadic partition used in the characterization of Besov-spaces by an equidistant one. It turned out that the characterization used in the abstract (here it would have been sufficient to assume that $g_0 \neq 0$ is any Schwartz function or $g_0 \in L^1$ with compactly supported Fourier transform, cf. [**10**], [**11**], [**20**], Kap. 5.2) is more elegant and admits shorter proofs of some of the basic properties of these spaces, such as the invariance of some of these spaces under the Fourier transform.

The most interesting among these spaces are $L^2(\mathbf{R}^m) = M^0_{2,2}(\mathbf{R}^m)$ and $S_0(\mathbf{R}^m) = M^0_{1,1}(\mathbf{R}^m)$. For $p = q = 2$, $s = 0$, $m = 1$ and g_0 being the Gauss function, our results may be considered as Gabor representation for f (cf. [**17**]), however, with an estimate on the coefficients in ℓ^2. This is in contrast to the classical situation (where the von Neumann lattice with $\alpha\beta = 2\pi$ is chosen; in that case the operators L_x and M_t involved commute, but unbounded coefficients may arise). For $p = q = 1$, $s = 0$ one obtains an improved atomic characterization for the Segal algebra $S_0(G)$ and for $p = q = \infty$ of its dual space (cf. [**8**], [**7**], [**13**]). A typical feature of our approach is the considerable freedom in the choice of g_0 as "basic" function.

As a corollary of the main result we shall have the following results:

COROLLARY 1. *Given $f \in \mathcal{S}(\mathbf{R}^m), s \in \mathbf{R}$ and $g_0 \in \mathcal{S}(\mathbf{R}^m)$, $g_0 \neq 0$, there exists $\alpha, \beta > 0$ (depending only on g_0) such that $f \in \mathcal{L}^2_s(\mathbf{R}^m)$ (the Bessel potential space,* cf. [**21**]) *if and only if*

$$f = \sum_{n,k} a_{n,k} M_{\beta n} L_{\alpha k} g_0$$

for a double sequence satisfying $[\sum_n \sum_k |a_{n,k}|^2 (1 + |n|)^{2s}]^{1/2} < \infty$.

COROLLARY 2. *Given $f \in L^1(\mathbf{R}^m)$ and $g_0 \in S_0(\mathbf{R}^m)$, $g_0 \neq 0$, there exists $\alpha, \beta > 0$ (depending only on g_0) such that $f \in S_0(\mathbf{R}^m)$ if and only if $f = \sum_{n,k} a_{n,k} M_{\beta n} L_{\alpha k} g_0$ for a double sequence satisfying $\sum_n \sum_k |a_{n,k}| < \infty$.*

As mentioned already in [**13**] the special choice g_0 =Gauss function implies a number of properties for $S_0(\mathbf{R}^m)$ (which are stated for general lca. groups in [**7, 8**], using a fairly different approach).

2. Basic properties of modulation spaces. We want to summarize here a few facts concerning modulation spaces which may be defined as inverse images under the Fourier transform of certain Wiener-type spaces (for basic facts cf. [**9**] and [**12**]). In order to describe these spaces we need the following conventions: We shall need the weighted L^q-spaces $L^q_s(\mathbf{R}^m)$, given by

$$L^q_s := \left\{ f \mid \| f | L^q_s(\mathbf{R}^m) \| := \left(\int_{\mathbf{R}^m} |f(x)|^q (q + |x|)^{sq} \right)^{1/q} < \infty \right\}$$

which are Banach spaces with respect to their natural norms for $1 \leq q \leq \infty$. Because $w = w_s : x \to (1 + |x|)^s$, $s > 0$, satisfies $w(xy) \leq Cw(x)w(y)$ for all $x, y \in \mathbf{R}^m$ it is a weight function on $\mathbf{R}^m$ in the sense of Reiter [**19**], and L^q_s is invariant under translation, given by $L_x f(z) := f(z - x)$. It also follows that L^1_s is a Banach convolution algebra (called *Beurling algebra*, cf. [**19**]) for $s > 0$ and one has $L^1_{|s|} * L^q_s \subseteq L^q_s$ for $1 \leq q \leq \infty$, together with the corresponding norm estimate. $C^\circ(\mathbf{R}^m)$ denotes the space of continuous, complex-valued functions vanishing at infinity, endowed with the sup-norm $\| \ \|_\infty$, and $M(\mathbf{R}^m)$ denotes the space of bounded, regular measures on $\mathbf{R}^m$, which is considered as the dual space to $C^\circ(\mathbf{R}^m)$. We denote by $\mathcal{F}L^p$ the image of L^p (considered as a subspace of $\mathcal{S}'(\mathbf{R}^m)$) under the Fourier transform and assume that it is endowed with its natural norm, i.e., $\|\hat{f}\|_{\mathcal{F}L^p} := \|f\|_{L^p}$. It is now clear from the basic properties of the Fourier transform that $\mathcal{F}L^p$ is a translation invariant Banach space of tempered distributions which is a pointwise Banach module over $\mathcal{F}L^1(\mathbf{R}^m)$. Consequently it is possible to define the *Wiener-type spaces* $W(\mathcal{F}L^p, L^q_s)$ (as introduced by the author in [**9**]) as follows: let $k \in \mathcal{D}(\mathbf{R}^m)$ be any nonzero *window-function* (one should think of a positive plateau-like function, satisfying $k(z) \cong 1$ on a compact set Q) and define the control function as follows:

$$K(f, k)(t) := \|(L_t k) f\|_{\mathcal{F}L^p} := \|M_t \hat{k} * \hat{f}\|_{L^p} \text{ for } t \in \mathbf{R}^m.$$

Then

$$W(\mathcal{F}L^p, L^q_s) := \{ f \in \mathcal{S}'(\mathbf{R}^m) | f \in (\mathcal{F}L^p)_{\text{loc}},\ K(f, k) \in L^q_s(\mathbf{R}^m) \},$$

endowed with its natural norm $\|f | W(\mathcal{F}L^p, L^q_s)\| := \|K(f, k) | L^q_s\|$.

Using this definition it is easy to verify that these Wiener-type spaces are translation invariant, but also invariant under multiplication with characters. More precisely, one has the following estimates for the operator norms of these operators:

$$\|L_x f | W(\mathcal{F}L^p, L^q_s)\| \leq (1 + |x|)^s \|f | W(\mathcal{F}L^p, L^q_s)\| \text{ for all } x \in \mathbf{R}^m,$$

and

$$\|M_t f | W(\mathcal{F}L^p, L^q_s)\| = \|f | W(\mathcal{F}L^p, L^q_s)\| \text{ for all } t \in \mathbf{R}^m.$$

An essential tool for the discrete way of describing these spaces (this is the original definition of these spaces) is based on the existence of suitable partitions of unity. Since we do not need the most general description in our situation we can stick to the following (restricted) definition of a bounded uniform partition of unity of size $\delta > 0$ (for short a δ-BUPU) in $\mathcal{F}L^1$:

DEFINITION. Given $\delta > 0$ any bounded family in the Banach space $\Psi = (\psi_n)_{n \in \mathbf{Z}^m}$ in $\mathcal{F}L^1(\mathbf{R}^m)$ is called a δ-BUPU in $\mathcal{F}L^1$ if the following properties hold:

(BP1) There is a lattice $\delta \mathbf{Z}^m$ in $\mathbf{R}^m$ (for some positive δ) such that $\mathrm{supp}\psi_n \subseteq B(\delta n, \delta)$ (the ball around δn with radius δ).

(BP2) $\sum_{n \in \mathbf{Z}^m} \psi_n(x) \equiv 1$.

Using BUPUs one can give the following discrete characterization: $f \in W(\mathcal{F}L^p, L^q_s)$ if and only if, for some BUPU, one has

$$\Big[\sum_{n \in \mathbf{Z}^m} \|f\psi_n\|^q_{\mathcal{F}L^p} (1 + |n|)^{sq} \Big]^{1/q} < \infty$$

(and this expression gives an equivalent norm, cf. [**9**]).

It is a consequence of this description that any window function k as described above (even any Schwartz function or any $k \in W(\mathcal{F}L^1, L^1_s)$ defines the same space $W(\mathcal{F}L^p, L^q_s)$ and gives an equivalent norm, cf. [**11**].

The modulation space $M^s_{p,q}(\mathbf{R}^m)$ can now be defined as inverse images of the spaces $W(\mathcal{F}L^p, L^q_s)$ under the Fourier transform. The invariance properties of Wiener-type spaces are easily translated into invariance properties of modulation spaces. Thus one has isometric translation invariance and the following estimates for the multiplication operators M_t : $\|M_t f | M^s_{p,q}\| \leq (1 + |t|)^s \|f | M^s_{p,q}\|$. In particular, the spaces $M^{|s|}_{1,1}(\mathbf{R}^m)$ are character invariant Segal algebras (i.e., they are dense, isometrically translation invariant spaces in $L^1(\mathbf{R}^m)$, complete with respect to their own norm, hence Banach ideals in $L^1(\mathbf{R}^m)$ (cf. [**19**; Chapter 6 & 2.2] for details about Segal algebras).

Since Fourier transformation is very well compatible with duality it is clear from the general results on decomposition spaces (of which Wiener-type spaces are a special case, cf. [**12**], [**11**], [**10**]) that modulation spaces show the natural behaviour with respect to duality, i.e., one has $\left(M^s_{p,q}(\mathbf{R}^n)\right)' = M^{-s}_{p',q'}(\mathbf{R}^n)$, for $1 \leq p, q < \infty$.

3. Atomic characterization for the modulation spaces. We want to prove the atomic characterization of modulations spaces indicated in the abstract. Apparently we have to verify two partial results, one on *synthesis* (i.e., that the expression in the atomic characterizations are actually convergent to elements in $M^s_{p,q}(\mathbf{R}^m)$) and, on the other hand, the *decomposition* result. We shall prove the last mentioned first. Because $M^{|s|}_{1,1}(\mathbf{R}^m)$ is a Segal algebra we shall write S for this space throughout the proof (fixing s).

VERFICATION OF THE MAIN THEOREM (stated in the abstract).

A) Let $g \in S$, $g \neq 0$ be given, and $f \in M^s_{p,q}(\mathbf{R}^m)$. We shall prove the decomposition result first with respect to functions $g \in S$ with the additional property that $\operatorname{supp} \hat{g} \subseteq K$, some compact subset of $\mathbf{R}^m$. Since it is possible to replace g by $M_{\beta n} g$, if necessary, we may assume that there exists $\delta > 0$ such that $\hat{g}(t) \neq 0$ for $|t| \leq \delta$. Applying Wiener's theorem on the inversion of the Fourier transform (cf. [**19**; Chapter 1 & 3.6] we find that there exists $h \in L^1(\mathbf{R}^m)$ such that $\hat{h}(t)\hat{g}(t) = 1$ for all $|t| \leq \delta$. Without loss of generality we may assume that $\hat{h}$ has compact support, e.g., $\hat{h}(t) = 0$ for $|t| \geq 2\delta$. Now let $\Phi = (\varphi_n)_{n \in \mathbf{Z}^m}$ be any *bounded, uniform spectral decomposition* of unity of size $\leq \delta$, i.e., a family given as inverse image under the Fourier transform of a δ-BUPU. Consequently we have $\sigma = \sum_{n \in \mathbf{Z}^m} \psi_n * \sigma$ for any $\sigma \in \mathcal{S}'(\mathbf{R}^m)$ (for example), where $\psi_n = M_{\beta n}\psi_0$ for some ψ_0 with supp $\hat{\psi}_0 \subseteq B(0,\delta) =: Q$. It is our aim to start with this spectral decomposition (at the moment we only have convergence in the weak topology, but part C) will show that one has norm convergence for $1 \leq p, q < \infty$). Since $\psi_n = M_{\beta n}(\psi_0 * g * h)$ we can write

$$f = \sum_{n \in \mathbf{Z}^m} \left(M_{\beta n}(\psi_0 * g * h)\right) * f =: \sum_{n \in \mathbf{Z}^m} M_{\beta n}(f_n * g),$$

with $f_n := (\psi_0 * h) * M_{-\beta n} f$. For later use let us fix the following constants:

(i) For any compact set $Q \subseteq \mathbf{R}^m$ there exists $C_Q > 0$ such that

$$||f|M|s_{p,q}|| \leq C_Q||f||_p$$

and (for later use)

$$||f|W(C^\circ, L^p)|| \leq C_Q||f||_p \text{ for all } f \in L^p(\mathbf{R}^m) \text{ with supp}\hat{f} \subseteq Q$$

(cf. [9], Theorem 5 for a proof of the last statement).

(ii) Using the discrete version of the norm on Wiener-type spaces (cf. [9]) we know that there exists $C_\Phi > 0$ such that

$$\sum_{n\in\mathbf{Z}^m} ||f_n||_p^q(1+|\beta n|)^{sq} \leq \sum_{n\in\mathbf{Z}^m} ||h||_1^q||M_{\beta n}\psi_0 * f||_p^q(1+|\beta n|)^{sq}$$
$$\leq C_\phi^p||h||_1^q||f|M_{p,q}^s(\mathbf{R}^m)||^q.$$

In the next step we apply a variant of Shannon's principle which will allow us to replace, given the f_n's, the convolution $f_n * g$ by a discrete sum of translates. Thus let us assume for the rest of this paragraph that f belongs to $L^p(\mathbf{R}^m)$ and supp$\hat{f} \subseteq Q$.

We proceed as follows, having a look on the Fourier transform side and using the notation $\sqcup\!\sqcup$ for the (translation bounded) Radon measure given as $\sqcup\!\sqcup := \sum_{k\in\mathbf{Z}^m} \delta_k$ and $\sqcup\!\sqcup_\rho := \sum_{k\in\mathbf{Z}^m} \delta_{\rho k}$. Since supp$\hat{g} \subseteq K$ (compact) there exists $\rho > 0$ such that $\hat{f}\hat{g} = \left[\sum_{k\in\mathbf{Z}^m}(L_{\rho k}\hat{f})\right]\hat{g} = (\sqcup\!\sqcup_\rho * \hat{f})\hat{g}$, or going back to the functions and using Poisson's formula, telling us that $\sqcup\!\sqcup$ is invariant under the Fourier transform

$$f * g = \left((\rho^{-m}\sqcup\!\sqcup_{1/\rho})f\right) * g = \rho^{-m}\sum_{k\in\mathbf{Z}^m} f(k/\rho)L_{k/\rho}g.$$

Applying the same argument to each f_n we have the following estimate for the sequence of coefficients $a_{n,k} := \rho^{-m}f_n(k/\rho)$:

$$\Big(\sum_{k\in\mathbf{Z}^m}|a_{n,k}|^p\Big)^{1/p} \leq C_\rho||f_n|W(C^\circ, L^p)|| \leq C_\rho C_Q||f_n||_p \text{ for all } n \in \mathbf{Z}^m,$$

which gives together with the previous estimates, the required summability properties for the double sequence $(a_{n,k})$.

B) Let us now consider the case of an arbitrary element g_1 in S. Since the Segal algebra $S \subseteq S_0(\mathbf{R}^m)$ is continuously embedded into

Wiener's algebra $W(\mathbf{R}^m) = W(C^0, L^1)$ (cf. [**8**]), hence into the Segal algebra $W(L^p, L^1)$ for $1 \leq p < \infty$, it is possible to approximate g_1 in the norm of $W(L^p, L^1)$ by elements g with compactly supported Fourier transform. In order to have the right constants (appropriate a priori estimates) let us note that we have the following facts at our disposition:

(iii) The family $\rho^{-m} \sqcup\!\sqcup\!\sqcup_{1/\rho}$ is uniformly bounded in the space $W(M, L^\infty)$.

(iv) There is a universal constant $C_p > 0$ (depending only on the norms used) such that the following estimates hold true (cf. [**9**], [**12**]):

$$\begin{aligned} ||g||_1 &\leq C_p ||g|W(L^p, L^1)|| \text{ for all } g \in W(L^p, L^1), \\ ||f\mu|W(M, L^p)|| &\leq C_p ||f|W(C^\circ, L^p)||\ ||\mu|W(M, L^p)|| \\ &\qquad \text{for } f \in W(C^\circ, L^p),\ \mu \in W(M, L^p), \\ ||\nu * g||_p &\leq C_p ||\nu|W(M, L^p)||\ ||g||W(L^p, L^1)|| \\ &\qquad \text{for } \nu \in W(M, L^p),\ g \in W(L^p, L^1). \end{aligned}$$

(v) Combining these estimates (with (i) above), we find some constant C_Q^1 (only dependent on the common support of $\hat{f}$ and p) such that

$$||(\rho^{-m} \underset{1/\rho}{\sqcup\!\sqcup\!\sqcup}) f|W(M, L^p)|| \leq C_Q^1 ||\ f\ ||_p \qquad \text{if supp } \hat{f} \subseteq Q.$$

Writing, for brevity, $D_\rho f$ for $(\rho^{-m} \sqcup\!\sqcup\!\sqcup_{1/\rho}) f$ (discrete version of f), we obtain the following estimate in L_p (still assuming supp$\hat{f} \subseteq Q$ and ρ chosen depending on the support of $\hat{g}$ as above):

$$\begin{aligned} &||f * g_1 - D_\rho f * g_1||_p \\ &\leq ||f * (g_1 - g)||_p + ||f * g - D_\rho f * g||_p + ||D_\rho f * (g - g_1)||_p \\ &\leq ||f||_p ||g - g_1||_1 + 0 + C_p ||D_\rho f|W(M, L^p)||\ ||g - g_1|W(L^p, L^1)|| \\ &\leq ||f||_p C_p ||g - g_1|W(L^p, L^1)||(1 + C_Q^1). \end{aligned}$$

Having this estimate (which does *not* depend on the support of $\hat{g}$) it is clear that we can choose g such that

$$||g - g_1|W(L^p, L^1)|| \leq \left(2C_p(+C_Q^1) C_Q C_\phi ||h||_1\right)^{-1},$$

hence

$$||f * g_1 - D_\rho f * g_1||_p \leq \left(2C_Q C_\phi ||h||_1\right)^{-1} ||f||_p.$$

This estimate, being valid for each $f = f_n$, we obtain, summing over n:

$$\begin{aligned}
&\Big\|f - \sum_{n\in\mathbf{Z}^m} M_{\beta n}(D_\Psi f_n * g)|M^s_{p,q}\Big\| \\
&= \Big\|\sum_{n\in\mathbf{Z}^m} M_{\beta n}\Big((f_n - D_\Psi f_n) * g\Big)|M^s_{p,q}\Big\| \\
&\le \sum_{n\in\mathbf{Z}^m} \Big\|M_{\beta n}|M^s_{p,q}\Big\| \|(f_n - D_\Phi f_n) * g|M^s_{p,q}\| \qquad \text{(by (i))} \\
&\le \sum_{n\in\mathbf{Z}^m} (1+|\beta n|)^{sq} C_Q \|(f_n - D_\Psi f_n) * g\|_p \\
&\le C_Q \sum_{n\in\mathbf{Z}^m} (1+|\beta n|)^{sq} \|f_n\|_p (2C_Q C_\Phi \|h\|_1)^{-1} \qquad \text{(by (ii))} \\
&\le (2C_\Phi C_Q \|h\|_1)^{-1} C_Q C_\Phi \|h\|_1 \|f|M^s_{p,q}\| = 1/2 \cdot \|f|M^s_{p,q}\|.
\end{aligned}$$

We have thus found a linear mapping $T_\Psi : f \to \sum_{n\in\mathbf{Z}^m} M_{\beta n}(D_\Psi f_n * g)$, such that $\mathrm{Id} - T_\Psi$ is a contraction on $M^s_{p,q}(\mathbf{R}^m)$ for $1 \le p, q < \infty$. Consequently T_Ψ is invertible on $M^s_{p,q}$ and we have

$$f = T_\Psi(T_\Psi^{-1} f) = T_\Psi\Big(\sum_{l=0}^{\infty}(Id - T_\Psi)^l(f)\Big) := T_\Psi(h), \text{ with } h \in M^s_{p,q}(\mathbf{R}^m).$$

Since $\|h|M^s_{p,q}\| \le C\|f|M^s_{p,q}\|$ we have

$$f = T_\Psi(h) = \sum_{n,k} a_{n,k} M_{\beta n} L_{\alpha k} g,$$

with $\Big[\sum_n \Big(\sum_k |a_{n,k}|^p\Big)^{q/p} (1+|\beta n|)^{sq}\Big]^{1/q} \le C_2 \|h|M^s_{p,q}\| \le$

$$C_2 C \ \|f|M^s_{p,q}\|,$$

and the proof is complete in this case.

C) We have now to discuss the synthesis problem, i.e., given any element in $S = M^{|s|}_{1,1}(\mathbf{R}^n)$ and a double sequence $(a_{n,k})$ satisfying the summability condition stated above, the corresponding Gabor sum defines an element of $M^s_{p,q}(\mathbf{R}^m)$.

Given now $g \in M^{|s|}_{1,1}(\mathbf{R}^n)$ we start splitting it by means of a uniform spectral decomposition (as used above), i.e., we write

$$g = \sum_{j\in\mathbf{Z}^m} \varphi_j * g = \sum_{j\in\mathbf{Z}^m} M_{\beta j} g_j, \text{ with } g_j := \varphi_0 * M_{-\beta j} g,$$

and

$$\sum_{j \in \mathbf{Z}^m} \|g_j\|_1 (1+|j|)^{|s|} \le C_1 \|g|M_{1,1}^{|s|}\| < \infty.$$

For later use let us note that the $\hat{g}_j$'s have common compact support Q. Consequently they belong to any Segal algebra (cf. [**19**; Chapter 6 & 2.2]), in particular to the Segal algebra $W(L^1, L^p)$. Moreover, there is a constant $C_2 < 0$ such that $\|g_j|W(L^p, L^1)\| \le C_2 \|g_j\|_1$ for all $j \in \mathbf{Z}^m$ (cf. [**9**, Theorem 5] for an alternative proof of this assertion). Calculating within $\mathcal{S}'(\mathbf{R}^m)$ we obtain, using the identity $L_x M_t = M_t L_x e^{ix\cdot t}$ for all $x, t \in \mathbf{R}^m$:

$$\begin{aligned}\sum_{n,k} a_{n,k} M_{\beta n} L_{\alpha k} g &= \sum_{n,k} a_{n,k} M_{\beta n} L_{\alpha k} (\sum_j M_{\beta j} g_j) \\ &= \sum_j M_{\beta j} \Big(\sum_{n,k} a^j_{n,k} L_{\alpha k} M_{\beta n} g_j \Big)\end{aligned}$$

with $a^j_{n,k} = \exp(i\alpha k \cdot \beta(n-j))$, hence $|a^j_{n,k}| = |a_{n,k}|$ for all $j, n, k \in \mathbf{Z}^m$. Rewriting the sum (in order to introduce some notations) one has

$$\begin{aligned} h :&= \sum_{n,k} a_{n,k} L_{\alpha k} M_{\beta n} g =: \sum_{j,n} M_{\beta j} h_{j,n} := \sum_j M_{\beta j} h_j, \\ h_{j,n} :&= \sum_k a^j_{n,k} L_{\alpha k} M_{\beta n} g_j \\ &= \Big(\sum_k a^j_{n,k} \delta_{\alpha k} \Big) * M_\beta j g_j \in W(M, L^p) * W(L^p, L^1) \subseteq L^p \end{aligned}$$

and (cf. [**9**, Theorem 3]) the estimate

$$\begin{aligned} \|h_{j,n}\|_p &\le C_3 \|\sum_{n,k} a^j_{n,k} \delta_{\alpha k} | W(M, L^p)\| \, \|g_j | W(L^p, L^1)\| \\ &\le C_4 \Big(\sum_k |a_{n,k}|^p \Big)^{1/p} \|g_j\|_1. \end{aligned}$$

Now $\operatorname{supp} \hat{h}_{j,n} \subseteq \beta n + \operatorname{supp} \hat{g}_j \subseteq \beta n + Q$ for all $n, j \in \mathbf{Z}^m$. In order to get an estimate for the sum over the n's we observe next that the family $(\beta n + Q)_{n \in \mathbf{Z}^m}$ constitutes an admissible uniform covering of $\mathbf{R}^m$ (this means essentially that is a covering of uniformly bounded height, cf. [**12**, Cor. 2.6]) and consequently (this is another characterization of Wiener-type spaces) there exists $C_5 > 0$ such that

$$\|h_j|M^s_{p,q}\| = \Big\|\sum_n h_{j,n}|M^s_{p,q}\Big\| = \Big\|\sum_n \hat{h}_{j,n}|W(\mathcal{F}L^p, L^q_s)\Big\|$$
$$\leq C_5\Big(\sum_n \|h_{j,n}\|_p^q(1+|\beta n|)^{sq}\Big)^{1/p}$$
$$\leq C_6\Big[\sum_n\Big(\sum_k |a_{n,k}|^p\Big)^{q/p}(1+|n|)^{sq}\Big]^{1/q}\|g_j\|_1.$$

We can now carry out the last step, i.e., summation over the $j's$ which yields immediately the required estimate, completing the proof.

$$\|h|M^s_{p,q}(\mathbf{R}^m)\|$$
$$\leq \sum_j \|M_{\beta j}h_j\| \leq \sum_j |||M_{\beta j}|||\ \|h_j|M^s_{p,q}\|$$
$$\leq C_6\Big[\sum_n\Big(\sum_k |a_{n,k}|^p\Big)^{q/p}(1+|n|)^{sq}\Big]^{1/q}\Big(\sum_j C_7(1+|j|)^{|s|}\|g_j\|_1$$
$$\leq C_8\Big[\sum_n\Big(\sum_k |a_{n,k}|^p\Big)^{q/p}(1+|n|)^{sq}\Big]^{1/q}\|g|M^{|s|}_{1,1}\|$$

Besides the Corollaries stated already in the introduction one has among others, the following useful consequence.

COROLLARY 3. (cf. [**13**]) *The Banach space* $M^s_{1,1}(\mathbf{R}^n)$ (*with* $s \geq 0$) *is the smallest among all Banach spaces satisfying the following conditions*:

a) *It is continuously embedded into* $\mathcal{S}'(\mathbf{R}^m)$,

b) *It has non-trivial intersection with* $\mathcal{S}(\mathbf{R}^m)$,

c) *It is isometrically translation invariant,*

d) *It satisfies* $|||M_t||| = O(1+|s|)$ *for* $t \to \infty$.

It is also possible to use the atomic characterization of the spaces $M^{|s|}_{1,1}(\mathbf{R}^m)$ in order to give an alternative proof showing the freedom in choice of the function g_0 used in the definition of $M^s_{p,q}(\mathbf{R}^m)$ in the abstract (Note that for all results presented so far we could have worked with a function $g_0 = \hat{k}$, with $k \in \mathcal{D}(\mathbf{R}^m)$). That we could have used any non-zero window function $k \in W(\mathcal{F}L^1, L^1_s)$ (which is equivalent to the use of $\hat{k} = g_0$ in $M^{|s|}_{1,1}(\mathbf{R}^m)$, thus in particular any Schwartz function

$g_0 \in \mathcal{S}(\mathbf{R}^m)$, so especially the Gauss function) in our definition can be shown as follows: if the intergral expression involving σ is finite for a given element $g_0 \in M_{1,1}^{|s|}(\mathbf{R}^m)$ (e.g., with $\operatorname{supp} \hat{g}_0$ compact) it is easily verified that it is also finite for g_0 replaced by $M_t L_x g_0$, for any $t, x \in \mathbf{R}^m$, and in the estimate only an additional factor $(1+|t|)^s$ arises. Now inserting any $g_1 \in M_{1,1}^{|s|}(\mathbf{R}^m)$, written in the atomic way based on g_0, it is clear that the integral expression (involving g_1 now instead of g_0) is finite as well. Thus *any* non-zero element $g_0 \in M_{1,1}^{|s|}(\mathbf{R}^m)$ gives another equivalent norm. This was proved using different methods already in [**11**].

4. Several remarks.

REMARK 1. We have not discussed the cases involving $p = \infty$ or $q = \infty$ in detail here. However it is clear that only marginal modifications are necessary in order to get the result also for these limiting cases. The only serious difference is the fact that Gabor sums don't neccessarily converge in the norm topology of that space (more precisely, they are norm convergent for a given f if and only if it belongs to the closure of $\mathcal{S}(\mathbf{R}^m)$ in the corresponding space. We leave the verification of details to the reader (thus $f \in S_0'(\mathbf{R}^m)$ - cf. [**7**] for applications invoilving this space - if and only if it has a Gabor representation involving bounded coefficients).

REMARK 2. The above proof shows that the mapping $f \rightarrow (a_{n,k})_{(n,k)\in \mathbf{Z}^2}$ is a bounded linear mapping from a modulation space to the corresponding mixed norm weighted l^p-space (with the same parameters), having as left inverse the (linear, bounded) operator from the sequence space associating to each double sequence the corresponding Gabor sum. Thus modulation spaces are retracts of these sequence spaces (cf. [**2**] for the terminology). As a consequence, the results concerning interpolation of modulation spaces (as given in [**11**], derived from the corresponding results for Wiener-type spaces) can be obtained as a consequence of our atomic characterization given above. One of the most interesting consequences is the fact that the modulation spaces $M_{p,p}^0(\mathbf{R}^m)$ are invariant under the Fourier transform.

As another consequence one could mention the possibility of using the atomic characterization in order to derive trace theorems for modulation spaces (similar to those of Besov spaces, cf. [**11**], [**21**]).

REMARK 3. An alternative approach to our result could be given, starting with the atomic characterization of the minimal spaces $M_{1,1}^{|s|}(\mathbf{R}^n)$ (which could be proved somewhat more easily than the general case), and to substitute for g as in part a) its Gabor sum with respect to the general element g_1 as in B). However, this method requires elementary but combersome calculations.

REMARK 4. It is clear that only slight modification would have been necessary in order to work with weighted modulation spaces (= modulation spaces based on weighted L^p-spaces of the form $L^p_w(\mathbf{R}^m)$, cf. [**5**] for a discussion of such spaces, e.g., $w = w_s$ in the most simple case on $\mathbf{R}^m$). Again one would have found that there are plenty of Banach spaces in this family (they can be shown to coincide with the spaces $W(\mathcal{F}L^p_s, L^p_s)$) which are invariant under the Fourier transform (cf. [**14**], [**11**] for details in this direction). Finally, we mention that the natural setting for these spaces (they don't need any dilation or differentiability structure of $\mathbf{R}^m$) would be that of lca. groups. Because, in this more general setting the space of tempered distributions is not so well known (and at least more complicated to work with) an alternative approach has to be chosen for the definition of the inverse Fourier transform of (quite general) Wiener-type spaces of the form $W(\mathcal{F}L^p_m, L^q_w)$. Such an approach has been presented in [**11**]. It is formulated in the most general form including also Banach spaces of Beurling-Björck ultra-distributions (which goes far beyond the nice family of spaces, $M^s_{p,q}(\mathbf{R}^m)$, treated here). In order to get atomic characterization of these spaces one has to modify our arguments again only in the expected way, using structure theory of lca. groups in order to get the fine BUPUs or arbitrary fine spectral decompositions, or alternatively the construction give in [**6**].

REMARK 5. A completely different approach to atomic decomposition, stressing the action of the Heisenberg group $\mathbf{R}^m \times \mathbf{R}^m \times \mathbf{T}$ on modulation spaces through $\pi((x,y,t)) := tM_xL_y$ is given in [**15**]. The approach given there does not explicitly make use of the Fourier transform in the way we have done. Moreover, that approach stresses the analogy between various situations where atomic decompositions arose in the literature, looking at them from a group theoretic point of view.

REMARK 6. In conclusion let us mention that, despite the great similarity between the situation one has in the atomic description of Besov spaces there is one big difference. Whereas Y. Meyer (see [**18**]) has found that in the case of Besov spaces, by means of a delicate construction, it is possible to find suitable functions g_0 such that the set of atoms actually forms a complete orthogonal system it has been shown (cf. [**1**]) that it is impossible to find any $g_0 \in S_0(\mathbf{R}^m) \supseteq \mathcal{S}(\mathbf{R}^m)$ such that the family $(M_{\beta n} L_{\alpha k} g_0)_{(n,k)\in \mathbf{Z}^2}$ forms a complete orthogonal system in $L^2(\mathbf{R}^m)$. On the other hand, as far as its aim is concerned our method is very closely related to the method of nonorthogonal expansions using frames as described by Daubechies/Grossmann/Meyer in [**3**].

REFERENCES

1. R. Balian, *Un princip d'incertitude fort en théorie du signal ou en mécanique quantique*, C.R. Acad. Sc. Paris, t. **292**, Série II, (1981), 1357-1362.

2. J. Bergh and J. Löfström, *Interpolation spaces, an introduction*, Grundl. math. Wiss. **223**, Springer Verlag, 1976.

3. I. Daubechies, A. Grossmann and Y. Meyer, *Painless nonorthogonal expansion*, J. Math. Phys. **27** (1986), 1271-1283.

4. H.G. Feichtinger, *A characterization of Wiener's algebra on locally compact groups*, Archiv f. Math. **29** (1977) 136-140.

5. ———, *Gewichtsfunktionen auf lokalkompakten Gruppen*, Sitzber. d. Österr. Akad. Wiss. **188** (1979), 451-471.

6. ———, *A characterization of minimal homogeneous Banach spaces*, Proc. Amer. Math. Soc. **81** (1981), 55-61.

7. ———, *Un espace de Banach de distributions tempérées sur les groupes localement compactes abelians.* Compt. Rend. Acad. Sci. Paris, Ser. A, **290**/17 (1980), A791-A794.

8. ———, *On a new Segal algebra*, Monatsh. Math. **92** (1981), 269-289.

9. ———, *Banach convolution algebras of Wiener's type*, Proc. Conf. "Functions, Series, Operators", Budapest, August 1980, Colloquia Math. Soc. J. Bolyai, North Holland Publ. Co., Amsterdam - Oxford - New York, 1983, 509-524.

10. ———, *A new family of functional spaces on Euclidean n-space* Proc. Conf. "Theory of Approximation of Functions", June 1983, Kiew, Teor. Priblizh, UDC 517.98, 493-497, zbl.505/46024.

11. ———, *Modulation spaces on locally compact abelian groups*, Techn. Report, Univ. Vienna, 1983.

12. ——— (with P. Gröbner), *Banach spaces of distributions defined by decomposition methods*, I. Math. Nachr. **123** (85), 97-120.

13. ———, *Minimal Banach spaces and atomic representations, Publ. Math. Debrecen*, (1987), announcement at Conf. in Debrecen Nov. 1984, Publ. Math. Debrecen **33** (1986), 167 and **34** (1987), 231-240.

14. ———, *Generalized amalgams, with applications to the Fourier transform*, Conf. Approximation theory and function spaces, Spring 1986, Warsaw (Poland), report.