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FULLY INVARIANT SUBMODULES OF p-LOCAL BALANCED PROJECTIVE GROUPS

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ABSTRACT. In this note we answer completely the question concerning the structure of fully invariant submodules of p-local balanced projective groups. In fact, every such submodule turns out to be a direct sum of an S-group and a balanced projective, and so its structure is completely determined by the well-known theories surrounding the S-groups and balanced projectives.

Ever since Warfield's initial work on balanced projective groups [**4**], there has been an open question concerning the structure of fully invariant subgroups of those groups. In this note we will provide a complete answer in the p-local case. We are able to show that every fully invariant submodule of a p-local balanced projective group is an SKT module (as introduced by Wick [**5**].) Some nice consequences of this result are that fully invariant submodules of p-local balanced projective groups are classified by a complete set of isomorphism invariants and that they satisfy general structural properties known for the class of isotype submodules of p-local balanced projective groups such as transitivity, full transitivity, and the equivalence of p^α-high submodules [**2**].

We will assume that all groups in this note are p-local and abelian; that is modules over the ring $\mathbf{Z}_p = \{\frac{m}{n} : m, n \in \mathbf{Z}$, the ring of integers, with $(n,p) = 1\}$. Fully invariant submodules of a p-local group G are simply those submodules which contain their image under any endomorphism of G. It will be necessary to highlight the properties balanced projective p-local groups share which will be especially fruitful in our present study. Recall that the *height* of an element $x \in G$ is the ordinal α if $x \in p^\alpha G/p^{\alpha+1}G$, and x has height ∞ if $x \in p^\alpha G$ for all ordinals α. We will write $|x|_G$ to denote the height of x computed in G, and we will suppress the index when G is understood. An exact

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sequence $0 \to A \to B \to C \to 0$ is said to be *balanced exact* provided the sequence $0 \to p^\alpha A \to p^\alpha B \to p^\alpha C \to 0$ is also exact for each ordinal α, and a p-local group G is said to be *balanced projective* if G satisfies the projective property with respect to all balanced exact sequences. Following Warfield [**3**], we will call a p-group G an S-group if G appears as the torsion submodule of a p-local balanced projective group. Both the class of S-groups and the class of balanced projectives are classified by known invariants [**3**] and [**4**], and Wick [**5**] has shown that the class of *SKT modules* (which consists of those p-local groups which are isomorphic to a direct sum of an S-group and a balanced projective group) can be classified by combining the invariants for the preceding two classes of groups.

Balanced projective p-local groups have many nice structural properties and can be characterized in several different fashions [**2**], but we will only require that balanced projectives can be decomposed into a direct sum of groups of torsion-free rank one and that they possess a K-basis. Precisely, an element $x \in G$ is said to be *free valuated* if $|p^k x| = |x| + k$ for each $k < \omega$, and if X if an independent set of free valuated elements $\{x_i\}$ such that $|\sum t_i x_i| = \min\{|t_i x_i|\}$ for $t_i \in Z_p$ and $G/\langle X\rangle$ is torsion, then we say that X is a K-*basis* for G. Following Warfield [**4**], a p-local group M is a λ-*elementary balanced projective group* (where λ is a limit ordinal) if $p^\lambda M \cong Z_p$ and $M/p^\lambda M$ is torsion totally projective. Recall that a totally projective group is simply a torsion balanced projective group. Warfield was able to show that every balanced projective group can be written as a direct sum of λ-elementary balanced projective groups for various limit ordinals λ, a totally projective p-group, and a divisible group. Hence, for every element x in a balanced projective group G, there exists some non-negative integer r such that $p^r x$ is free valuated.

A *p-height sequence*, written $\overline{\alpha} = \{\alpha_n\}_{n<\omega}$, is a strictly increasing sequence of ordinals, α_n, where we follow the standard conventions that $\infty + n = \infty$ for all $n < \omega$ and that $\infty < \infty$. If $x \in G$, then a p-height sequence can be generated by x be setting $\alpha_n = |p^n x|$. Hence, for every p-height sequence $\overline{\alpha}$, there is a fully invariant submodule

$$G(\overline{\alpha}) = \{x \in G \,|\, |p^n x| \geq \alpha_n \text{ for every } n < \omega\}.$$

It is an unpublished result attributed to E. Walker and L. Fuchs that every fully invariant subgroup of a totally projective p-group G has the form $G(\overline{\alpha})$ for some p-height sequence $\overline{\alpha}$ and that every such subgroup is totally projective with $G/G(\overline{\alpha})$ also totally projective.

The success mentioned above with the study of fully invariant subgroups of totally projective p-groups has led Warfield [**4**] and others to search for a description of fully invariant submodules of balanced projective modules. The näive conjecture that these submodules also have the form $G(\overline{\alpha})$ (for some appropriate p-height sequence $\overline{\alpha}$) and that they are again balanced projective leads to quick counter-examples. The torsion part of a mixed balanced projective group is indeed fully invariant and is not necessarily balanced projective, and if the torsion part $T \subseteq G(\overline{\alpha})$ for some p-height sequence $\overline{\alpha}$, then there may appear elements of infinite order in $G(\overline{\alpha})$ which make the equality impossible.

THEOREM 1. *If $\overline{\alpha}$ is a p-height sequence and G is balanced projective, then $G(\overline{\alpha})$ is also balanced projective.*

PROOF. Decompose $G = \oplus_\lambda M_\lambda \oplus T$, where each M_λ is a λ-elementary balanced projective group for some limit ordinal λ and T is a torsion totally projective group. We will ignore divisible parts, since if D is the divisible part of G, then D will also be the divisible part of $G(\overline{\alpha})$ for any p-height sequence $\overline{\alpha}$. Now $G(\overline{\alpha})$ also has the direct decomposition $G(\overline{\alpha}) = \oplus_\lambda M_\lambda(\overline{\alpha}) \oplus T(\overline{\alpha})$, and since $T(\overline{\alpha})$ is totally projective, the proof will be complete if we can show $M_\lambda(\overline{\alpha})$ is balanced projective for a fixed but arbitrary limit ordinal λ.

We will begin by showing that $p^\omega M_\lambda(\overline{\alpha}) = p^\gamma M_\lambda$, where $\gamma = \sup\{\alpha_n\}$. Suppose that $x \in p^\omega M_\lambda(\overline{\alpha})$. The, for a fixed but arbitrary $n < \omega, x \in p^n M_\lambda(\overline{\alpha})$, which implies that $x = p^n y$ for $y \in M_\lambda(\overline{\alpha})$. Hence $x \in p^{\alpha_n} M_\lambda$ for any nonnegative integer n, and so $x \in \cap_{n<\omega} p^{\alpha_n} M_\lambda = p^\gamma M_\lambda$. Now suppose $x \in p^\gamma M_\lambda$. Then $x \in \cap_{n<\omega} p^{\alpha_n} M_\lambda$, and so, for any fixed but arbitrary non-negative integer $n, x = p^n y$ for some $y \in p^{\alpha_n} M_\lambda$. Let m be a non-negative integer. If $m \leq n$, then $|p^m y| \geq \alpha_n + m \geq \alpha_m$ since $\overline{\alpha}$ is increasing, and if $m > n$, then write $m = n + t$ for $t > 0$ and note that $|p^m y| = |p^t(p^n y)| = |p^t x| \geq \gamma + t > \alpha_m$. Thus, for each $n < \omega$, there exists $y \in M_\lambda(\overline{\alpha})$ such that $x = p^n y$. We conclude that $x \in p^\omega M_\lambda(\overline{\alpha})$, and the claim is proved.

Suppose $\lambda \geq \gamma$ and write $\lambda = \gamma + \beta$ for some ordinal β. In this case,

$$p^{\lambda} M_{\lambda} = p^{\beta}(p^{\gamma} M_{\lambda}) = p^{\beta}(p^{\omega} M_{\lambda}(\overline{\alpha})) = p^{\omega+\beta} M_{\lambda}(\overline{\alpha})$$

is balanced projective. Now $M_{\lambda}/p^{\lambda}M_{\lambda}$ is torsion and totally projective, and so (since $p^{\lambda}M_{\lambda}$ is nice in M_{λ})$[M_{\lambda}/p^{\lambda}M_{\lambda}](\overline{\alpha}) = M_{\lambda}(\overline{\alpha})/p^{\lambda}M_{\lambda}$ is also totally projective. Thus $M_{\lambda}(\overline{\alpha})/p^{\omega+\beta}M_{\lambda}(\overline{\alpha})$ is balanced projective, and it follows that $M_{\lambda}(\overline{\alpha})$ is balanced projective by Theorem 5.1 in [**3**]. Finally, suppose $\lambda < \gamma$. Since $\lambda < \sup\{\alpha_n\}$, there must exist some integer m such that $\alpha_m > \lambda$. Then

$$p^{m} M_{\lambda}(\overline{\alpha}) \subseteq p^{\alpha_m} M_{\lambda} \subseteq p^{\lambda} M_{\lambda} \cong Z_p,$$

and, consequently, the Z_p-module $p^m M_{\lambda}(\overline{\alpha})$ is either 0 or else isomorphic to Z_p. Hence the torsion part of $M_{\lambda}(\overline{\alpha})$ is bounded by p^m, and so $M_{\lambda}(\overline{\alpha}) = tM_{\lambda}(\overline{\alpha}) \oplus F$, where $F \cong Z_p$ or 0 and $tM_{\lambda}(\overline{\alpha})$ is a direct sum of cyclic p-groups of order $\leq p^m$. We conclude that $M_{\lambda}(\overline{\alpha})$ is balanced projective. □

COROLLARY 1. *Every fully invariant subgroup of an S-group is an S-group.*

The above corollary is possible by Kaplansky's delicate argument for the description of fully invariant subgroups of fully transitive p-groups [**1**]. Recall that a p-local group G is fully transitive if, whenever x and y are elements of G with $|p^m x| \leq |p^m y|$ for each non-negative integer m, then there is an endomorphism of G carrying x onto y. It was shown in [**2**] that every isotype submodule of a p-local balanced projective group is fully transitive, but it is sufficient for the present study to be content with the fact that balanced projectives are fully transitive. We will, however, carry over Kaplansky's argument to the most general setting possible. If $\{\sigma_i\}$ is a strictly increasing sequence of ordinals, then a *gap* refers to the situation where $\sigma_i + 1 < \sigma_{i+1}$. We will write tG to denote the torsion part of a p-local group G.

LEMMA 1. (KAPLANSKY [**1**]). *Suppose G is a reduced p-local group and $\sigma_0, \sigma_1, \ldots, \sigma_{n-1}, \sigma_n = \infty$ is a strictly increasing sequence of* *ordinals. Then there exists some* $g \in G$ *of order* p^n *such that* $|p^i g| = \sigma_i$ *for* $i = 0, 1, \ldots, n-1$ *if and only if the following condition holds:*

(*) *if there is a gap between* σ_i *and* σ_{i+1}*, then there exists a nonzero element of the group* $p^{\sigma_i}G[p]/p^{\sigma_i+1}G[p]$.

LEMMA 2. *Suppose* G *is a reduced p-local group which satisfies the fully transitive property. Suppose also that* H *is a fully invariant submodule of* G *and* $\overline{\alpha} = \{\alpha_n\}_{n<\omega}$ *is the p-height sequence* $\alpha_n = \min\{|p^n h| : h \in H\}$. *If* $g \in G(\overline{\alpha})$ *and there exists an* $h \in H$ *with a nonnegative integer* k *such that* $p^k h$ *is free valuated and* $|p^k h| \leq |p^k g|$, *then* $g \in H$.

PROOF. We establish the existence of an $\overline{h} \in H$ such that $|p^m \overline{h}| \leq |p^m g|$ for every $m < \omega$. Once this has been accomplished, we can apply full transitivity to yield an endomorphism of G carrying $\overline{h}$ onto g, and this will imply that $g \in H$ since H is fully invariant. Our argument for the existence of such an $\overline{h}$ will be modeled after Kaplansky [**1**].

Write $k = \min\{i :$ there exists $h \in H$ with $p^i h$ free valuated and $|p^i h| \leq |p^i g|\}$. Because of the hypotheses, such a k exists in our setting. Suppose $\alpha_0, \ldots, \alpha_k$ has no gaps and choose $h_1 \in H$ such that $|p^k h_1| = \alpha_k$. By the way $\overline{\alpha}$ and k were chosen, $p^k h_1$. Suppose $|p^{k+r} h_1| > \alpha_{k+r-1} + 1$ is the first gap. There exists $g' \in G$ such that $|g'| > \alpha_{k+r-1}$ and $p^{k+r} h_1 = pg'$, and so $p^{k+r-1} h_1 - g'$ has order p with height α_{k+r-1}. By Lemma 1, there exists some $\overline{g} \in G$ with $p^{k+r}\overline{g} = 0$ and $|p^i \overline{g}| = \alpha_i$ for each $i = 0, 1, \ldots, k+r-1$. Since $|p^k h| > \alpha_k$ and $p^k h$ is free-valuated, it follows that $|p^m(\overline{g} + h)| \leq |p^m g|$ for each $m < \omega$. But since $|p^i \overline{g}| \geq |p^i h_1|$ for each $i < \omega, \overline{g} \in H$, and so $\overline{h} = \overline{g} + h \in H$ with $|p^m \overline{h}| \leq |p^m g|$ for each $m < \omega$.

Now suppose there are some gaps in the sequence $\alpha_0, \ldots, \alpha_k$ and suppose the first gap occurs between α_{j-1} and α_j. There is an $h_j \in H$ such that $|p^i h_j| = \alpha_i$ for all $i = 0, \ldots, j-1$. If $|p^{j-1}(h_j + h)| > \alpha_{j-1}$, then $|p^i h| = \alpha_i$ for all $i = 0, \ldots, j-1$ and we can replace $h_j + h$ by h. If $p^j h_j \neq 0$, then $|p^j h_j| \geq \alpha_j > \alpha_{j-1} + 1$ by definition, and so $p^j h_j = p\overline{g}$ for some $\overline{g}$ of height $> \alpha_{j-1}$. Hence $p^{j-1} h_j - \overline{g}$ has order p and height α_{j-1}, and so by Lemma 1, there exists a $g' \in G$ with $p^j g' = 0$ and $|p^i g'| = \alpha_i$ for all $i = 0, 1, \ldots, j-1$. Since $|p^m h_j| \leq |p^m g'|$ for each $m < \omega$, we can apply full transitivity and the fully invariant nature of H again to conclude that $g' \in H$. Hence we can assume that $p^j h_j = 0$ and $|p^i(h_j + h)| = \alpha_i$ for each $i = 0, 1, \ldots, j-1$. If the second gap occurs between α_{l-1} and α_l, then there is an $h' \in H$ with $|p^{l-1}h'| = \alpha_{l-1}$, and this implies that $|p^i h'| = \alpha_i$ for all $i = j, \ldots, l-1$. If $|p^{l-1}h| = \alpha_{l-1}$, then $|p^i h| = \alpha_i$ for $i = j, \ldots, l-1$ and we can replace $h' + h$ by h. By Lemma 1 again, we can choose $h_l \in G$ such that $p^l h_l = 0, |p^i h_l| \geq \max\{|p^i h'|, \alpha_i + 1\}$ for each $i = 0, 1, \ldots, j-1$, and $|p^i h_l| = |p^i h'|$ for $i = j, \ldots, l-1$. Clearly $h_l \in H$ since $|p^i h'| \leq |p^i h_l|$ for each $i < \omega$. Hence $|p^i(h_j + h_l + h)| = \alpha_i$ for $i = 0, 1, \ldots, l-1$. Continuing in this fashion, for each gap that occurs between α_{r-1} and α_r for $r \leq k$, we construct $h_r \in H$ with $p^r h_r = 0$ and $|p^i(h_j + h_l + \cdots + h_r + h)| = \alpha_i$ for each $i = 0, 1, \ldots, r-1$. If the last gap occurs between α_{r-1} and α_r and $r < k$, then choose $h_k \in H$ such that $|p^k h_k| = \alpha_k$. By the way k and $\overline{\alpha}$ were chosen, $p^k h_k$ is not free valuated, and so there exists a gap in the p-height sequence of $p^k h_k$. Suppose $|p^{k+s}h_k| > \alpha_{k+s-1} + 1$ is the first such gap. Thus there exists an element of order p in G of height α_{k+s-1}, and so by Lemma 1, there is some $g_k \in G$ with $p^{k+s}g_k = 0, |p^i g_k| \geq \max\{|p^i h_k|, \alpha_i + 1\}$ for $i = r, r+1, \ldots, k+s-1$. Since $|p^i h_k| \leq |p^i g_k|$ for all $i < \omega$, we conclude that $g_k \in H$, and so $\overline{h} = h_j + h_l + \cdots + h_r + g_k + h$ is an element of H satisfying $|p^m \overline{h}| \leq |p^m g|$ for each $m < \omega$. □

THEOREM 2. *If H is a fully invariant submodule of the reduced p-local balanced projective group G, then H is an SKT module.*

PROOF. Let $\overline{\alpha} = \{\alpha_n\}_{n<\omega}$, where $\alpha_n = \min\{|p^n h| : h \in H\}$, and let $\delta = \min\{|x| : x \in H \text{ and } x \text{ is free valuated}\}$. Several cases must be considered depending upon the nature of δ and H. If $\delta = \infty$, then H is torsion and $H = tG(\overline{\alpha})$ by Kaplansky's argument [**1**]. By Theorem 1, H is an S-group, and so we can assume $\delta \neq \infty$.

Case 1. Suppose there exists $y \in H$ with $|p^n y| = \delta t_n$ for every $n < \omega$ with $|y|_H = 0$. We will first show that $H = tG(\overline{\alpha}) + p^\delta G$. It is clear that $tG(\overline{\alpha}) = tH$, and if $g \in p^\delta G$, then g satisfies Lemma 2 with $k = 0$, which implies that $g \in H$. Now suppose $h \in H$ is an element of infinite order and choose $r < \omega$ such that $p^r h$ is free valuated. We can assume that $\delta > \alpha_0$ for the remainder of the argument in this case, since if $\delta = \alpha_0$, then $H = p^\delta G$. If $p^r h \in p^{\delta + r}G$, then $p^r h = p^r g$ for some

$g \in p^\delta G$, and so $h = g + z$, where $z \in tG(\overline{\alpha})$. If $p^r h \notin p^{\delta+r}G$, then since $y - p^r h \in p^\delta G$, we can at least write $y = p^r h + \overline{g}$ for some $\overline{g} \in p^\delta G$. Since $\alpha_0 < \delta$, there exists some $g \in p^{\alpha_0}G$ such that $\overline{g} = pg$, and thus $|p^r g| \geq \delta + r - 1$. Therefore $|p^r g| \geq |p^r h|$, where $g \in G(\overline{\alpha})$, and so $g \in H$ by Lemma 2. But then $y = p(p^{r-1}h + g)$, and this contradicts $|y|_H = 0$. We conclude that it is impossible for $p^r h \notin p^{\delta+r}G$, and so $H = tG(\overline{\alpha}) + p^\delta G$.

If we decompose $G = \oplus_\lambda M_\lambda \oplus T$, where each M_λ is a λ-elementary balanced projective and T is totally projective, then $tG(\overline{\alpha}) + p^\delta G = \oplus_\lambda (tM_\lambda(\overline{\alpha}) + p^\delta M_\lambda) \oplus (T(\overline{\alpha}) + p^\delta T)$. Since $T(\overline{\alpha}) + p^\delta T = T(\overline{\alpha})$ is totally projective, it suffices to show that, for a fixed but arbitrary limit ordinal $\lambda, H_\lambda = tM_\lambda(\overline{\alpha}) + p^\delta M_\lambda$ is an SKT module. If $\delta \geq \lambda + \omega$, then $H_\lambda = tM_\lambda(\overline{\alpha})$ is an S-group, and if $\lambda \leq \delta < \lambda + \omega$, then $H_\lambda = tM_\lambda(\overline{\alpha}) \oplus p^\delta M_\lambda$ is an SKT module. Suppose now that $\delta < \lambda$ and let $x \in M_\lambda(\overline{\alpha})$. In this case $|p^r x| \geq \lambda$ for some $r < \omega$, and so $p^r x = p^r g$ for some $g \in p^\delta M_\lambda$ since λ is a limit ordinal. Hence $x - g \in tM_\lambda(\overline{\alpha})$, and we conclude that $M_\lambda(\overline{\alpha}) = tM_\lambda(\overline{\alpha}) + p^\delta M_\lambda$, which implies by Theorem 1 that H_λ is balanced projective. Combining the above cases we have that $H = tG(\overline{\alpha}) + p^\delta G$ is an SKT module.

Case 2. Suppose there exists $y \in H$ with $|p^n y| = \delta + n$ for every $n < \omega$ with $0 < |y|_H = k$ for some $k < \omega$. With $\overline{\beta} = (\alpha_0, \alpha_1, \ldots, \alpha_{k-1}, \delta, \delta+1, \ldots)$, we will show that $H = tG(\overline{\alpha}) + G(\overline{\beta})$. We begin by proving that either δ is a limit ordinal or else we return to Case 1. Suppose δ is not a limit ordinal and write $\delta = \gamma + m$ for some positive integer m. If $m \leq k$, then $y = p^m \overline{g}$ for some $\overline{g} \in p^\gamma G$, which implies that $p^k h = p^m \overline{g}$ for some $h \in H$. Hence $p^m(p^n h - \overline{g}) = 0$, where $m + n = k$, and so $p^n h - \overline{g} \in tG(\overline{\alpha}) \subseteq H$. Now $\overline{g}$ must be free valuated with $|\overline{g}| = \gamma < \delta$, and this contradicts the minimality of $|y| = \delta$. Suppose now that $m > k$. Since $p^k h \in p^{\gamma+m}G$, it follows that $h - p^{m-k}\overline{g} \in tG(\overline{\alpha}) \subseteq H$ for some $\overline{g} \in p^\gamma G$, and so $|p^{m-k}\overline{g}| < \delta$ with $p^{m-k}\overline{g}$ free valuated, and this also contradicts the minimality of δ. Hence in this situation, we would either contradict δ not being a limit ordinal or else we would return to Case 1. We conclude that δ is a limit ordinal. We can also assume $\delta > \alpha_k$; for otherwise $H = G(\overline{\beta})$ is balanced projective by Lemma 2 and Theorem 1.

As in Case 1, $tG(\overline{\alpha}) = tH$, and if $x \in G(\overline{\beta})$,then $|p^k x| \geq |p^k h|$, where $h \in H$ is chosen so that $y = p^k h$. Since $p^k h$ is free valuated, we have that $x \in H$ by Lemma 2. Suppose now that $\overline{h}$ is an element of infinite order in H. If $\overline{h} \notin G(\overline{\beta})$, then it must be true that $|p^k\overline{h}| < \delta$. Indeed, if $m < \omega$ is chosen so that $|p^{k+m}\overline{h}| < \delta + m$, then by the nature of heights we force $|p^k\overline{h}| < \delta$. Now choose l so that $p^{k+l}\overline{h}$ is free valuated. Clearly $l > 0$ since $\overline{h} \notin G(\overline{\beta})$. If $p^{k+l}\overline{h} \in p^{\delta+l}G$, then $p^l(p^k\overline{h}) = p^lg$ for some $g \in p^\delta G$. Since δ is a limit ordinal, it follows that we can choose $g' \in p^{\alpha_{k-1}}G$ such that $p^kg' = g$. Hence $g' \in G(\overline{\beta}) \subseteq H$ and $p^l(p^k\overline{h}) = p^l(p^kg')$, and then $\overline{h} = g' + z$ for some $z \in tG(\overline{\alpha})$. Finally, if $\delta \leq |p^{k+l}\overline{h}| < \delta + l$, then $y = p^{k+l}\overline{h} + g_1$ for some $g_1 \in p^\delta G$. Since δ is a limit ordinal, there exists $g_2 \in p^{\alpha_k}G$ such that $p^{k+1}g_2 = g_1$. Thus $|p^{k+l}g_2| \geq \delta + l - 1 \geq |p^{k+l}\overline{h}|$, which implies that $g_2 \in H$ by Lemma 2. Hence $y = p^{k+1}(p^{l-1}\overline{h} + g_2)$, and this contradicts $|y|_H = k$. We conclude that $p^{k+l}\overline{h} \in p^{\delta+l}G$, and the claim that $H = tG(\overline{\alpha}) + G(\overline{\beta})$ is proved.

As before, we decompose $G = \oplus_\lambda M_\lambda \ominus T$, and write $tG(\overline{\alpha}) + G(\overline{\beta}) = \oplus_\lambda(t_\lambda M_\lambda(\overline{\alpha}) + M_\lambda(\overline{\beta})) \oplus (T(\overline{\alpha}) + T(\overline{\beta}))$. Since $T(\overline{\alpha}) + T(\overline{\beta}) = T(\overline{\alpha})$ is totally projective, it suffices to show that $H_\lambda = tM_\lambda(\overline{\alpha}) + M_\lambda(\overline{\beta})$ is an SKT module for a fixed but arbitrary limit ordinal λ. If $\delta \geq \lambda + \omega$, then $H_\lambda = tM_\lambda(\overline{\alpha})$, which is an S-group. Since δ is a limit ordinal, either $\delta < \lambda$ or else $\delta = \lambda$. If $\delta = \lambda$, then $p^ktM_\lambda(\overline{\beta}) \subseteq tp^\lambda M_\lambda = 0$, and so $tM_\lambda(\overline{\beta})$ is a bounded pure subgroup of $M_\lambda(\overline{\beta})$. Hence $M_\lambda(\overline{\beta}) = tM_\lambda(\overline{\beta}) \oplus F(\overline{\beta})$, where $F(\overline{\beta} \cong M_\lambda(\overline{\beta})/tM_\lambda(\overline{\beta})$ is torsion-free and isomorphic to 0 or Z_p. In this case $H_\lambda = (tM_\lambda(\overline{\alpha}) + tM_\lambda(\overline{\beta})) \oplus F(\overline{\beta})$, which is an SKT module. Finally, suppose $\delta < \lambda$ and pick $x \in M_\lambda(\overline{\alpha})$ with some $r < \omega$ such that $|p^rx| \geq \lambda$. Since λ is a limit ordinal $p^rx \in p^{\delta+r}M_\lambda$, and so $p^rx = p^rg$ where $g \in p^\delta M_\lambda$. Hence $x - g \in tM_\lambda(\overline{\alpha})$, and so $x \in tM_\lambda(\overline{\alpha}) + M_\lambda(\overline{\beta})$ since $p^\delta M_\lambda \subseteq M_\lambda(\overline{\beta})$. Therefore, $H_\lambda = M_\lambda(\overline{\alpha})$ is balanced projective, and so $H = tG(\overline{\alpha}) + G(\overline{\beta})$ is an SKT module.

Case 3. Suppose there exists $y \in H$ with $|y| = \delta, y$ free valuated, and $|y|_H \geq \omega$. In this case δ must be a limit ordinal, and so we can decompose $G = G_\delta \oplus N_\delta$, where G_δ is chosen so that $G_\delta \cap p^\delta G = 0$ and N_δ is chosen so that if $x \in N_\delta$ and x is free valuated, then $|x|_g \geq \delta$. Clearly such a decomposition is possible by Warfield's decomposition of balanced projectives [**4**]. Upon showing that $H = tG_\delta(\overline{\alpha}) \oplus N_\delta(\overline{\alpha})$ and realizing that $G(\overline{\alpha}) = G_\delta(\overline{\alpha}) \oplus N_\delta(\overline{\alpha})$, it will be clear that H is an SKT module.

It is evident that $tG_\delta(\overline{\alpha}) \subseteq H$, so consider $a \in N_\delta(\overline{\alpha})$. There is some non-negative integer r such that p^ra is free valuated, and so $|p^ra| \geq \delta$. Since $|y|_H \geq \omega$, there must be some $h \in H$ such that $|p^rh| \leq |p^ra|$, and hence $a \in H$ by Lemma 2. Suppose now that $h \in H$, and we can assume that h has infinite order. We write $h = g + x$, where $g \in G_\delta$ and $x \in N_\delta$. Since $h \in G(\overline{\alpha})$, it follows that $g \in G_\delta(\overline{\alpha})$ and $x \in N_\delta(\overline{\alpha})$, and since G is balanced projective, we can choose m large enough so that $p^m h, p^m g$, and $p^m x$ are all free valuated. It follows that $p^m g \in G_\delta \cap p^\delta G = 0$, which implies that $g \in tG_\delta(\overline{\alpha})$. This concludes the proof of Case 3 and also of Theorem 2. □

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