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CONVERGENCE OF A CLASS OF DEFICIENT INTERPOLATORY SPLINES

S.S. RANA*

1. Introduction. The most popular choice for reasonably efficient approximating functions still continues to be in favor of cubic splines (see de Boor [**3**, p. 49]). The interpolation problems of matching a cubic spline at one intermediate point and a deficient cubic spline at two intermediate points between the successive mesh points have been studied by Meir and Sharma [**6**]. For this case, further studies in the direction of the results proved in [**6**] have been made in [**1**, **4** and **5**]. Instead of supplicating the additional degrees of freedom by prescribing values at two intermediate points for deficient cubic splines, our object is to study deficient cubic splines having two interpolatory conditions, one of which is the matching condition at intermediate points of the dividing intervals while the other is matching of derivatives at intermediate points. Interesting studies for sharp convergence properties for such spline interpolants when $f \in C^3$ or $f \in C^4$ have been made in the last section of the paper. It may be mentioned here that analysis to obtain the error bounds for interpolating deficient complex cubic splines is given by Chien-Kel Lu in [**2**].

2. Existence and uniqueness. Let a mesh on $[0,1]$ be given by

$$P : 0 = x_0 < x_1 < \cdots < x_n = 1$$

with $p = x_i - x_{i-1}$ for $i = 1, 2, \ldots, n$. For a positive integer m, π_m denotes the set of algebraic polynomials of degree not greater than m. For a function s defined over P, we denote the restriction of s over $[x_{i-1}, x_i]$ by s_i. The class of deficient cubic splines defined over P is given by

$$S(3,P) = \{s : s \in C^1[0,1], s_i \in \pi_3 \text{ for } i = 1, 2, \ldots, n\}.$$

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Writing $t_i = x_{i-1} + \alpha p$ and $y_i = x_{i-1} + \beta p$ with $0 < \alpha < 1,\ 0 \le \beta \le 1$, and considering a function f, we propose

PROBLEM A. Suppose f' exists over P. Then under what restrictions on α and β does there exist a unique spline interpolant $s \in S(3,P)$ of f which satisfies the interpolatory conditions

$$s(y_i) = f(y_i) \tag{2.1}$$

$$s'(t_i) = f'(t_i) \tag{2.2}$$

for $1 \le i \le n$ where $f(y_i)$'s and $f'(t_i)$'s are given functional values and derivatives respectively?

In order to investigate Problem A, we observe that, since $s(x) \in S(3,P)$, $s'(x)$ is piecewise quadratic. Therefore, for the interval $[x_{i-1}, x_i]$, we may write

$$p^2 s'(x) = p(x - x_{i-1})m_i + p(x_i - x)m_{i-1} + (x_i - x)(x - x_{i-1})c_i \tag{2.3}$$

where $m_i = s'(x_i)$ and the c_i's are appropriate constants which have to be determined. Integrating (2.3), we get

$$\begin{aligned} 6p^2 s(x) = {} & 3p(x - x_{i-1})^2 m_i - 3p(x_i - x)^2 m_{i-1} \\ & + (x - x_{i-1})^2(3(x_i - x) + (x - x_{i-1}))c_i + 6p^2 d_i. \end{aligned} \tag{2.4}$$

We are now set to answer Problem A in

THEOREM 1. *Let f be* 1*-periodic. Then there exists a unique* 1*-periodic spline $s(x)$ in the class $S(3,P)$, satisfying the interpolatory conditions* (2.1) *and* (2.2) *if* (i) $\alpha = 1/3$ *and* $\beta \in [0, .84]$ *or if* (ii) $\alpha = 2/3$ *and* $\beta \in [.16, 1]$, *or if* (iii) $\alpha = 1/2$ *and* $\beta \in (0,1)$.

PROOF. Using the interpolatory conditions (2.1) and (2.2) in (2.4), we have,

$$6f(y_i) = 3\beta^2 p m_i - 3(1-\beta)^2 p m_{i-1} + (3 - 2\beta)\beta^2 p c_i + 6d_i \tag{2.5}$$

and

$$f'(t_i) = \alpha m_i + (1-\alpha)m_{i-1} + \alpha(1-\alpha)c_i. \tag{2.6}$$

Eliminating c_i, d_i between the equations (2.5) - (2.6) and using the requirement $s \in C$, we get

$$R_i m_{i-1} - (R_i + R_i^* + 1 - 6\alpha + 6\alpha^2)m_i + R_i^* m_{i+1} = F_i, \tag{2.7}$$

where $R_i = (1-\alpha)(1-\beta)^2(3\alpha - 2\beta - 1), R_i^* = \alpha\beta^2(2\beta - 3\alpha)$ and

$$F_i = (6\alpha(1-\alpha)(f(y_{i+1})-f(y_i))/p) - \beta^2(3-2\beta)(f'(t_{i+1})-f'(t_i)) - f'(t_i).$$

In order to prove Theorem 1, we consider all the cases of Theorem 1 separately. For case (i) in which $\alpha = 1/3$ and $\beta \in [0, .84]$, the coefficient of m_{i+1} is nonpositive for $0 \le \beta \le .5$ and nonnegative for $.5 \le \beta \le .84$. Further, the coefficients of m_{i-1} and m_i are always nonpositive and nonnegative respectively. Thus, the excess of the positive value of the coefficient of m_i over the sum of the positive values of the coefficients of m_{i-1} and m_{i+1} is $1/3$ for $0 \le \beta \le .5$ and $(1+2\beta^2-4\beta^3)/3 = J_1(\beta)$, say, for $.5 \le \beta \le .84$. For the case (ii) in which $\alpha = 2/3$ and $\beta \in [.16, 1]$, we observe that the coefficient of m_{i-1} is nonnegative for $.16 \le \beta \le .5$ and nonpositive for $.5 \le \beta \le 1$. However, the coefficients of m_{i+1} and m_i are always nonpositive and nonnegative respectively. Thus, the excess of the positive value of the coefficient of m_i over the sum of the positive values of the coefficients of m_{i-1} and m_{i+1} is $(-1+8\beta-10\beta^2+4\beta^3)/3 = J_2(\beta)$, say, for $.16 \le \beta \le .5$ and $1/3$ for $.5 \le \beta \le 1$. Considering $\alpha = 1/2$ and $\beta \in (0,1)$ for the last case of Theorem 1, we see that the coefficient of m_{i+1} is nonpositive for $\beta \in (0, 3/4]$, nonnegative for $\beta \in [3/4, 1)$ and the coefficient of m_{i-1} in (2.7) is nonnegative for $\beta \in (0, 1/4]$, nonpositive for $\beta \in [1/4, 1)$. Further, the coefficient of m_i is nonnegative for $\beta \in (0,1)$. Thus, the excess of the positive value of the coefficient of m_i over the sum of the positive values of the coefficients of m_{i-1} and m_{i+1} in (2.7) is $\beta(6-9\beta+4\beta^2)/2 = J_3(\beta)$, say, for $\beta \in (0, 1/4], 1/2$ for $\beta \in [1/4, 3/4]$ and $(1+3\beta^2-4\beta^3)/2 = J_4(\beta)$, say, for $\beta \in [3/4, 1)$. It may be seen easily that $J_1(\beta), J_2(\beta), J_3(\beta)$ and $J_4(\beta)$ are positive for corresponding values of β. Thus, the coefficient matrix A of the system of equations (2.7) is invertible and its row-max norm for $\alpha = 1/3$ (or $2/3$), that is,

$$||A^{-1}|| \le K_1 \tag{2.8}$$

where $K_1 = \max_\beta\{3, J_1^{-1}(\beta),\ J_2^{-1}(\beta)\}$. □

Starting with the equation (2.7) and following closely the foregoing proof of Theorem 1, we can also prove a similar result for other choices of α and β.

THEOREM 2. *Let f be 1-periodic. Then there exists a unique 1-periodic spline $s(x)$ in the class $S(3, P)$ satisfying the interpolatory conditions* (2.1) *and* (2.2) *if* (i) $\beta = 1/3$ *and* $\alpha \in [2/9, .74]$ *or if* (ii) $\beta = 2/3$ *and* $\alpha \in [.22, 7/9]$ *or if* (iii) $\beta = 1/2$ *and* $\alpha \in [.23, 7/9)$.

3. Error bounds. In this section, we obtain the bounds of the function $e = s - f$, where s is the deficient cubic spline interpolant of f in $S(3, P)$. In what follows, we shall use the notation that, for the function $f(x), f(x_i) = f_i$ and $w(f, p)$ is the modulus of continuity of f. We shall prove

THEOREM 3. *Let f be a 1-periodic function in the class $C^3[0, 1]$ and $s(x)$ be its spline interpolant in $S(3, P)$ of Theorem* 1. *Then, for* $r = 0, 1$,

$$||(s - f)^{(r)}(x)| \leq Kp^{3-r\omega}(f''', p) \tag{3.1}$$

where K is some positive constant.

PROOF. Writing $m_i - f'_i = e'_i$, we have from (2.7)

$$A(e'_i) = (F_i) - A(f'_i) = (D_i), \text{ say,} \tag{3.2}$$

where A is the coefficient matrix of the system of equations (2.7). In order to get the derivative of the error function, first we estimate (D_i). Using the results that $f(x) = f_j + (x - x_j)f'_j + (x - x_j)^2 f''_j/2 + (x - x_j)^3 f'''(\beta_j)/6$ and $f'(x) = f'_j + (x - x_j)f''_j + (x - x_j)^2 f'''(\alpha_j)/2$, where

α_j and β_j lie in the appropriate intervals, we see that

$$\begin{aligned} D_i =&(p^2/2)((1-\alpha)^2 - (3-2\beta+2\alpha\beta)\beta^2|(f'''(\eta_i) - f'''(\alpha_i)) \\ &+ |2\alpha(1-\alpha) - 3\alpha\beta(2-2\alpha+\alpha\beta)|(f'''(\beta_i) - f'''(\eta_i)) \\ &+ |2a(1-a)\beta^3 - 3\alpha(2-\alpha)\beta^2|(f'''(\alpha_i) - f'''(\beta_i)) \\ &+ 2\alpha\beta^3(f'''(\beta_{i+1}) - f'''(\theta_{i+1})) \\ &+ 2\alpha^2\beta^3(f'''(\alpha_{i+1}) - f'''(\beta_{i+1})) \\ &+ 3\alpha^2\beta^2(f'''(\theta_{i+1}) - f'''(\alpha_{i+1}))), \end{aligned}$$

where $z_i \in [x_{i-1}, x_i]$ for $z = \alpha, \beta, \theta, \eta$. Thus, in view of conditions of Theorem 1, we have, for $\alpha = 1/3, 2/3$,

$$|D_i| \leq J(\alpha)p^2 w(f''', p), \tag{3.3}$$

where $J(1/3) = 1.75$ and $J(2/3) = 5.12$. Now, following the standard arguments based on the diagonal dominant property and using (2.8), (3.3) in (3.2), we get

$$||(e_i)'|| \leq J(\alpha)K_1 p^2 w(f''', p) \tag{3.4}$$

where K_1 is given by (2.8). Now combining (2.6) with (2.3), we replace $s'(x)$ by $e'(x)$ and m_i by e_i' in (2.3) and adjust suitably the additional terms by using the result of Taylor's Theorem to see that

$$8||e'(x)|| \leq 17||(e_i)'|| + 4p^2 w(f''', p). \tag{3.5}$$

Thus, using the estimate (3.4) in (3.5), we have

$$8||e'(x)|| \leq (4 + 17K_1 J(\alpha))p^2 w(f''', p). \tag{3.6}$$

This proves (3.1) for $r = 1$. The other inequality follows directly by an application of Rolle's Theorem. □

In order to study the convergence properties of deficient cubic spline interpolants for $\alpha = 1/2$, we consider $f \in C^4[0,1]$ and, using the results that $f(x) = f_j + (x - x_j)f_j' + (x - x_j)^2 f_j''/2 + (x - x_j)^3 f_j'''/6$

$+(x-x_j)^4 f^{(4)}(\beta_j)/24$ and $f'(x) = f'_j + (x-x_j)f''_j + (x-x_j)^2 f'''_j/2 + (x-x_j)^3 f^{(4)}(\alpha_j)/6$, we have (from (3.2))

$$\begin{aligned} D_i =& (p^3/48)(2|1-6\beta+9\beta^2-4\beta^3|(f^{(4)}(\eta_i)-f^{(4)}(\beta_i)) \\ &+ (f^{(4)}(\alpha_i)-f^{(4)}(\beta_i)) + 3\beta^4(f^{(4)}(\beta_{i+1})-f^{(4)}(\beta_i)) \\ &+ \beta^2(3-2\beta)((f^{(4)}(\theta_{i+1})-f^{(4)}(\alpha_i)) + f^{(4)}(\theta_{i+1})-f^{(4)}(\alpha_{i+1})) \\ &+ 4\beta^3(f^{(4)}(\beta_i)-f^{(4)}(\theta_{i+1}))), \end{aligned}$$

where $z_i \in [x_{i-1}, x_i]$ for $z = \alpha, \beta, \theta, \eta$. Thus,

$$|D_i| \le (5/24)p^3 w(f^{(4)}, p). \tag{3.7}$$

Writing K_2 for $\max_\beta\{J_3^{-1}(\beta), 2, J_4^{-1}(\beta)\}$, where $J_3(\beta)$ and $J_4(\beta)$ have already been defined in §2, and, again using the standard arguments, we have

$$||(e_i)'|| \le (5/24)K_2 p^3 w(f^{(4)}, p). \tag{3.8}$$

Hence, by a reasoning used already in the proof of Theorem 3, we have

$$||e'|| \le K_3 p^3 w(f^{(4)}, p), \tag{3.9}$$

where K_3 is some positive constant. We have thus proven

THEOREM 4. *Let f be a 1-periodic function in the class $C^4[0,1]$ and $s(x)$ be its spline interpolant in $S(3,P)$. Then, for $j = 0, 1$ and $\alpha = 1/2$,*

$$||(s-f)^{(j)}(x)|| \le p^{4-j} K_3 w(f^{(4)}, p), \tag{3.10}$$

where K_3 is a positive constant.

(3.9) proves Theorem 4 for $j = 1$. The other inequality for $j = 0$ follows directly by integration. □

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DEPARTMENT OF P.G. STUDIES AND RESEARCH IN MATHEMATICS, R.D. UNIVERSITY, JABALPUR, INDIA