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# TWO CONCRETE NEW CONSTRUCTIONS OF THE REAL NUMBERS

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ABSTRACT. Two new methods are put forward for constructing the complete ordered field of real numbers out of the ordered field of rational numbers. The methods are motivated by some known theorems on so-called Engel and Sylvester series. Amongst advantages of the methods are the facts that they do not require an arbitrary choice of "base", or any equivalence classes or similar constructs.

**Introduction**. By old theorems of Lambert (1770) and Engel (1913) (see Perron [**2**]), every real number $A$ has a unique representation as the sum of a series

$$A = a_0 + \frac{1}{a_1} + \frac{1}{a_1 a_2} + \cdots + \frac{1}{a_1 a_2 \ldots a_n} + \cdots = (a_0, a_1, a_2, \ldots),$$

say, where the $a_i$ are integers such that $a_{i+1} \geq a_i \geq 2$ for $i \geq 1$. Further, $A$ is rational if and only if $a_{i+1} = a_i$ for all sufficiently large $i$.

An analogous representation (see [**2**]) of Lambert (1770) and Sylvester (1880) states that every real

$$A = a_0 + \frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n} + \cdots = ((a_0, a_1, a_2, \ldots)),$$

say, where the $a_i$ are integers defined uniquely by $A$, such that $a_1 \geq 2$ and $a_{i+1} \geq a_i(a_i - 1) + 1$ for $i \geq 1$. Further, $A$ is rational if and only if $a_{i+1} = a_i(a_i - 1) + 1$ for all sufficiently large $i$.

In certain ways, these representations may be compared with that by "simple" continued fractions, and are even simpler than the latter. The main purpose of this note is to justify this remark by deriving some elementary further properties of the Engel-Lambert and Sylvester-Lambert representations, and (with these and the above-mentioned

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results as initial motivation) then developing two new methods for constructing the real number system from the ordered field of rational numbers. These methods are partly similar to one recently introduced by G.J. Rieger [**3**] for constructing the real numbers via continued fractions, but in certain respects they are less complicated than that of present methods. However, an important advantage which the present methods do share with that of Rieger over other standard ones is that they do not require an arbitrary choice of a "base", or the use of (infinite) equivalence classes or similar construct.

In principle, one could adapt the methods below so as to use properties of so-called Lüroth series (cf. [**2**]) as yet another starting motivation, but the latter series are perhaps less elegant than the above ones and so will not be treated at this stage.

**1. Further properties of the representations.** For the reader's convenience, we recall the algorithms which lead to the series representations considered above:

Given any real number $A$, write it as $A \doteq a_0 + A_1$ where $a_0$ is an integer and $0 < A_1 \leq 1$. Then inductively define the "digit"

$$a_n = 1 + \frac{1}{A_n} \geq 2, \text{ for } n \geq 1,$$

where

$$A_{n+1} = \begin{cases} a_n A_n - 1 & \text{in the Engel-Lambert case,} \\ A_n - \frac{1}{a_n} & \text{in the Sylvester-Lambert case.} \end{cases}$$

In particular, when $a_{n+1} = a_n$ for $n \geq m$ in the Engel-Lambert case, we may write

$$\begin{aligned} A &= (a_0, a_1, \cdots) = (a_1, \cdots, a_{m-1}, a_m), \text{ say} \\ &= a_0 + \frac{1}{a_1} + \frac{1}{a_1 a_2} + \cdots + \frac{1}{a_1 a_2 \cdots a_{m-1}} + \frac{1}{a_1 a_2 \cdots a_m} \sum_{r=0}^{\infty} \frac{1}{a_m^r} \\ &= a_0 + \frac{1}{a_1} + \frac{1}{a_1 a_2} + \cdots + \frac{1}{a_1 a_2 \cdots a_{m-1}} + \frac{1}{a_1 a_2 \cdots a_{m-1}(a_m - 1)}. \end{aligned}$$

On the other hand, when $a_{n+1} = a_n(a_n - 1) + 1$ for $n \geq m$ in the Sylvester-Lambert case, we shall write

$$\begin{aligned} A &= (a_0, a_1, \dots) = (a_0, \dots, a_{m-1}, \ddot{a}_m), \text{ say} \\ &= a_0 + \frac{1}{a_1} + \cdots + \frac{1}{a_{m-1}} + \sum_{n=m}^{\infty} \frac{1}{a_n} \\ &= a_0 + \frac{1}{a_1} + \cdots + \frac{1}{a_{m-1}} + \frac{1}{a_m - 1}, \end{aligned}$$

since, here, $1/(a_n - 1) - 1/a_n = 1/(a_{n+1} - 1)$ for $n \geq m$. It may be verified that an integer $k$ has the representation $(k-1, \dot{2})$, or $((k-1, \ddot{2}))$, according to the case being considered.

PROPOSITION (1.1). *Let* $A = (a_0, a_1, \dots) \neq B = (b_0, b_1, \dots)$, *or* $A = (a_0, a_1, \dots) \neq B = (b_0, b_1, \dots)$. *In both of these cases, the condition* $A < B$ *is equivalent to*:

(i) $a_0 < b_0$, if $a_0 \neq b_0$,

(ii) $a_i > b_i$ for the first $i \geq 1$ such that $a_i \neq b_i$, if $a_0 = b_0$.

PROOF. In both cases, if $a_0 < b_0$, then

$$A = a_0 + A_1 \leq a_0 + 1 \leq b_0 < b_0 + B_1 = B,$$

using the initial notation for the two algorithms above. Therefore part (i) follows in both cases.

Now suppose that $a_0 = b_0$, and $a_i > b_i$ for the first $i \geq 1$ such that $a_i \neq b_i$. In the Engel-Lambert case, we then have

$$\begin{aligned} A &= a_0 + \frac{1}{a_1} + \cdots + \frac{1}{a_1 a_2 \cdots a_i}\left(1 + \frac{1}{a_{i+1}} + \frac{1}{a_{i+1}a_{i+2}} + \cdots\right) \\ &= X_i + \frac{1}{m_i a_i}\left(1 + \frac{1}{a_{i+1}} + \frac{1}{a_{i+1}a_{i+2}} + \cdots\right), \end{aligned}$$

say, where $m_1 = 1$ and $m_i = a_1 a_2 \cdots a_{i-1}$ for $i > 1$. Since $a_i \geq b_i + 1$ and $a_{n+1} \geq a_n$, we then have

$$\begin{aligned} A &\leq X_i + \frac{1}{m_i(b_i + 1)} \sum_{r=0}^{\infty} \frac{1}{(b_i + 1)^r} = X_i + \frac{1}{m_i b_i} \\ &< X_i + \frac{1}{m_i b_i}\left(1 + \frac{1}{b_{i+1}} + \frac{1}{b_{i+1}b_{i+2}} + \cdots\right) = B. \end{aligned}$$

In the Sylvester-Lambert case, it is easier to proceed with the defining algorithm for the digits (a method which could also have been used above):

From $a_n = 1 + [\frac{1}{A_n}]$ for $n \geq 1$, we get $a_n - 1 \leq \frac{1}{A_n} < a_n$, and thus $1/(a_n - 1) \geq A_n > 1/a_n$. Then the equation $A_{n+1} = A_n - 1/a_n$ gives

$$\frac{1}{a_n - 1} \geq \frac{1}{a_n} + A_{n+1} = \frac{1}{a_n} + \frac{1}{a_{n+1}} + A_{n+2} = \cdots .$$

Since $A_k \to 0$ as $k \to \infty$, it follows that

$$\frac{1}{a_n - 1} \geq \frac{1}{a_n} + \frac{1}{a_{n+1}} + \cdots ,$$

and so

$$\frac{1}{b_i} \geq \frac{1}{a_i - 1} \geq \frac{1}{a_i} + \frac{1}{a_{i+1}} + \cdots .$$

Therefore

$$\begin{aligned} A &= a_0 + \frac{1}{a_1} + \cdots + \frac{1}{a_i} + \frac{1}{a_{i+1}} + \cdots \leq a_0 + \frac{1}{a_1} + \cdots \frac{1}{a_{i-1}} + \frac{1}{b_i} \\ &= b_0 + \frac{1}{b_1} + \cdots + \frac{1}{b_i} < B. \end{aligned}$$

Thus part (ii) follows in both cases. □

Note that the uniqueness of the digits in the above representations is established in Perron [**2**]; however, the above argument also re-proves this in the Engel-Lambert case.

**2. Constructions and order properties.** In the constructions below, standard facts about the ordered field **Q** of all rational numbers are taken as understood. With the above representations and Proposition 1.1 as initial motivation, now define two sets $\mathcal{E}$ and $\mathcal{S}$ and order relations on them as follows:

Let $\mathcal{E}$ be the set of all formal infinite sequences $A = (a_0, a_1, a_2, \ldots)$ of integers $a_i$ such that $a_{i+1} \geq a_i \geq 2$ for $i \geq 1$. Also, let $\mathcal{S}$ be the set of all formal infinite sequences $A = (a_0, a_1, a_2, \ldots)$ of integers $a_i$ such that $a_1 \geq 2$ and $a_{i+1} \geq a_i(a_i - 1) + 1$ for $i \geq 1$.

In both cases, we shall use corresponding lower-case letters to denote the "digits" of elements of $\mathcal{E}$ and $\mathcal{S}$, and we define $A < B$ if and only if

(i) $a_0 < b_0$, if $a_0 \neq b_0$, or

(ii) $a_i > b_i$ for the first $i \geq 1$ such that $a_i \neq b_i$, if $a_0 = b_0$.

LEMMA 2.1. *In both cases, $<$ is a "total ordering" relation, i.e., it is transitive and satisfies the trichotomy law.*

PROOF. The same argument works for both cases. Firstly, trichotomy is obvious. Next, let $A < B$ and $B < C$. If $a_0 < b_0 < c_0$, or $a_0 = b_0 < c_0$, $a_0 < b_0 = c_0$, it follows directly that $A < C$.

Now suppose that $a_0 = b_0 = c_0$, and that $a_r = b_r$ for $r < i, a_i > b_i$, and $b_r = c_r$ for $r < j, b_j > c_j$. Then:

(i) If $i < j$, then $a_r = c_r$ for $r < i$ and $a_i > b_i = c_i$.

(ii) If $i = j$, then $a_r = c_r$ for $r < j$ and $a_i > b_i > c_i$.

(iii) If $i > j$, then $a_r = b_r = c_r$ for $r < j$ and $a_j = b_j > c_j$.

Thus $A > C$ in each case. □

We may now introduce symbols $\leq, >$ and $\geq$, and define (*least*) *upper bounds* and (*greatest*) *lower bounds*, in the usual way.

LEMMA 2.2. *Every non-empty subset of $\mathcal{E}$ (respectively, $\mathcal{S}$ which is bounded above has a least upper bound (supremum).*

PROOF. First consider a non-empty subset $X$ of E, which is bounded above by a sequence $B = (b_0, b_1, \dots)$. Then $A \leq B$ and $a_0 \leq b_0$ for all $A \in X$, since otherwise $A > B$. If $d_0$ is the maximum value of $a_0$ for $A \in X$, we may then assume $d_0 = b_0$ since otherwise $(d_0, \dot{2})$ will also be an upper bound for $X$, where $\dot{a}$ denotes the constant infinite sequence $a, a, \dots$. We may also assume that $B \notin X$, since otherwise there is nothing to prove.

Now $A < B$ for every $A \in X$, and there is a largest index $k$ such that every $A \in X$ with $a_0 = b_0$ has $a_1 = b_1, \dots, a_k = b_k$. Then define $c_0 = b_0, \dots, c_k = b_k$, and let $c_{k+1}$ be the least possible value for the digit $a_{k+1}$ of any $A \in X$ with $a_0 = b_0$.

Next let $c_{k+2}$ be the least possible value for the digit $a_{k+2}$ of an element of $X$ of the form $(c_0, \ldots, c_{k+1}, a_{k+2}, a_{k+3}, \ldots)$. Continue inductively, to define $c_{k+i+1}$ as the least possible value for the digit $a_{k+i+1}$ of an element of $X$ of the form $(c_0, \ldots, c_{k+i}, a_{k+i+1}, a_{k+i+2}, \ldots)$. This process then defines a sequence $C = (c_0, c_1, c_2, \ldots)$ with $c_{i+1} \geq c_i \geq 2$ for $i \geq 1$. Also, if $C \neq A \in X$, then $C > A$, since either $c_0 > a_0$, or $c_0 = a_0$ and $c_i < a_i$ for the first $i > k$ such that $c_i \neq a_i$ (by the minimality of $c_i$).

Lastly, $C = \sup X$, since otherwise $X$ has an upper bound $D < C$. In that case, $d_o = c_0$, since otherwise $D < A$ for some $A \in X$, by the method of definition of $c_0$ above. Thus $d_m > c_m$ for the first $m \geq 1$ such that $d_m \neq c_m$. Hence every element of the form $A = (c_0, \ldots c_m, a_{m+1}, a_{m+2}, \ldots)$ in $X$ satisfies $D < A \leq D$, a contradiction.

The argument for $\mathcal{S}$ is almost identical, except that at an early stage we need to consider $((d_0, \ddot{2}))$, where $\ddot{a}$ denotes the infinite sequence $t_1, t_2, \ldots$ with $t_1 = a$ and $t_{i+1} = t_i(t_i - 1) + 1$. Also the sequence $C = ((c_0, c_1, \ldots))$ defined inductively will now satisfy $c_1 \geq 2$ and $c_{i+1} \geq c_i(c_i - 1) + 1$ for $i \geq 1$, since it is now define via suitable elements of $\mathcal{S}$. □

**3. Embedding and density of rationals.** We now consider natural embeddings of $\mathbf{Q}$ into $\mathcal{E}$ and S:

PROPOSITION 3.1. *The Engel-Lambert and Sylvester-Lambert algorithms define* $1-1$ *order-preserving maps*

$$\rho_E : \mathbf{Q} \to \mathcal{E} \text{ and } \rho_S : \mathbf{Q} \to \mathcal{S},$$

*whose images are dense in* $\mathcal{E}$ *and* $\mathcal{S}$, *respectively.*

PROOF. It is an immediate consequence of the results quoted earlier that the two algorithms define 1-1 maps $\rho_E : \mathbf{Q} \to \mathcal{E}$ and $\rho_S : \mathbf{Q} \to \mathcal{S}$. By Proposition 1-1 and the definition of order in $\mathcal{E}$ and $\mathcal{S}$, these maps are then order-preserving.

Now let $A < B$ in $\mathcal{E}$. If $a_0 < b_0$, let $C = (b_0, \dot{c})$ where $c = b_1 + 1$. Then $A < C < B$ and $C \in \rho_E(\mathbf{Q})$. On the other hand, if $a_0 = b_0$, then

$a_m > b_m$ for the first $m$ such that $a_m \neq b_m$. In that case, $A < D < B$ where $D = (b_0, \ldots, b_m, \dot{b}), b = b_{m+1} + 1$. Thus $\rho_E(\mathbf{Q})$ is dense in $\mathcal{E}$. A similar argument works for $\mathcal{S}$, using $\ddot{a}$ instead of $\dot{a}$.

It may be noted that the earlier rewriting of the series for rationals as finite sums suggests a way of expressing elements of $\rho_E(\mathbf{Q})$ and $\rho_S(\mathbf{Q})$ as *finite* sequences. Although this is possible and intuitively quite desirable, it involves some extra formalities with the meaning of order for finite sequences, which can be avoided if we continue to work only with infinite sequences at this stage.

From now on, we shall regard $\mathbf{Q}$ as an actual *subset* of $\mathcal{E}$ or $\mathcal{S}$, whenever convenient.

APPROXIMATION LEMMA 3.2. *Given any element $A$ of $\mathcal{E}$ (respectively, $\mathcal{S}$, there exist rationals $A^{(n)}$, $A_{(n)}$ for $n \geq 1$ such that*

(i) $A_{(m)} < A_{(n)} < A \leq A^{(n)} \leq A^{(m)}$*for* $m < n$,

(ii) $A = \sup A_{(n)} = \inf A^{(n)}$,

(iii) $A^{(n)} - A_{(n)} \leq 2^{-n}$.

PROOF. Given $A = (a_0, a_1, \ldots) \in \mathcal{E}$, define the *upper* and *lower* rational approximation to $A$ by $A^{(n)} = (a_0, \ldots, a_{n-1}, a\dot{a}_n)$, and $A_{(n)} = (a_0, \ldots, a_{n-1}, \dot{a}'_n)$ where $a'_n = a_n + 1$, $n \geq 1$. Then part (i) follows. Next suppose that $A < B \leq A^{(n)}$ for all $n$. In that case, we must have $a_0 = b_0$ and $a_m > b_m$ for the first $m$ such that $a_m \neq b_m$. This gives the contradiction $A^{(m)} < B \leq A^{(m)}$. Thus $A = \inf A^{(n)}$. Similarly suppose that $A_{(n)} \leq C < A$ for all $n$. Then we must have $a_0 = c_0$ and $c_m > a_m$ for the first $m$ such that $c_m \neq a_m$. This yields the contradiction $A_{(m+1)} \leq C < A_{(m+1)}$. Hence $A = \sup A_{(n)}$.

The same argument leads to parts (i) and (ii) for $\mathcal{S}$, except that we now use $\ddot{a}_n$ and $\ddot{a}'_n$ in order to define $A^{(n)}$ and $A_{(n)}$.

For part (iii) in $\mathcal{E}$, a formula for Engel-Lambert series of rationals in

§1 leads to

$$A^{(n)} - A_{(n)} = \frac{1}{a_1 a_2 \cdots a_{n-1}(a_n - 1)} - \frac{1}{a_1 a_2 \cdots a_{n-1} a_n}$$
$$= \frac{1}{a_1 a_2 \cdots a_n(a_n - 1)} \leq \frac{1}{2^n},$$

since $a_i \geq 2$. For $\mathcal{S}$, the corresponding formula for Sylvester-Lambert series of rationals gives instead

$$A^{(n)} - A_{(n)} = \frac{1}{a_n - 2} - \frac{1}{a_n} = \frac{1}{a_n(a_n - 1)} < \frac{1}{2^n},$$

since $a_1 \geq 2$ and $a_{i+1} > a_i(a_i - 1)$. □

In passing, we note that the representation $(k-1, \dot{2})$ or $((k-1, \ddot{2}))$ for any *integer* $k$ shows that (in both $\mathcal{E}$ and $\mathcal{S}$) an element $A > 0$ if and only if $a_0 \geq 0$. Also $A < 0$) if and only if: $a_0 < -1$, or else $a_0 = -1$ and some $a_i$ is not in the sequence $\dot{2}$ (or $\ddot{2}$).

**4. Algebraic operations in $\mathcal{E}$ and $\mathcal{S}$.** In defining algebraic operations on $\mathcal{E}$ and $\mathcal{S}$, it is particularly convenient to regard $\mathbf{Q}$ as an actual subset of each of these sets. It also simplifies the discussion considerably if we now *re-define*.

$$A^{(n)} = A_{(n)} = A \quad (n \geq 1)$$

for any *rational* $A$.

For any $A, B \in \mathcal{E}$ (or any $A, B \in \mathcal{S}$) now *define*

$$A + B = \sup(A_{(n)} + B_{(n)}), \quad -A = \sup(-A^{(n)}),$$

which exist in $\mathcal{E}$ (respectively, $\mathcal{S}$) because

$$A_{(n)} + B_{(n)} \leq A^{(1)} + B^{(1)}, \qquad -A^{(n)} \leq -A_{(1)}.$$

LEMMA 4.1. *The above operations make $\mathcal{E}$ (respectively, $\mathcal{S}$) into an abelian group containing $(\mathbf{Q},+)$ as a dense subgroup. Further,*

(i) $A < B \Rightarrow A + C < B + C$,

(ii) $A < B < \Leftrightarrow -A < -B$.

PROOF. Obviously, $A + B = B + A$ and $A + 0 = A$. Also, the definition of order and rational approximations shows that $A < B \Rightarrow A_{(n)} < B_{(n)}$, $A^{(n)} < B^{(n)}$ for $n$ sufficiently large. Hence $A < B \Rightarrow A + C \leq B + C$ (with strict inequality to be shown later).

If addition is not associative, we now obtain a contradiction as follows: Suppose that $X = A + (B + C), Y = (A + B) + C$ and $X < Y$. Then, by Proposition 3.1, there exist rationals $r, s$ such that $X < r < s < Y$. Hence, by the weak monotone law above, $x_n < r < s < y_n$ where

$$x_n = A_{(n)} + B_{(n)} + C_{(n)}, y_n = A^{(n)} + B^{(n)} + C^{(n)}.$$

Thus

$$0 < s - r < y_n - x_n < \frac{3}{2^n},$$

by Lemma 3.2 (iii). This is impossible since $n$ is arbitrary. Similarly, $X > Y$ is impossible.

Next note that $-A^{(n)} \leq -A_{(m)}$ for all $m, n$. Thus

$$-A^{(m)} \leq \sup(-A^{(n)}) \leq -A_{(m)} \text{ for all } m.$$

Hence the weak monotone law gives $A_{(m)} - A^{(m)} \leq A + (-A) \leq A^{(m)} - A_{(m)}$ for all $m$. Therefore $-2^{-m} \leq A + (-A) \leq 2^{-m}$ for all $m$, by Lemma 3.2 (iii). This leads to a contradiction if $A + (-A) \neq 0$, because in that case Proposition 3.1 implies that there exists a rational $r$ independent of $m$ such that $0 < r < A + (-A)$, or $A + (-A) < r < 0$.

It now follows that $\mathcal{E}$ (respectively, $\mathcal{S}$) forms an abelian group with $(\mathbf{Q},+)$ as a dense subgroup. Then $A + C = B + C \Rightarrow A = B$, and hence the *strict* monotone law (i) now follows from the weak one.

Lastly, let $A < B$. Then $A^{(n)} < B^{(n)}$, or $-A^{(n)} > -B^{(n)}$, for $n$ sufficiently large. Thus $-A = \sup(A^{(n)}) \geq \sup(-B^{(n)}) = -B$, giving $-A > -B$ since $A \neq B$. Hence (ii) follows, since $-(-X) = X$.

Next, for any $A, B \in \mathcal{E}$*(respectively,* $\mathcal{S}$*), define*

$$A.B = \begin{cases} \sup(A_{(n)}.B_{(n)}) & \text{if } A \geq 0,\ B \geq 0, \\ (-A).(-B) & \text{if } A \leq 0,\ B \leq 0, \\ -((-A).B) & \text{if } A \leq 0,\ B \geq 0, \\ -(A.(-B)) & \text{if } A \geq 0,\ B \leq 0. \end{cases}$$

Also *define*

$$A^{-1} = \begin{cases} \sup((A^{(n)})^{-1}) & \text{if } A > 0, \\ -((-A)^{-1}) & \text{if } A < 0. \end{cases}$$

These definitions are unambiguous, because firstly

$$A_{(n)}.B_{(n)} \le A^{(1)}.B^{(1)}, \qquad (A^{(n)})^{-1} \le (A_{(1)})^{-1},$$

when $A > 0, B > 0$, since then $a_0 \ge 0$, $b_0 \ge 0$ and all the rational approximations are positive. Secondly, in order to cover all cases, we use the fact that $A < 0$ if and only if $-A > 0$, by Lemma 4.1 (ii). □

LEMMA (4.2) *The above, together with the earlier operations, make $\mathcal{E}$ (respectively, $\mathcal{S}$) into a field containing* **Q** *as a dense subfield. Further,*

$$A < B,\ C > 0 \Rightarrow A.C < B.C.$$

PROOF. Clearly $A.B = B.A$ and $A.1 = A$. Then, in order to show that $\mathcal{E}(\mathcal{S})$ forms a field, it remains only to verify that $\cdot$ is associative, and distributive relative to $+$, and that $A^{-1}.A = 1$ for $A \ne 0$. After dealing first with positive elements only, this reduces the problem to straightforward algebraic manipulation of the different "sign" cases. For the sake of brevity, we shall therefore consider only positive elements. Further, since the associative and distributive laws are quite similar, we consider only the latter.

Firstly note that earlier remarks on rational approximations lead easily to the weak monotone law

$$0 < A < B,\ C > 0 \Rightarrow A.C \le B.C$$

(with strict inequality to be shown later).

For positive $A, B, C$, now suppose that $X = A.(B + C)$, $Y = A.B + A.C$ and $X < Y$. Then there exist rationals $r, s$ with $X < r < s < Y$, and thus the weak monotone laws imply that $x_n < r < s < y_n$, where $x_n = A_{(n)}.(B_{(n)} + C_{(n)})$ and $y_n = A^{(n)}.B^{(n)} + A^{(n)}.C^{(n)}$. Now, Lemma 3.2 implies that

$$\begin{aligned} 0 &< s - r < y_n - x_n \\ &\le (A_{(n)} + 2^{-n})B_{(n)} + C_{(n)} + 2^{1-n}) - A_{(n)} \cdot (B_{(n)} + C_{(n)}) \\ &\le 2^{-n}(2A^{(1)} + B^{(1)} + C^{(1)} + 2^{1-n}). \end{aligned}$$

This is impossible since $n$ is arbitrary. Similarly, $X > Y$ is impossible.

Next, for $A > 0$, note that $0 < A_{(m)} \leq A^{(n)}$ for all $m, n$. Thus $(A_{(m)})^{-}1 \geq (A^{(n)})^{-1}$ for all $m, n$, which implies that $A^{-1} \leq (A_{(m)})^{-1}$ for all $m$. Hence

$$\begin{aligned} \frac{A_{(m)}}{A_{(m)} + 2^{-m}} &\leq (A^{(m)})^{-1}.A_{(m)} \\ &\leq A^{-1}.A \leq (A_{(m)})^{-1}.A^{(m)} \\ &\leq \frac{2^{-m} + +A_{(m)}}{A_{(m)}} \end{aligned}$$

for all $m$. Therefore, for all $m$,

$$\frac{-2^{-m}}{A^{(1)} + 2^{-m}} \leq \frac{-2^{-m}}{A_{(m)} + 2^{-m}} \leq A^{-1}.A - 1 \leq \frac{2^{-m}}{A_{(m)}} \leq \frac{2^{-m}}{A_{(1)}}.$$

This leads to a contradiction if $A^{-1}.A \neq 1$, in the same way as $A + (-A) \neq 0$ led to a contradiction earlier.

The strict monotone law for multiplication of positive elements now follows from the weak one and the law $A.C = B.C \Rightarrow A = B$ (for positive elements). Then Lemma 4.2 follows after suitable (tedious) algebraic checking of the various "sign" cases.

The above discussion has shown that both $\mathcal{E}$ and $\mathcal{S}$ form *ordered* fields with the *least upper bound* property. By standard theorems, treated for example in Chapter 5 of Cohen and Ehrlich [**1**], it then follows that $\mathcal{E}$ and $\mathcal{S}$ form concrete new models for the *real number* system $\mathbf{R}$.

**5. Another new construction of R.** By using some known theorem [**2**] on so-called *Cantor* products as a new initial motivation, it is possible to construct another concrete but quite different model for $\mathbf{R}$. This is treated in [**4**].

## REFERENCES

**1.** L.W. Cohen and G. Ehrlich. *The Structure of the Real Number System*, New York, Krieger Publ. Co., 1977.

**2.** O. Perron, *Irrationalzahlen*, Chelsea Publ. Co., New York, 1951.

**3.** G.J. Rieger, *A new approach to the real numbers* (motivated by continued fractions). Abh. Braunschweig. Wiss. Ges. **33** (1982), 205-217.

**4.** A. Knopfmacher and J. Knopfmacher, *A new construction of the real numbers* (*via infinite products*), Nieuw Arch. Wisk.

**5.** (1987), 19-31.

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