ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 18, Number 4, Summer 1988

WEAKLY CLOSE-TO-CONVEX FUNCTIONS

ALBERT E. LIVINGSTON

ABSTRACT. We obtain coefficient bounds and integral means inequalities for the class of multivalent weakly close to convex functions. We also consider the integral means problem for the subclass of multivalent convex functions.

Introduction. Let $S(p)$ denote the class of functions f, analytic in $\Delta = \{z : |z| < 1\}$, with p zeros, counting multiplicity, in Δ and such that there exists $\delta > 0$ so that $\operatorname{Re}[zf'(z)/f(z)] > 0$ for $\delta < |z| < 1$. Functions in $S(p)$ are called p-valent starlike functions. Hummel [**11**] extended $S(p)$ to the class of weakly starlike functions and this was further extended by Styer [**21**]. Following Styer, we say that a function is a member of $S_{wc}(p)$ if and only if there exists a sequence of functions f_n in $S(p)$ such that f_n converges to f uniformly on compact subsets of Δ. Equivalently [**21**], f is in $S_{wc}(p)$ if and only if there exists h in $S(1)$, with $h(0) = h'(0) - 1 = 0$, such that

$$(1.1) \qquad f(z) = [h(z)]^p \prod_{j=1}^{p} \frac{(z-\alpha_j)(1-\overline{\alpha}_j z)}{z}, \quad |\alpha_j| \leq 1.$$

The fundamental difference between Styer's definition and Hummel's is that Hummel's requires $|\alpha_j| < 1$.

The author [**13**] studied the class of close to convex functions $K(p)$. A function F, analytic in Δ with $F(0) = 0$, is in $K(p)$ if and only if there exists f in $S(p)$ with $f(0) = 0$ and a $\delta > 0$ such that $\operatorname{Re}[zF'(z)/f(z)] > 0$ for $\delta < |z| < 1$. Styer [**22**] extended this class to the class of weakly close to convex functions $K_w(p)$, giving several equivalent characterizations of $K_w(p)$. A function F, not identically zero in Δ, is in $K_w(p)$ if and only if there is a sequence of functions F_n in $K(p)$ such that F_n converges to F uniformly on compact subsets of Δ. Equivalently F is in $K_w(p)$ if and only if $F(0) = 0$ and there is a function f in $S_{wc}(p)$ with $f(0) = 0$, such that $\operatorname{Re}[zF'(z)/f(z)] > 0$ in

Received by the editors on March 18, 1986, and in revised form on July 25, 1986.

Δ. Recently Lyzzaik [**17**] has proven the following striking result. If f is in $K_w(p)$, then there exists a polynomial P of degree at most p and a function ϕ in S, the class of functions analytic and univalent in Δ with $f(0) = f'(0) - 1 = 0$, such that $f(z) = P(\phi(z))$.

One of the main unsolved problems in the theory of multivalent functions is the problem of finding sharp estimates on the coefficients of the power series expansion in Δ. There are two conjectures concerning this problem. One conjecture gives an upper bound which depends on the magnitude of the first p coefficients and was first conjectured by Goodman [**6**], and the other, also conjectured by Goodman [**8**], gives a bound which is dependent on the magnitudes of the non-zeros of the function. Some progress has been made by the author on the first conjecture in $K(p)$ [**14, 16**]. In §2 we will verify that the second conjecture holds in $K_w(p)$. We also solve the integral means problem for f in $K_w(p)$.

2. Coefficient inequalities. In this section we will use the symbol $f(z) << g(z)$ to mean that if $f(z) = \sum a_n z^n$ and $g(z) = \sum b_n z^n$ in Δ, then $|a_n| \leq |b_n|$ for all n. Goodman [**8**] has conjectured that if $f(z) = z^q + q_{q+1}z^{q+1} + \cdots$ is at most p-valent in Δ and if $f(\beta_j) = 0,\ 0 < |\beta_j| < 1,\ j = 1, 2, \ldots s$, then $f(z) << F(z)$ where

$$F(z) = \frac{z^q}{(1-z)^{2p}}(1+z)^{2(p-s-q)} \prod_{j=1}^{s} \left(1 + \frac{z}{|\beta_j|}\right)(1 + |\beta_j| z) \tag{2.1}$$

and when $f(z) \neq 0$ for $0 < |z| < 1$, that $f(z) << F(z)$ where

$$F(z) = \frac{z^q(1+z)^{2(p-q)}}{(1-z)^{2p}}. \tag{2.2}$$

We note [**8**] that $F(z) = Q(K(z))$ where $K(z) = z/(1-z)^2$ and

$$Q(z) = z^q(1+4z)^{(p-s-q)} \prod_{j=1}^{s} \left(1 + \frac{(1+|\beta_j|)^2}{|\beta_j|} z\right).$$

Also $F(z)$ is given by (1.1), up to a constant factor, with $h(z) = z/$

$(1-z)^2, \alpha_j = -|\beta_j|$ for $j = 1, 2, \ldots, s$, $\alpha_j = -1$ for $j = (s+1) \cdots (p-q)$ and $\alpha_j = 0$ for $j = (p-q+1), \ldots, p$.

The conjecture is known to be true in $S(p)$ [**7**] and hence in $S_{wc}(p)$. Until recently there was no device for handling the problem in $K_w(p)$. One could obtain bounds which involved the zeros of the derivative but until the result of Lyzzaik [**17**] there was no representation of f in terms of its zeros. However, with Lyzzaik's result and with the recent proof of the Bieberbach conjecture by DeBranges [**3**], the problem is easily solved in $K_w(p)$.

THEOREM 1. *If* $f(z) = z^q + a_{q+1}z^{q+1} + \cdots$ *is in* $K_w(p)$, *then* $f(z) << F(z)$, *where* F *is defined by* (2.1) *or* (2.2).

PROOF. According to the result of Lyzzaik [**17**], there exists a polynomial $P(z) = z^q + b_{q+1}z^{q+1} + \cdots + b_t z^t, b_t \neq 0$ and $s + q \leq t \leq p$ and there exists ϕ in S such that $f(z) = P(\phi(z))$. Since $P(\phi(\beta_j)) = 0$ for $j = 1, 2, \ldots s$, there exists $\alpha_j \in C - \phi(\Delta)$, $j = 1, 2, \ldots, (t - s - q)$, such that

$$P(z) = z^q \prod_{j=1}^{s} \left(1 - \frac{z}{\phi(\beta_j)}\right) \prod_{j=1}^{t-s-q} \left(1 - \frac{z}{\alpha_j}\right).$$

Thus,

$$f(z) = [\phi(z)]^q \prod_{j=1}^{s} \left(1 - \frac{\phi(z)}{\phi(\beta_j)}\right) \prod_{j=1}^{(t-s-q)} \left(1 - \frac{\phi(z)}{\alpha_j}\right), \tag{2.3}$$

where $\prod_{j=1}^{s}(1 - \phi(z)/\phi(\beta_j))$ is taken to be identically one and $s = 0$, if $f(z) \neq 0$ for $0 < |z| < 1$. According to DeBranges [**3**], $\phi(z) << z/(1 - z)^2$. Thus we obtain

$$[\phi(z)]^q << \frac{z^q}{(1 - z)^{2q}} \tag{2.4}$$

(2.5)
$$\left(1 - \frac{\phi(z)}{\phi(\beta_j)}\right) << \left(1 + \frac{1}{|\phi(\beta_j)|} \frac{z}{(1 - z)^2}\right) << \left(1 + \frac{(1 + |\beta_j|)^2}{|\beta_j|} \frac{z}{(1 - z)^2}\right)$$
$$= \frac{(z + |\beta_j|)(1 + |\beta_j| z)}{|\beta_j|(1 - z^2)}$$

$$(2.6)\quad \left(1-\frac{\phi(z)}{\alpha_j}\right) << \left(1+\frac{1}{|\alpha_j|}\frac{z}{(1-z)^2}\right) << 1+\frac{4z}{(1-z)^2} = \left(\frac{1+z}{1-z}\right)^2$$

where in (2.5) we used the inequality $|\phi(z)| \geq |z|/(1+|z|)^2$ and in (2.6) we used the 1/4 Theorem. Combining (2.3) through (2.6) gives

$$f(z) << \frac{z^q}{(1-z)^{2q}}\left(\frac{1+z}{1-z}\right)^{2(t-s-q)} \prod_{j=1}^{s} \frac{(z+|\beta_j|)(z+|\beta_j|z)}{|\beta_j|(1-z)^2}.$$

Since $t \leq p$, this yields $f(z) << F(z)$. □

COROLLARY 1. *If* $f(z) = z^q + a_{q+1}z^{q+1} + \cdots$ *is in* $K_w(p)$ *with* $f(\beta_j) = 0, 0 < |\beta_j| < 1, j = 1, 2, \ldots, s$, *then*

$$|a_n| \leq \sum_{k=1}^{p} \frac{2k}{(p-k)!(p+k)!(n-p-1)!(n^2-k^2)}|b_k|$$

for $n > p$, *where* $F(z) = z^q + b_{q+1}z^{q+1} + b_{q+2}z^{q+2} + \cdots$ *is given by* (2.1).

PROOF. As we have already noted, $F(z) = Q(K(z))$ where Q is a polynomial of degree p and $K(z) = z/(1-z)^2$. We therefore have by results of Lyzzaik and Styer [**18**] that, for $n > p$,

$$(2.7)\qquad b_n = \sum_{k=q}^{p} E(K(z), p, k, n)b_k$$

where $E(K(z), p, k, n)$ is a certain determinant involving the coefficients of $K(z)$ and its powers, which satisfies [**18**]

$$|E(K(z), p, k, n)| = \frac{2k}{(p-k)!(p+k)!(n-p-1)!(n^2-k^2)}.$$

Using the fact that $|a_n| \leq |b_n|$, the corollary follows. □

Inequality (2.7) is similar to the inequality conjectured by Goodman [**6**]. The $|b_k|$ would be replaced by $|a_k|$ in Goodman's conjecture. A similar inequality involving the zeros of $f'(z)$ is proven in [**10**].

THEOREM 2. *If* $f(z) = z^q + a_{q+1}z^{q+1} + \cdots$ *is in* $K_w(p)$, *then*

$$|F(-r)| \leq |f(z)| \leq F(r)$$

where F *is defined by* (2.1) or (2.2).

PROOF. The inequality $|f(z)| \leq F(r)$ follows from (2.3) and the inequalities $|\phi(z)| \leq r/(1-r)^2$; $|\phi(\beta_j)| \geq |\beta_j|/(1+|\beta_j|)^2, j = 1, 2, \ldots, s$, and $|\alpha_j| \geq 1/4, j = 1, 2, \ldots, (t-s-q)$. To obtain the lower bound from (2.3) we use $|\phi(z)| \geq r/(1+r)^2$ and note the following facts. For each j, the function $1 - \phi(z)/\alpha_j$ is a member of S_0, the class of nonzero univalent functions in Δ. Thus, according to Duren and Schober [**4**], $|1 - \phi(z)/\alpha_j| \geq |(1-r)/(1+r)|^2$. If we let $h(z) = 1 - \phi(z)/\phi(\beta_j)$ and $\delta = \arg \beta_j$, then $1/h(e^{i\delta}z)$ is a member of $\sum(|\beta_j|)$, a class studied by the author [**15**]. According to Theorem 3 [**15**], we obtain

$$1/|h(e^{i\delta}z)| \leq (1+r)^2/||\beta_j|| - r(1 - |\beta_j|r).$$

Thus

$$|1 - \phi(z)/\phi(\beta_j)| \geq ||\beta_j| - r|(1 - |\beta_j|r)/|\beta_j|(1+r)^2.$$

The inequality $|f(z)| \geq |F(-r)|$ now follows.

We note that Theorem 2 is known for $S(p)$ [**7**] and hence for $S_{wc}(p)$.

3. Integral means. In this section we use a technique of Baernstein [**1**] as employed by Leung [**12**], to obtain bounds on the integral means of functions in $K_w(p)$. For this purpose we review some known facts. Let $g(x)$ be a real valued integrable function on $[-\pi, \pi]$ and define for $0 \leq \theta \leq \pi$,

$$g^*(\theta) = \sup_{|E|=2\theta} \int_E g,$$

where $|E|$ denotes the Lebesgue measure of E in $[-\pi, \pi]$.

LEMMA 1. [**1**]. *For* g *and* h *in* $L^1[-\pi, \pi]$ *the following statements are equivalent:*

(a) *for every convex non decreasing function* Φ *on* $(-\infty,\infty)$,

$$\int_{-\pi}^{\pi} \Phi(g(x))dx \leq \int_{-\pi}^{\pi} \Phi(h(x))dx;$$

(b) *for every* t *in* $(-\infty,\infty)$,

$$\int_{-\pi}^{\pi} [g(x)-t]^{+}dx \leq \int_{-\pi}^{\pi} [h(x)-t]^{+}dx;$$

(c) $g^*(\theta) \leq h^*(\theta)$ *for* $0 \leq \theta \leq \pi$.

LEMMA 2. [**1**]. *If* f *is in* S, *then for each* r *in* $(0,1)$

$$(\pm \log|f(re^{i\theta})|)^* \leq (\pm\log|K(re^{i\theta})|)^*$$

where $K(z) = z/(1-xz)^2, |x| = 1$.

LEMMA 3. *If* ϕ *is in* S *and* $|\beta| < 1$, *then, for* r *in* $(0,1), r \neq |\beta|$,

$$\left(\pm\log\left|1-\frac{\phi(re^{i\theta})}{\phi(\beta)}\right|\right)^* \leq \left(\pm\log\left|\frac{\left(1+\frac{re^{i\theta}}{|\beta|}\right)(1+|\beta|re^{i\theta})}{(1-re^{i\theta})^2}\right|\right)^*.$$

PROOF. Let $h(z) = 1-\phi(z)/\phi(\beta)$ and let $\delta = \arg(\beta)$. The function $1/h(e^{i\delta}z)$ is a member of $\sum(|\beta|)$ which was studied by the author in [**15**]. By a result proven in [**15**] we have for any non decreasing convex function Φ on $(-\infty,\infty)$,

$$\int_{-\pi}^{\pi} \Phi\left(\pm\log\left|\frac{1}{h(e^{i\delta}z)}\right|\right)d\theta \leq \int_{-\pi}^{\pi} \Phi\left(\pm\log\left|\frac{(1-re^{i\theta})^2}{\left(1+\frac{re^{i\theta}}{|\beta|}\right)(1+|\beta|re^{i\theta})}\right|\right)d\theta.$$

Thus

$$\int_{-\pi}^{\pi} \Phi(\pm\log|h(re^{i\theta})|)d\theta \leq \int_{-\pi}^{\pi} \Phi\left(\pm\log\left|\frac{\left(1+\frac{re^{i\theta}}{|\beta|}\right)(1+|\beta|re^{i\theta})}{(1-re^{i\theta})^2}\right|\right)d\theta.$$

The Lemma now follows from the equivalence of (a) and (c) in Lemma 1. □

LEMMA 4. *If ϕ is in S and α is in $C - \phi(\Delta)$, then, for each r in* $(0,1)$,

$$\left(\pm\log\left|1-\frac{\phi(re^{i\theta})}{\alpha}\right|\right)^* \leq \left(\pm\log\left|\frac{1+re^{i\theta}}{1-re^{i\theta}}\right|^2\right)^*.$$

PROOF. The function $1-\phi(z)/\alpha$ is a member of S_0, the class of univalent and nonzero functions in Δ. The result for S_0 is stated in [**4**]. □

LEMMA 5. [**12**]. *For g and h in $L^1[-\pi,\pi]$,*

$$[g(\theta)+h(\theta)]^* \leq g^*(\theta)+h^*(\theta).$$

Equality holds if g and h are both symmetric on $[-\pi,\pi]$ and non increasing on $[0,\pi]$.

THEOREM 3. *Let $f(z) = z^q + a_{q+1}z^{q+1} + \cdots$ be a member of $K_w(p)$. If Φ is any non decreasing convex function on $(-\infty,\infty)$, then, for any r in $(0,1)$, $r \neq |\beta_j|$, $j = 1,2,\ldots,s$,*

$$\int_{-\pi}^{\pi}\Phi(\pm\log|f(re^{i\theta})|)d\theta \leq \int_{-\pi}^{\pi}\Phi(\pm\log|F(re^{i\theta})|)d\theta$$

where F is defined by (2.1)*or* (2.2).

PROOF. Using Lemmas 2 through 5 and the representation (2.3) we have

$$\begin{aligned}(\log|f(re^{i\theta})|)^* &\le (\log|\phi(re^{i\theta})|^q)^* \\ &\quad+\sum_{j=1}^{s}\left(\log\left|1-\frac{\phi(re^{i\theta})}{\phi(\beta_j)}\right|\right)^* \\ &\quad+\sum_{j=1}^{t-s-q}\left(\log\left|1-\frac{\phi(re^{i\theta})}{\alpha_j}\right|\right)^* \\ &\le\left(\log\left|\frac{re^{i\theta}}{(1-re^{i\theta})^2}\right|^q\right)^* \\ &\quad+\sum_{j=1}^{s}\left(\log\left|\frac{\left(1+\frac{re^{i\theta}}{|\beta_j|}\right)(1+|\beta_j|re^{i\theta})}{(1-re^{i\theta})^2}\right|\right)^* \\ &\quad+(t-s-q)\left(\log\left|\frac{1+re^{i\theta}}{1-re^{i\theta}}\right|^2\right)^*.\end{aligned}$$

Since $t \le p$, this yields

$$(\log|f(re^{i\theta})|)^* \le (\log|F(re^{i\theta})|)^*,$$

where in the next to last step we have used the fact that $(\log|(1+re^{i\theta})/(1-re^{i\theta})|^* \ge 0$ and in the last step we have used the statement on equality in Lemma 5. Using the equivalence of (a) and (c) in Lemma 1, gives

$$\int_{-\pi}^{\pi}\Phi(\log|f(re^{i\theta})|)d\theta \le \int_{-\pi}^{\pi}\Phi(\log|F(re^{i\theta})|)d\theta.$$

To obtain the other inequality, we note that replacing x by -1 in

Lemma 2 and replacing θ by $\theta+\pi$ in Lemmas 3 and 4 we obtain

$$\begin{aligned}
(-\log|f(re^{i\theta})|)^* \leq{}& \left(-\log\left|\frac{re^{i\theta}}{(1+re^{i\theta})^2}\right|^q\right)^* \\
&+\sum_{j=1}^{s}\left(-\log\left|\frac{\left(1-\frac{re^{i\theta}}{|\beta_j|}\right)(1-|\beta_j|re^{i\theta})}{(1+re^{i\theta})^2}\right|\right)^* \\
&+(p-s-q)\left(-\log\left|\frac{1-re^{i\theta}}{1+re^{i\theta}}\right|^2\right)^* \\
={}& \left(\log\left|\frac{(1+re^{i\theta})^{2q}}{(re^{i\theta})^q}\prod_{j=1}^{s}\right.\right. \\
&\left.\left.\frac{(1+re^{i\theta})^2}{\left(1-\frac{re^{i\theta}}{|\beta_j|}\right)(1-|\beta_j|re^{i\theta})}\left(\frac{1+re^{i\theta}}{1-re^{i\theta}}\right)^{2(p-s-q)}\right|\right)^*
\end{aligned}$$

where again we have used the equality statement in Lemma 5. We thus obtain from Lemma 1

$$\begin{aligned}
\int_{-\pi}^{\pi}\Phi(-\log|f(re^{i\theta})|)d\theta &\leq \int_{-\pi}^{\pi}\Phi(-\log|F(-re^{i\theta})|)d\theta \\
&= \int_{-\pi}^{\pi}\Phi(-\log|F(re^{i\theta})|)d\theta.
\end{aligned}$$

□

We immediately obtain

COROLLARY 2. *If f is in $K_w(p)$, then, for $-\infty<\lambda<\infty$ and r in* $(0,1), r\neq|\beta_j|$,

$$\int_{-\pi}^{\pi}|f(re^{i\theta})|^\lambda d\theta \leq \int_{-\pi}^{\pi}|F(re^{i\theta})|^\lambda d\theta$$

where F is defined by (2.1) *or* (2.2).

We also note that letting $\lambda\to\pm\infty$ in the corollary gives another proof of Theorem 2.

COROLLARY 3. *Let* $f(z) = z^q + \cdots$ *be in* $K_w(p)$ *with* $f'(\alpha_j) = 0$, $0 < |\alpha_j| < 1$, $j = 1, 2, \ldots, t$, *and* $q + t \leq p$. *If* Φ *is any non decreasing convex function on* $(-\infty, \infty)$, *then for* $r \in (0,1), r \neq |\alpha_j|$,

$$\int_{-\pi}^{\pi} \Phi(\log |f'(re^{i\theta})|)d\theta \leq \int_{-\pi}^{\pi} \Phi(\log |G'(re^{i\theta})|)d\theta$$

where G *in* $K_w(p)$ *is defined by*

$$zG'(z) = q\left(\frac{1+z}{1-z}\right)F(z)$$

and F *is given by* (2.1) *with* $\beta_j = \alpha_j$ *and* $s = t$ *and* F *is given by* (2.2) *if* $f'(z) \neq 0$ *for* $0 < |z| < 1$.

PROOF. There exists g in $S_{wc}(p)$ and a β such that, for z in Δ,

$$\operatorname{Re}\left[e^{i\beta}\frac{zf'(z)}{g(z)}\right] > 0.$$

Let $P(z) = e^{i\beta}zf'(z)/g(z) = qe^{i\beta} + \cdots$. It follows from Corollary 1 of Leung [**12**] that

$$(\log |P(re^{i\theta})|)^* \leq \left(\log q\left|\frac{1+re^{i\theta}}{1-re^{i\theta}}\right|\right)^*.$$

Also we must have $g(\alpha_j) = 0, j = 1, 2, \ldots, t$. Since $g \in S_{wc}(p) \subset K_w(p)$, it follows by Theorem 2 that

$$(\log |g(re^{i\theta})|)^* \leq (\log |F(re^{i\theta})|)^*.$$

Writing $e^{i\beta}f'(z) = P(z)g(z)$ and proceeding as in Theorem 2 gives the corollary. □

If we single out the case $q = p$ we obtain

COROLLARY 4. *If* $f(z) = z^p + \cdots$ *is in* $K_w(p)$ *and* Φ *is a nondecreasing convex function on* $(-\infty, \infty)$ *then, for* r *in* $(0,1)$,

$$\int_{-\pi}^{\pi} \Phi(\log |f'(re^{i\theta})|)d\theta \leq \int_{-\pi}^{\pi} \Phi(\log |F'(re^{i\theta})|)d\theta$$

where $F(z) = z^p/(1-z)^{2p}$.

4. Convex functions. Hallenbeck and the author [**9**] considered integral means problems for several classes of multivalent functions. In particular we considered the class of multivalent convex functions $C(p)$, consisting of functions f, analytic in Δ, for which there exists ρ so that $\operatorname{Re}(1 + zf''(z)/f'(z)) > 0$ for $\rho < |z| < 1$. Functions in $C(p)$ are at most p-valent in Δ and $f'(z)$ has $(p-1)$ zeros in Δ. There is a connection with α starlike functions first considered by Mocanu [**19**], namely $f(z) = z^p + \cdots$ is in $C(p)$ if and only if $[f(z)]^{1/p}$ is α starlike with $\alpha = 1/p$. In what follows we will use the symbol $f \prec g$ to mean that f is subordinate to g. That is, there exists $w(z)$, analytic in Δ with $w(0) = 0$ and $|w(z)| < 1$ in Δ, so that $f(z) = g(w(z))$.

LEMMA 6. *If* $f(z) = z^p + \cdots$ *is in* $C(p)$, *then*

a) $\frac{f'(z)}{z^{p-1}} \prec \frac{F'(z)}{z^{p-1}}$,

b) $\frac{f(z)}{z^p} \prec \frac{F(z)}{z^p}$,

where $F(z) = p\int_0^z \frac{\xi^{p-1}}{(1-\xi)^{2p}}d\xi$.

PROOF. To obtain (a) we note that $(1/p)zf'(z) = z^p + \cdots$ is in $S(p)$. It is well known that such a function is the p-th power of a univalent starlike function. Thus $(1/p)zf'(z) = h(z)^p$ where $h(z) = z + \cdots$ is a member of $S(1)$. Using the fact that $h(z)/z \prec 1/(1-z)^2$ [**20**], gives part (a).

To prove part (b) we apply Theorem 1 in [**2**] to $[f(z)]^{1/p}$ with $\alpha = 1/p, d = 0, M = \infty$ to obtain the existence of $G(z) = z^p + \cdots$ in $C(p)$ such that $f(z)/z^p \prec G(z)/z^p$ for all f in $C(p)$. Let $f(z) = z^p + a_{p+1}z^{p+1} + \cdots, F(z) = z^p + A_{p+1}z^{p+1} + \cdots$, and $G(z) = z^p + b_{p+1}z^{p+1} + \cdots$. It is known [**7**] that, for all f in $C(p), |a_{p+1}| \le |A_{p+1}|$, with equality if and only if $f(z) = (1/x^p)F(xz)$ for some x with $|x| = 1$. Since G is in $C(p)$, we have $|b_{p+1}| \le |A_{p+1}|$. But $F(z)/z^p \prec G(z)/z^p$ implies $|A_{p+1}| \le |b_{p+1}|$. Thus $|b_{p+1}| = |A_{p+1}|$. Therefore, there exists $x, |x| = 1$, such that $G(z) = 1/x^pF(xz)$. Part (b) now follows.

For $p=1$, part (b) is a result of Strohhäcker [**20**]. It was proven for $p=2$ and $p=3$ in [**9**]. □

LEMMA 7. [**12**]. *If g and h are subharmonic in Δ and $g \prec h$, then, for each r in $(0,1)$ and $0<\theta<\pi$,*

$$g^*(re^{i\theta}) \le h^*(re^{i\theta}).$$

THEOREM 4. *If $f(z) = z^p + \cdots$ is in $C(p)$ and if Φ is any non-decreasing convex function on $(-\infty,\infty)$, then, for any r in $(0,1)$:*

a) $\int_{-\pi}^{\pi} \Phi\Big(\pm \log\Big|\frac{f(re^{i\theta})}{(re^{i\theta})^p}\Big|\Big)d\theta \le \int_{-\pi}^{\pi} \Phi\Big(\pm \log\Big|\frac{F(re^{i\theta})}{(re^{i\theta})^p}\Big|\Big)d\theta,$

b) $\int_{-\pi}^{\pi} \Phi\Big(\pm \log\Big|\frac{f'(re^{i\theta})}{(re^{i\theta})^{p-1}}\Big|\Big)d\theta \le \int_{-\pi}^{\pi} \Phi\Big(\pm \log\Big|\frac{F'(re^{i\theta})}{(re^{i\theta})^{p-1}}\Big|\Big)d\theta,$

c) $\int_{-\pi}^{\pi} \Phi\Big(\pm \log\Big|\frac{f''(re^{i\theta})}{(re^{i\theta})^{p-2}}\Big|\Big)d\theta \le \int_{-\pi}^{\pi} \Phi\Big(\pm \log\Big|\frac{F''(re^{i\theta})}{(re^{i\theta})^{p-2}}\Big|\Big)d\theta,$

where

$$F(z) = p\int_0^z \frac{\xi^{p-1}}{(1-\xi)^{2p}}d\xi.$$

PROOF. It follows from Lemma 6 that $\log|f(z)/z^p| \prec \log|F(z)/z^p|$ and $\log|f'(z)/z^{p-1}| \prec \log|F'(z)/z^{p-1}|$. An application of Lemma 7 and Lemma 1 gives (a) and (b). To prove part (c), we note that from Lemma 6, there exists $\phi(z)$ with $\phi(0)=0$ and $|\phi(z)|<1$ such that

$$f'(z) = \frac{pz^{p-1}}{(1-\phi(z))^{2p}}.$$

Also, since $P(z) = 1 + zf''(z)/f'(z) = p+\cdots$ has positive real part in Δ, there exists $w(z)$, analytic in Δ with $w(0)=0$ and $|w(z)|<1$, such that

$$P(z) = p\Big(\frac{1+w(z)}{1-w(z)}\Big).$$

Making use of a result of Feng and MacGregor [**5**, Lemma 5], we can write

$$\begin{aligned} zf''(z) = f'(z)(P(z)-1) &= \frac{pz^{p-1}[(p-1)+(p+1)w(z)]}{(1-\phi(z))^{2p}(1-w(z))} \\ &= \frac{pz^{p-1}[(p-1)+(p+1)w(z)]}{(1-\psi(z))^{2p+1}} \end{aligned}$$

where ψ is analytic in Δ, with $\psi(0) = 0$ and $|\psi(z)| < 1$. Using Lemmas 5 and 7 we obtain

$$\left(\log\left|\frac{f''(re^{i\theta})}{(re^{i\theta})^{p-2}}\right|\right)^* \leq [\log p|(p-1)+(p+1)re^{i\theta}|]^* + \left[\log\frac{1}{|1-re^{i\theta}|^{2p+1}}\right]^*$$
$$= \left[\log\left|\frac{p((p-1)+(p+1)re^{i\theta})}{(1-re^{i\theta})^{2p+1}}\right|\right]^*.$$

Thus, from Lemma 1, we obtain

$$\int_{-\pi}^{\pi}\Phi\left(\log\left|\frac{f''(re^{i\theta})}{(re^{i\theta})^{p-2}}\right|\right)d\theta \leq \int_{-\pi}^{\pi}\Phi\left(\log\left|\frac{p[(p-1)+(p+1)re^{i\theta}]}{(1-re^{i\theta})^{2p+1}}\right|\right)d\theta$$
$$= \int_{-\pi}^{\pi}\Phi\left(\log\left|\frac{F''(re^{i\theta})}{(re^{i\theta})^{p-2}}\right|\right)d\theta.$$

The other inequality is obtained in a similar way.

COROLLARY. *If $f(z) = z^p + \cdots$ is in $C(p)$, then, for r in $(0,1)$ and $-\infty < \lambda < \infty$,*

a) $\int_{-\pi}^{\pi}|f(re^{i\theta})|^\lambda d\theta \leq \int_{-\pi}^{\pi}|F(re^{i\theta})|^\lambda d\theta$,

b) $\int_{-\pi}^{\pi}|f'(re^{i\theta})|^\lambda d\theta \leq \int_{-\pi}^{\pi}|F'(re^{i\theta})|^\lambda d\theta$,

c) $\int_{-\pi}^{\pi}|f''(re^{i\theta})|^\lambda d\theta \leq \int_{-\pi}^{\pi}|F''(re^{i\theta})|^\lambda d\theta$,

where

$$F(z) = p\int_0^z \frac{\xi^{p-1}}{(1-\xi)^{2p}}d\xi.$$

REFERENCES

1. A Baernstein II, *Integral means, univalent functions and circular symmetrization,* Acta. Math. **133** (1974), 139-169.

2. R. Barnard and J.L Lewis, *Subordination theorems for some classes of starlike functions*, Pac. J. of Math. **56** (1975), 333-366.

3. L DeBranges, *A proof of the Bieberbach conjecture*, Acta. Math. **154** (1985), 137-152.

4. P Duren and G. Schober, *Nonvanishing univalent functions*, Math. Z. **170** (1980), 195-216.

5. J Feng and T.H. MacGregor, *Estimates on integral means of the derivatives of univalent functions*, J D'Analyse Math. **29** (1976), 203-321.

6. A.W. Goodman, *On some determinants related to p-valent functions,* Trans. Amer. Math. Soc. **63** (1948), 175-192.

7. ——— , *On the Schwarz-Christoffel transformation and p-valent functions,* Trans. Amer. Math. Soc. **68** (1950), 204-223.

8. ———, *Typically real functions with assigned zeros,* Proc. Amer. Math. Soc. **2** (1951), 349-357.

9. D.J. Hallenbeck and A.E.Livingston, *Applications of Extreme point theory to classes of multivalent functions,* Trans. Amer. Math. Soc. **221** (1976), 339-359.

10. ——— and ———, *A coefficient estimate for multivalent functions,* Proc. Amer. Math. Soc. **54** (1976), 201-206.

11. J.A. Hummel, *Multivalent starlike functions,* J.D'Analyse Math. **18** (1967), 133-160.

12. Y.J. Leung, *Integral means of the derivatives of some univalent functions,* Bull. London Math. Soc. **11** (1979), 289-294.

13. A.E. Livingston, *p-valent close-to-convex functions,* Trans. Amer. Math. Soc. **115** (1965), 161-179

14. ———, *The coefficients of multivalent close-to-convex functions,* Proc. Amer. Math. Soc. **21** (1969), 545-552.

15. ———, *On the integral means of univalent, meromorphic functions,* Pac. J. of Math. **72** (1977), 167-180.

16. ———, *A coefficient inequality for functions of positive real part with an application to multivalent functions,* Pac. J. of Math. **120** (1985), 139-151.

17. A. Lyzzaik, *Multivalent linearly accessible functions and close-to-convex functions,* Proc. London Math. Soc. **44** (1982), 178-192.

18. ——— and D. Styer, *Goodman's conjecture and the coefficients of univalent functions,* Proc. Amer. Math. Soc. **69** (1978), 111-114.

19. P. Mocanu, *Une propriété de convexité généralisée dans la représentation conforme,* Mathematica (Cluj) **34** (1969), 127-133.

20. E. Strohhäcker, *Beiträge zur Theorie der schlichten Funktionen,* Math. Z **37** (1933), 356-380.

21. D. Styer, *On weakly starlike multivalent functions,* J. D'Analyse Math. **26** (1973), 217-233.

22. ———, *Close-to-convex multivalent functions with respect to weakly star-like functions,* Trans. Amer. Math. Soc. **169** (1972), 105-112.

UNIVERSITY OF DELAWARE, NEWARK, DE 19717