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A COEFFICIENT ESTIMATE FOR NONVANISHING H^p FUNCTIONS

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ABSTRACT. The Krzyz conjecture asserts that if $f(z) = a_0 + a_1 z + a_2 z^2 + \cdots$ is a nonvanishing analytic function with $|f| \leq 1$ in $|z| < 1$, then $|a_n| \leq 2/e$ $(n = 1, 2, \ldots)$. Hummel, Scheinberg and Zalcman more generally conjectured that $|a_n| \leq (2/e)^{1/2}$ for all nonvanishing $f \in H^p$ with $||f||_p \leq 1 (1/p + 1/q = 1, 1 < p < \infty)$. We prove the latter conjecture for $n = 2$ and $n = 3$ for a natural subclass of nonvanishing H^p functions. We also point out a relationship between the two conjectures for this subclass. Our main tool in this investigation is the Pontryagin Maximum Principle.

1. Introduction. Let B_p denote the set of all nonvanishing H^p functions $f(z) = a_0 + a_1 z + a_2 z^2 + \cdots$ with $||f||_p \leq 1$. Hummel, Scheinberg and Zalcman [4] conjectured that

$$\sup_{B_p} |a_n| = \left(\frac{2}{e}\right)^{1/q}, \text{ for all } n \geq 1, \tag{1}$$

where $1 < p < \infty$ and $1/p + 1/q = 1$. If true, the bound is attained by

$$H_n(z) = \left(\frac{(1+z^n)^2}{2}\right)^{\frac{1}{p}} \left(\exp\left(\frac{z^n - 1}{z^n + 1}\right)\right)^{\frac{1}{q}}$$

and its rotations $e^{i\nu} H_n(e^{i\mu} z)$, where $\nu \mu \in \mathbf{R}$. To date, the only evidence supporting (1) is given in [1] where the conjecture was verified for $n = 1$ and for arbitrary $n \geq 2$ provided $a_m = 0$ for all $1 \leq m < (n+1)/2$. In this paper we prove the conjecture for $n = 2$ and $n = 3$ for a certain natural subclass of nonvanishing H^p functions which we now describe.

It is well-known (see [2] for example) that if f is a nonvanishing H^p function then

$$f(z) = e^{i\lambda} \Omega(z) I(z), \tag{2}$$

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where $\Omega(z)$ and $I(z)$ are the outer and inner factors of f, respectively, and $\lambda \in \mathbf{R}$. For $n \geq 2$, let $B_p(n)$ denote those functions $f \in B_p$ of the form (2) with $\Omega'(0) = \cdots = \Omega^{(n-1)}(0) = 0$. Observe that $H_n \in B_p(n)$ for each $n \geq 2$ and that the classes $B_p(n)$ are nested in B_p. We can now state our main result.

THEOREM 1. *Let* $1 < p < \infty$.

(a) *If* $f \in B_p(2)$, *then* $|a_2| \leq (\frac{2}{e})^{1/2}$.

(b) *If* $f \in B_p(3)$, *then* $|a_3| \leq (\frac{2}{e})^{1/q}$.

Equality holds only for H_2 *and* H_3, *respectively, and their rotations.*

The conjecture (1) is a generalization of the Krzyz conjecture which asserts that

$$\sup_{B_\infty} |a_n| = \frac{2}{3}, \text{ for all } n \geq 1. \tag{3}$$

This conjecture has been proved only for $n = 1, 2, 3, 4$ [**4, 6, 7**]. We point out a connection between the conjecture (1) for $B_p(n)$ and the Krzyz conjecture. Subordination methods maybe used to study the Krzyz conjecture but seem to be of little use in studying the more general problem (1). We will make use of the Pontryagin Maximum Principle here. This appears to be the most effective tool in considering the conjecture (1).

2. Preliminaries. Any function $f \in B_p$ has the form (2). Since the inner function I belongs to B_∞ in order to prove our theorem we need sharp estimates for coefficients of functions in B_∞. Suppose $g(z) = b_0 + b_1 z + b_2 z^2 + \cdots \in B_\infty$ and $b_0 = e^{-t}$ for some $0 \leq t < \infty$. In [**4**], or directly, we obtain

$$\left|\frac{b_2}{b_0}\right| \leq \begin{cases} 2t & 0 \leq t \leq 2 \\ 2(t^2 - t) & 2 \leq t < \infty. \end{cases} \tag{4}$$

Bounds for $|b_3/b_0|$were also obtained in [**4**], but not all were sharp.

Prokhorov and Szynal [**6**] obtained the following sharp bounds:
(5)
$$\left|\frac{b_3}{b_0}\right| \le \begin{cases} 2y \equiv F_1(t) & 0 \le t \le t_1 \\ \frac{\sqrt{8}}{3}(2t-1)^{3/2} \equiv F_2(t), & t_1 \le t \le t_2 \\ \frac{\sqrt{8}}{3}(2t^2-6t+3)(t-2)^{3/2}(t-3)^{-1/2} \equiv F_3(t), & t_2 \le t \le t_3 \\ \frac{\sqrt{8}}{3}t(2t-3)^{3/2}(-t^2 6t-6)^{-1/2} \equiv F_4(t), & t_3 \le t \le t_4 \\ \frac{2}{3}t(2t^2-6t+3) \equiv F_5(t), & t_4 \le t < \infty, \end{cases}$$

where $t_1 = 1.65495\cdots, t_2 = 3.22474\cdots, t_3 = 3.47568\cdots$, and $t_4 = 3.82287\cdots$ are roots of

$$16t^3 - 33t^2 + 12t - 2 = 0$$

$$2t^2 - 8t + 5 = 0$$

$$2t^3 - 12t^2 + 21t - 12 = 0$$

$$2t^2 - 10t + 9 = 0$$

respectively.

Finally, we mention that the outer and inner functions in (2) have the form

$$\Omega(z) = \exp\left(\frac{1}{2\pi}\int_0^{2\pi} \frac{e^{i\theta}+z}{e^{i\theta}-z}\log\omega(\theta)d\theta\right) \tag{6}$$

and

$$I(z) = \exp\left(-\int_0^{2\pi} \frac{e^{i\theta}+z}{e^{i\theta}-z}d\mu(\theta)\right), \tag{7}$$

where ω is a nonnegative measurable function with $\log\omega \in L^1$, $||\omega||_{L^p} = ||f||_p$ and μ is a bounded nondecreasing function with $\mu'(\theta) = 0$ a.e.. Note also that $\omega(\theta) = |f(e^{i\theta})|$ a.e..

3. Proof of Theorem 1. For any fixed $n \ge 2$, the class $B_p(n) \cup \{0\}$ is compact and if f belongs to $B_p(n)$, then so does $e^{i\nu}f(e^{i\mu}z)$. Hence it is enough to consider

$$\max_{B_p(n)} \operatorname{Re}\{a_n\} \equiv J_p(n) \tag{8}$$

and we may assume $f(0) = a_0 > 0$. Let $F_n(z) = \sum_{k=0}^{\infty} A_k z^k$ be an extremal function for (8). Then clearly $||F_n||_p = 1$. Indeed, if $||F_n||_p < 1$, then there exists a $\lambda > 1$ such that $||\lambda F_n||_p = 1$. Hence $\lambda F_n \in B_p(n)$ with larger coefficients. Summarizing, it suffices to consider (8) over all $f(z) = \Omega(z)I(z) \in B_p(n)$ with $f(0) > 0$ and $||f||_p = 1$. Let $B_p^*(n)$ denote this class. Hence we have

$$J_p(n) = \max_{B_p^*(n)} \operatorname{Re}\{a_n\}. \tag{9}$$

For convenience put

$$\frac{1}{2\pi}\int_0^{2\pi} \frac{e^{i\theta}+z}{e^{i\theta}-z} \log\omega(\theta)d\theta = \sum_{k=0}^{\infty} c_k z^k \tag{10}$$

and

$$I(z) = b_0 + b_1 z + b_2 z^2 + \cdots, \tag{11}$$

with $b_0 = e^{-t}$ for some $t \geq 0$. Hence we have

$$c_0 = \frac{1}{2\pi}\int_0^{2\pi} \log\omega d\theta \tag{12}$$

and

$$c_k = \frac{1}{2\pi}\int_0^{2\pi} 2e^{-ik\theta}\log\omega(\theta)d\theta, \quad k \geq 1. \tag{13}$$

If $f(z) = a_0 + a_1 z + a_2 z^2 + \cdots = \Omega(z)I(z) \in B_p^*(n)$, then

$$a_0 = e^{c_0} b_0 = e^{c_0 - t} \tag{14}$$

and

$$a_n = e^{c_0 - 1}\left(c_n + \frac{b_n}{b_0}\right). \tag{15}$$

(Since $\Omega'(0) = \cdots = \Omega^{(n-1)}(0) = 0$ we have $c_1 = \cdots = c_{n-1} = 0$.)

We want to maximize Re $\{a_n\}$ given by (15). Now as $\Omega(z)$ and $I(z)$ can vary independently, it is apparent from (15) that sharp bounds on $|\frac{b_n}{b_0}|$, when $b_0 = e^{-t}$, are needed. This is precisely what is done in resolving the conjecture (3).

Let M denote the set of all nonnegative functions ω with $\log \omega \in L^1$ satisfying

$$\frac{1}{2\pi}\int_0^{2\pi} \omega^p d\theta = 1 \tag{16}$$

and

$$c_k = \frac{1}{2\pi}\int_0^{2\pi} 2e^{-ik\theta}\log\omega(\theta)d\theta = 0, \quad 1 \le k \le n-1. \tag{17}$$

It follows from the above and (15) that

$$J_p(n) = \max_{t>0} \max_{\substack{I\in B_\infty \\ I(0)=e^{-t}}} \max_{\omega\in M} \text{Re}\,\{e^{c_0 - t}(c_n + b_n/b_0)\}, \tag{18}$$

where $c_0 = \frac{1}{2\pi}\int_0^{2\pi} \omega(\theta)d\theta$.

We consider the inner maximum first. Since there exists an extremal function F_n for (9) there exists a function ω_n maximizing the inner functional (indeed, $\omega_n(\theta) = |F_n(e^{i\theta})|$. a.e.). Our goal is to identify the form an extremal ω_n must have. Fix such a function ω_n and let

$$c_0^* = \frac{1}{2\pi}\int_0^{2\pi} \log\omega_n(\theta)d\theta$$

and

$$c_k^* = \frac{1}{2\pi}\int_0^{2\pi} 2e^{-ik\theta}\log\omega_n(\theta)d\theta, \quad k \ge 1.$$

($C_k^* = 0$ for $\le k \le n-1$). Now, since ω_n maximizes the inner functional in (18) we have

$$\text{Re}\Big\{e^{c_0}\Big(c_n + \frac{b_n}{b_0}\Big)\Big\} \le \text{Re}\,\Big\{e^{c_0^*}\Big(c_n^* + \frac{b_n}{b_0}\Big)\Big\}.$$

In fact, if we let M^* denote all functions $\omega \in M$ such that $c_0 = 1.2\pi\int_0^{2\pi}\log\omega(\theta)d\theta = c_0^*$, then ω_n is also extremal for the simple problem

$$\max_{\omega\in M^*} \text{Re}\,\{c_n\}. \tag{19}$$

The extremal problem (19) leads to the following isoperimetric problem:

$$\max\left\{\frac{1}{2\pi}\int_0^{2\pi}\cos n\theta\log\omega(\theta)d\theta\right\}$$

such that

$$\int_0^{2\pi}\cos k\theta\log\omega(\theta)d\theta = 0 \quad (1\le k\le n-1)$$

$$\int_0^{2\pi}\sin k\theta\log\omega(\theta)d\theta = 0 \quad (1\le k\le n-1)$$

$$\frac{1}{2\pi}\int_0^{2\pi}\log\omega(\theta)d\theta = c_0^*$$

$$\frac{1}{2\pi}\int_0^{2\pi}\omega^p(\theta)d\theta = 1,$$

where the maximum is taken over all nonnegative measurable functions ω satisfying the stated constraints. The vanishing of the $2n-2$ integrals above are the constraints (17). This problem, which has a solution ω_n, is a simple problem in control theory. Let us define the functions $x_j(t)$ as follows:

$$\begin{aligned}
x_0(t) &= -\frac{1}{\pi}\int_0^t \cos n\theta\log\omega(\theta)d\theta\\
x_k(t) &= \int_0^t \cos k\theta\log\omega(\theta)d\theta \quad (1\le k\le n-1)\\
x_{k+n-1}(t) &= \int_0^t \sin k\theta\log\omega(\theta)d\theta \quad (1\le k\le n-1)\\
x_{2n-1}(t) &= \frac{1}{2\pi}\int_0^t \log\omega(\theta)d\theta\\
x_{2n}(t) &= \frac{1}{2\pi}\int_0^t \omega^p(\theta)d\theta.
\end{aligned}$$

Then we have the problem of determining a control ω of the system of equations

$$\begin{aligned}
\dot{x}_0(t) &= -\frac{1}{\pi}\cos nt \log\omega(t) \equiv G_0(\omega, t)\\
\dot{x}_k(t) &= \cos kt \log\omega(t) \equiv G_k(\omega, t) \quad (1 \le k \le n-1)\\
\dot{x}_{k+n-1}(t) &= \sin kt \log\omega(t) \equiv G_{k+n-1}(\omega, t) \quad (1 \le k \le n-1)\\
\dot{x}_{2n-1}(t) &= \frac{1}{2\pi}\log\omega(t) \equiv G_{2n-1}(\omega, t)\\
\dot{x}_{2n}(t) &= \frac{1}{2\pi}\omega^p(t) \equiv G_{2n}(\omega, t)
\end{aligned}$$

that takes it from the initial point $(x_0(0), \ldots, x_{2n}(0)) = \bar{0}$ to the final point $(x_0(2\pi), x_1(2\pi), \ldots, x_{2n}(0))$ under the condition that $x_0(t)$ attains its minimum at the final time $t = 2\pi$. We can apply the Pontryagin Maximum Principle which asserts that an optimal ω_n must satisfy $\frac{dH}{d\omega}|_{\omega_n} = 0$, where $H = \sum_{k=0}^{2n} \psi_k G_k$ and where ψ_k are solutions to the conjugate system

$$\dot{\psi} = -\frac{\partial H}{\partial x_k}, \quad 0 \le k \le 2n.$$

(See [**5, 8, 1**].) Since the functions G_k (hence H) are independent of $x_0, x_1, \ldots, x_{2n}$ we see that ψ_k are all constants. Hence in our case

$$\begin{aligned}
H = \lambda_0(\cos nt \log\omega) &+ \sum_{k=1}^{n-1}(\lambda_k \cos kt + \mu_K \sin kt)\log\omega\\
&+ \lambda_n\omega + \mu_n\omega^p,
\end{aligned}$$

where λ_k and μ_k are constants. Now since $\frac{dH}{d\omega}|_{\omega} = 0$,it follows that an extremal ω_n must have the form

$$\omega_n^p(t) = \alpha_0 + \beta_0 \cos nt + \sum_{k=1}^{n-1}(\alpha_k \cos kt + \beta_k \sin kt), \tag{20}$$

for some $\alpha_k, \beta_k \in \mathbf{R}$. Now as $\omega_n \ge 0$, from (20) we see that ω_n^p is a nonnegative trigonometric polynomial of degree n, and, invoking an old result of Fejer and Riesz, we conclude that

$$\omega_n^p(t) = |\sum_{k=0}^{n} \gamma_k^{e^{ikt}}|^2, \tag{21}$$

for some $\gamma_k \in \mathbf{C}$. Note that our extremal function $F_n(z) = \Omega_n(z)I_n(z)$ and that

$$\Omega_n(z) = \exp\left\{\frac{1}{2\pi}\int_0^{2\pi}\frac{e^{i\theta}+z}{e^{i\theta}-z}\log\omega_n(\theta)d\theta\right\}$$
$$= \Big(\sum_{k=0}^{n} d_k z^k\Big)^{2/p}$$

for some $d_k \in \mathbf{C}$. Now, since $||\omega_n||_{L_p} = 1$ we get $\sum_{k=0}^{n}|\gamma_k|^2 = 1$ and so $\sum_{k=0}^{n}|d_k|^2 = 1$. Since $\Omega_n'(0) = \Omega''(0) = \cdots = \Omega_n^{(n-1)}(0) = 0$, we must have $d_1 = d_2 = \cdots = d_{n-1} = 0$. Also note that $\Omega_n(0) = e^{c_0} = d_0^{2/p} > 0$. Hence we obtain

$$\Omega_n(z) = (d_0 + d_n z^n)^{2/p}, \tag{22}$$

where $d_0^2 + |d_n|^2 = 1$ and $|d_n| \leq d_0$. The last result follows since $\Omega_n(z) \neq 0$ in $|z| < 1$.

If $F_n(z) = \sum_{k=0}^{\infty} A_k z^k = \Omega_n(z)I_n(z)$ is an extremal function for (9) with $I_n(z) = b_0 + b_1 z + b_2 z^2 + \cdots$ ($b_0 = e^{-t}$ for some $t \geq 0$) and Ω_n given by (22), then

$$J_p(n) = \text{Re}\,\{A_n\} = \text{Re}\left\{d_0^{2/p}e^{-t}\Big(\frac{2}{p}\frac{d_n}{d_0} + \frac{b_n}{b_0}\Big)\right\}. \tag{23}$$

It remains to maximize the right-hand side of (23) over d_0, d_n with $|d_n| \leq d_0, d_0^2 + |d_n|^2 = 1$, and over functions in B_∞ with $b_0 = e^{-t}$ for some $0 \leq t < \infty$. The connection with the conjecture (3) is now clear.

At the present there are only sharp bounds for $|\frac{b_2}{b_0}|$ and $|\frac{b_3}{b_0}|$ as pointed out in §2. Hence we now restrict our attention to $n = 2$ and $n = 3$.

For $n = 2$ we see that (23) becomes

$$J_p(2) = \text{Re}\left\{d_0^{2/p}e^{-t}\Big(\frac{2}{p}\Big(\frac{d_2}{d_0}\Big) + \Big(\frac{b_2}{b_0}\Big)\Big)\right\}. \tag{24}$$

Let $u = |\frac{d_2}{d_0}|$ and note that since $d_0^2 + |d_2|^2 = 1$ and $|d_2| \leq d_0$, we have $d_0^{2/p} = (1+u^2)^{-1/p}$ and $0 \leq 1$. Using (4) we see that (24) implies

$$J_p(2) \leq (1+u^2)^{-1/p}e^{-t}\Big(\frac{2u}{p} + \psi(t)\Big) \equiv \phi(u,t), \tag{25}$$

where

$$\psi(t) = \begin{cases} 2t, & 0 \le t \le 2 \\ 2(t^2 - t), & 2 \le t < \infty \end{cases}.$$

For $0 \le t \le 2$, it is easy to see that $\phi(u,t) \le (2/e)^{1/q}$ with equality if and only if $u = 1$ and $t = 1/q$. Suppose next that $2 \le t \le 3$. Hence we have

$$\phi(u,t) = (1+u^2)^{-1/p} e^{-t}\Big(\frac{2u}{p} + 2(t^2 - t)\Big). \tag{26}$$

Now since $t^2 - t \le 3.7t - 5$ when $2 \le t \le 3$ we obtain

$$\phi(u,t) \le (1+u^2)^{-1/p} e^{-t}\Big(\frac{2u}{p} + 7.4 - 10\Big) \equiv \phi^*(u,t).$$

It follows that $\phi^*(u,t) \le \phi^*(u,t_0)$, where $t_0 = \frac{87}{37} - \frac{u}{(3.7p)}$ and hence

$$\begin{aligned} \phi(u,t) &\le \phi^*(u,t) \le \phi^*(u,t_0) \\ &= e^{-87/37}(7.4)\Big(\frac{e^{u/3.7}}{(1+u^2)}\Big)^{1/p} \\ &< (0.71)\Big(\frac{e^{0.14/3.7}}{1+(0.13)^2}\Big)^{1/p} \\ &< 0.73 < \frac{2}{e} < \Big(\frac{2}{e}\Big)^{1/q}. \end{aligned}$$

Finally, suppose $t \ge (3+\sqrt{5}/2 = 2.61803\cdots$. Note that this overlaps the interval $[2,3]$ just considered. It is easy to see that $\phi(u,t)$ given by (26) is a decreasing function of t for $t \ge (3+\sqrt{5})/2$. Hence we obtain

$$\phi(u,t) \le \phi\Big(u, \frac{3+\sqrt{5}}{2}\Big) \le \max_{2 \le t \le 3} \phi^*(u,t) < \Big(\frac{2}{e}\Big)^{1/q}.$$

This completes the proof for the case $n = 2$.

The case $n = 3$ is similar to $n = 2$ and so, as above from (23), we have the estimate

$$J_p(3) \le (1+u^2)^{-1/p} e^{-t}\Big(\frac{2u}{p} + \psi(t)\Big) \equiv \phi(u,t), \tag{27}$$

where $\psi(t) = F_k(t), t_{k-1} \leq t < t_k$ for $1 \leq k \leq 5$, and the functions F_k and constants t_k are given by (5) ($t_0 = 0, t_5 = \infty$).

If $0 \leq t \leq t_1$ then, as above, it is easy to check that $\phi(u,t) \leq (\frac{2}{e})^{1/q}$ with equality only for $u = 1$ and $t = 1/q$. Since the functions F_k are a little involved we have to consider smaller intervals.

If $t_1 \leq t \leq 1.8$, then we get $F_2(t) < 3.05t - 1.5$, and so

$$\begin{aligned}\phi(u,t) &\leq (1+u^2)^{-1/p}e^{-t}\Big(\frac{2u}{p} + (3.05t - 1.5)\Big)\\ &< (1+u^2)^{-1/p}e^{-1.65}\Big(\frac{2u}{p} + 3.55\Big) \equiv M_1.\end{aligned}$$

For $1.8 \leq t \leq 1.9$, then $F_2(t) < 3.16t - 1.57$, and so a calculation gives

$$\phi(u,t) < (1+u^2)^{-1/p}e^{-1.8}\Big(\frac{2u}{p} + 4.12\Big) \equiv M_2.$$

If $1.9 \leq t \leq 2$, then $F_2(t) \leq \sqrt{8/3}(2t-1)$ and hence

$$\phi(u,t) < (1+u^2)^{-1/p}e^{-1.9}\Big(\frac{2u}{p} + 4.58\Big) \equiv M_3.$$

If $2 \leq t \leq t_2$, then

$$\phi(u,t) \leq \phi(u,2) < (1+u^2)^{-1/p}e^{2}\Big(\frac{2u}{p} + 4.9\Big) \equiv M_4.$$

For $t_2 \leq t \leq t_3$, we have the simple estimate $F_3(t) < 17.2$ and so

$$\phi(u,t) < (1+u^2)^{1/p}e^{-3.22}\Big(\frac{2u}{p} + 17.2\Big) \equiv M_5.$$

Now, for $t_3 \leq t \leq t_4$, we see that, since $(2y-3)^3/(-t^2+6t-6)$ is increasing, we get $F_4(t) < \sqrt{8}/3t(6.62) < 6.3t$. Thus we obtain

$$\phi(u,t) < (1+u^2)^{-1/p}e^{-3.47}\Big(\frac{2u}{p} + 22\Big) \equiv M_6.$$

If $t_4 \leq t \leq 4.4$, we can easily estimate $F_5(t)$ as follows:

$$F_5(t) < \begin{cases} 30, & t_4 \leq t \leq 4 \\ 37, & 4 \leq t \leq 4.2 \\ 45, & 4.2 \leq t \leq 4.4. \end{cases}$$

Hence, in these appropriate intervals, we conclude that

$$\phi(u,t) < \begin{cases} (1+u^2)^{-1/p}e^{3.82}(\frac{2u}{p}+30) \equiv M_7 \\ (1+u^2)^{-1/p}e^{-4}(\frac{2u}{p}+37) \equiv M_8 \\ (1+u^2)^{-1/p}e^{-4.2}(\frac{2u}{p}+45) \equiv M_9. \end{cases}$$

Finally, if $4.4 \leq t < \infty$ we see that $\phi(u,t)$ is a decreasing function of t and so $\phi(u,t) < M_9$.

If we now let $\beta(u,p) = (1+u^2)^{-1/p}e^{-c}(2u/p+m)$, where c and m are fixed constants and $m > 2$, then it is easy to check that β attains its maximum when $p = 1$ and $u = (-m+\sqrt{m^2+4})/2$. Thus we see that $M_k < 0.733$ for $1 \leq k \leq 9$, and, since $J_p(3) \leq \max_{1 \leq k \leq 0} M_k < 0.733 < \frac{2}{e} < (2/e)^{1/q}$, we are done.

The statement of equality follows from the fact that in (4), for $0 \leq t \leq 2$, equality occurs only for $\exp(-t((1-z^2)/(1+z^2))$ and its rotations; while in (5) equality holds when $0 \leq t \leq t_1$ only for $\exp(-t((1-z^3)/(1+z^3)))$ and its rotations. (See [**4** and **6**].) □0

4. Remarks. Finally we should point out that the Pontryagin Maximum Principle can be used to investigate the more general problem

$$\max_{B_p} \operatorname{Re}\{a_n\}, \quad n \geq 2, 1 < p < \infty. \tag{28}$$

In fact, we are again led to a certain isoperimetric problem and we conclude that any extremal function F must have the form

$$F(z) = \Big(\sum_{k=0}^{n} d_k z^k\Big)^{2/p} I(z),$$

where $d_k \in \mathbf{C}$ satisfy $\sum_{k=0}^{n} |d_k|^2 = 1$ and $\sum_{k=0}^{n} d_k z^k \neq 0$ in $|z| < \infty$. Since the inner function $I(z)$ has the form

$$I(z) = \exp\Big(-\int_0^{2\pi} \frac{e^{i\theta}+z}{e^{i\theta}-z} d\mu(\theta)\Big), \tag{29}$$

for some bounded non-decreasing μ with $\mu'(\theta) = 0$ a.e., the Goluzin variation [**3**] (see also [**4**]) may be applied to (29) for the problem (28) with fixed $\Omega(z)$. One then easily obtains

THEOREM 2. *If F is an extremal function for* (28), *then*

$$F(z) = \Big(\sum_{k=0}^{n} d_k z^k\Big)^{2/p} \exp\Big(-t\sum_{k=1}^{n} \lambda_k\Big(\frac{e^{i\theta_k}+z}{e^{i\theta_k}-z}\Big)\Big),$$

where $d_k \in \mathbf{C}$ satisfy $\sum_{k=0}^{n} |d_k|^2 = 1$ and $\sum_{k=0}^{n} d_k z^k \neq 0$ in $|z| < 1, t \geq 0, \lambda \geq 0$ with $\sum \lambda_k = 1$ and $\theta_k \in \mathbf{R}$.

This result, when $p = \infty$, is given in [**4**]. Although our result gives the form of an extremal function F for the conjecture (1), hence greatly simplifying the problem, there are difficulties that arise in trying to maximize the coefficients of F. Even the case $n = 2$ presents some difficulties. Nevertheless, it is hoped that Theorem 2 in conjection with other results will eventually lead to a proof of the conjecture (1).

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