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BOUNDARY VALUE PROBLEMS FOR SEMILINEAR ELLIPTIC EQUATIONS OF ARBITRARY ORDER IN UNBOUNDED DOMAINS

MARTIN SCHECHTER

ABSTRACT. We study boundary value problems for equations of the form $Au = f(x,u)$, where A is an elliptic operator of order $2m$. If A has suitable properties, we can allow $f(x,u)$ to grow in u to an arbitrarily high power. It is allowed to have exponential growth even when $2m < n$.

1. Introduction. We shall be concerned with boundary value problems of the form

$$A(x,D)u = f(x,u) \text{ in } \Omega, \tag{1.1}$$

$$B_j(x,D)u = 0 \text{ on } \partial\Omega,\ 1 \le j \le m, \tag{1.2}$$

where $A(x,D)$ is a uniformly elliptic operator of order $2m$ in a (bounded or unbounded) domain $\Omega \subset \mathbf{R}^n$, and the operators (1.2) cover it on $\partial\Omega$, the boundary of Ω(cf. [10, p. 224]). If the coefficients of $A(x,D)$ and the $B_j(x,D)$ as well as $\partial\Omega$ are sufficiently regular, then for any $1 < p < \infty$ the estimate

$$||u||_{2m,p} \le C(||A(x,D)u||_p + ||u||_p) \tag{1.3}$$

holds for $u \in H^{2m,p}(\Omega)$ satisfying (1.2), where $||u||_{k,p}$ is the norm in the Sobolev space $H^{k,p}(\Omega)$ and $||u||_p$ is the $L^p(\Omega$ norm (cf. Agmon-Douglis-Nirenber [**1**]). We shall require more: that $A(x,D)$ is a bijective map of those $u \in H^{2m,p}(\Omega)$ satisfying (1.2) onto $L^p(\Omega)$. Sufficient conditions for this to hold can be found in [**2, 3, 6, 8, 15-17**]. We shall show that it is true for the Dirichlet problem for constant coefficient operators for which the corresponding polynomial does not vanish on $\mathbf{R}^n$ (cf. §2).

Concerning the function $f(x,u)$ we shall assume that

$$|f(x,u)| \le \sum_{k=1}^{\infty} V_k(x)|u|^{b_k}, \quad b_k \ge 0 \tag{1.4}$$

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where the b_k are restricted only by the inequality

$$(1/p - 2m/n)b_k < 1/p \tag{1.5}$$

In particular, if $n \le 2mp$, we can have $b_k \to \infty$. The $V_k(x)$ are required to be in certian spaces which were introduced elsewhere [**11**] (definitions are given in §2). These spaces depend on n, m, p and b_k. A series corresponding to the right hand side of (1.4) is required to converge. In the case $n < 2mp$ we can even allow

$$|f(x,u)| \le V(x)e^{C|u|} \tag{1.6}$$

provided $V(x)$ is in $L^p(\Omega)$. In particular, we can solve the Dirichlet problem in unbounded domains for equations such as

$$[(-\Delta)^m + 1]u = V(x)e^{C|u|} \tag{1.7}$$

provided $V(x) \in L^p(\Omega)$ for some $p > n/2m$ and $||V||_p$ is bounded by a constat depending on m, n, C and Ω.

Our results have the advantage that strong solutions are obtained, i.e., solutions in $H^{2m,p}(\Omega)$ are found. The restrictions on $f(x,u)$ are extremely mild. Usually one is permitted growth in u only up to order $(n+2m)/(n-2m)$ when $n > 2m$. We can obtain nonvanishing solutions as well (Theorems 2.6 and 2.10). For instance if Ω is bounded and $n < 2p$, assume that

$$0 \le \alpha(u) \le f(x,u) \le V(x)e^{C|u|}, \ u \ge 0 \tag{1.8}$$

where $V(x)$ is a function in $L^p(\Omega)$ and $\alpha(t)$ is a nondecreasing function defined for $t \ge 0$ such that $\alpha(t)/t$ is bounded away from 0 on any bounded interval. Then there exists a solution $u(x) > 0$ in Ω of the Dirichlet problem

$$-\Delta u = \lambda f(x,u) \text{ in } \Omega, \ u = 0 \text{ on } \partial\Omega \tag{1.9}$$

for some $\lambda > 0$.

Our main results are stated in Section 2. Proofs are given in §3.

2. The main results. In stating our hypothese we shall use a family of norms depending on three parameters. Put

$$\begin{aligned} w_\alpha(x) &= |x|^{\alpha-n},\ 0<\alpha<n \\ &= 1-|\log|x||, \alpha = n \\ &= 1, \alpha > n \end{aligned}$$

For a function $V(x)$ defined on $\mathbf{R}^n$ we define

$$\begin{aligned} M_{\alpha,r,t}(V) &= \left(\int\left(\int_{|x-y|<1}|V(x)|^r w_\alpha(x-y)dx\right)^{t/\gamma}dy\right)^{1/t}, \\ 1\le t<\infty &= \sup_y\left(\int_{|x-y|<1}(V(x)^r w_\alpha(x-y)dx\right)^{1/r},\ t=\infty, \\ M_{o,r,t}(V) &= ||V||_t \equiv \text{ the } L^t(\mathbf{R}^n) \text{ norm of } V. \end{aligned} \tag{2.1}$$

We let $M_{\alpha,r,t}$ be the set of those V such that $M_{\alpha,r,t}(V)<\infty$. The space $H^{s,p}$ is defined as the completion of test functions (smooth with compact supports) with respect to the norm

$$||u||_{s,p} = ||\overline{F}(1+|\xi|^2)Fu||_p, \tag{2.2}$$

where F denotes the Fourier transform, ξ its argument and $\overline{F}$ its inverse. When s is a positive integer and $1<p<\infty$, this norm is equivalent to the sum of the L^p norms of u and its derivatives up to order s. Let Ω be an arbitrary domain (bounded or unbounded) in $\mathbf{R}^n$. We shall consider a function $f(x,u)$ which is measurable in x for each u and continuous in u for almost every x. Our assumption on $f(x,u)$ will be

$$|f(x,u)| \le V_0(x) + \sum_{k=1}^N V_k(x)|u|^{b_k} \tag{2.3}$$

where $V_k(x)\in M_{\alpha_k,r_k,t_k}$, and the parameters satisfy

$$1/b_k \le q \le r_k,\ 1/q \le b_k/p + 1/t_k,\ 1\le t_k \le \infty, \tag{2.4}$$

$$0 \le \alpha_k/nr_k < sb_k/n + 1/q - b_k/p - 1/t_k \tag{2.5}$$

for some s, p, q such that $s > 0$ and $1 < p, q < \infty$. If $t_k = \infty$, we make the additional assumption

$$\int_{|x-y|<1} |V_k(x)|^{r_k} dx \to 0 \text{ as } |y| \to \infty. \tag{2.6}$$

The functions $V_k(x)$ are to vanish outside Ω. Thus (2.6) is unnecessary if Ω is bounded. We assume $V_0(x)$ is in $L^q = L^q(\mathbf{R}^n)$. Later we shall remove the requirement that N be finite.

We let $H^{s,p}(\Omega)$ denote the restrictions to Ω of functions in $H^{s,p}$. Under the norm

$$||u||_{s,p}^{\Omega} = \inf ||v||_{s,p}, v = u \text{ in } \Omega \tag{2.7}$$

it becomes a Banach space. Our first result is

THEOREM 2.1. *Let A be any continuous linear bijective map of $D(A) \subset H^{s,p}(\Omega)$ to $L^q(\Omega)$. Then for each $R > 0$ either*

$$Au = f(x, u), u \in D(A), \ ||u||_{s,p}^{\Omega} \le R \tag{2.8}$$

has a solution or there is a λ such that $0 < \lambda < 1$ and

$$Au = \lambda f(x, u), u \in D(A), \ ||u||_{s,p}^{\Omega} = R \tag{2.9}$$

has a solution.

In order to allow N to be infinite in (2.3) we shall need the following result proved in [**11**].

THEOREM 2.2. *If $s, b > 0, 1 < p < \infty, 1 \le t \le \infty$,*

$$1/b \le q \le r < \infty, \ 1/q \le b/p + 1/t \tag{2.10}$$

$$0 \le \alpha/nr < sb/n + 1/q - b/p - 1/t \tag{2.11}$$

then there is a constant $C(n, s, p, q, b, \alpha, r, t) < \infty$ such that

$$\Big(\int |V(x)|^q |u(x)|^{qb} dx\Big)^{1/q} \le C(n, s, p, q, b, \alpha, r, t) M_{\alpha,r,t}(V) ||u||_{s,p}^b. \tag{2.12}$$

If $t < \infty$, then multiplication by $|V(x)|^{1/b}$ is compact operator from $H^{s,p}$ to L^{qb}. If $t = \infty$, the same will be true if we assume in addition that

$$\int_{|x-y|<1} |V(x)|^r dx \to 0 \text{ as } |y| \to \infty. \tag{2.13}$$

If we make use of this theorem, we can replace (2.3) with

$$|f(x,u)| \leq V_0(x) + \sum_{k=1}^{\infty} V_k(x)|u|^{b_k} \tag{2.14}$$

provided (2.4)-(2.6) hold and there is an $R > 0$ such that

$$W(R) = \sum_{k=1}^{\infty} C_k M_{\alpha_k, r_k, t_k}(V_k) R^{b_k} < \infty, \tag{2.15}$$

where

$$C_k = C(n, s, p, q, b_k, \alpha_k, r_k, t_k,). \tag{2.16}$$

We have

THEOREM 2.3. *If* (2.3) *is replaced by* (2.14), *then the conclusions of Theorem* 2.1 *hold for any* $R > 0$ *satisfying* (2.15).

THEOREM 2.4. *If*

$$||u||_{s,p}^{\Omega} \leq C_0 ||Au||_q^{\Omega}, \qquad u \in D(A), \tag{2.17}$$

and there is an $R < \infty$ *such that*

$$C_0[||V||_q + W(R)] \leq R \tag{2.18}$$

then (2.8) *has a solution.*

THEOREM 2.5. *For every* $\lambda > 0$ *sufficiently small there exists an* $R \geq 0$ *such that* (2.9) *has a solution.*

THEOREM 2.6. *Assume, in addition, that there is an* $R > 0$ *such that*

$$\inf ||f(\cdot, u)||_q > 0 \text{ for } ||u||_{s,p} = R. \tag{2.19}$$

Then there is a $\lambda > 0$ *such that* (2.9) *has a solution*

Next we turn our attention to an elliptic boundary value problem. Let

$$A(x,D) = \sum_{|\mu| \leq 2m} a_\mu(x) D^\mu$$

be an elliptic partial differential operator of order $2m$ in Ω, where $\mu = (\mu_1, \mu_2, \ldots, \mu_n)$ is a multi-index of length $|\mu| = \mu_1 + \cdots + \mu_n$ and

$$D^\mu = D_1^\mu 1 \ldots\ldots D_n^{\mu_n}, \qquad D_j = \frac{-i\partial}{\partial x_j}.$$

We assume that $A(x,D)$ is uniformly elliptic, i.e., that

$$|\sum_{|\mu|=2m} a_\mu(x) \xi^\mu| \geq C_0 |\xi|^{2m}, \xi \in \mathbf{R}^n$$

for some $C_0 > 0$. We assume also that there is a system of m boundary operators of the form

$$B_j(x,D) = \sum_{|\mu| \leq m_j} b_{j\mu}(x) D^\mu, \qquad 1 \leq j \leq m$$

which cover $A(x,D)$ (cf [**10**]). Let $1 < p < \infty$, and let $D(A)$ denote the set of those u in $H^{2m,p}$ such that

$$B_j(x,D)u = 0 \text{ on } \partial\Omega, \qquad 1 \leq j \leq m.$$

We let A designate the restriction of $A(x,D)$ to $D(A)$. We assume that A is bijective from $D(A)$ to $L^p(\Omega)$ (for sufficient conditions cf. [2,3,6,8,15-17]). We have

THEOREM 2.7. *Assume that* (2.4)-(2.6), (2.14), (2.15) *hold with* $s = 2m$ *and* $q = p$. *Then for any* $R > 0$ *satisfying* (2.15) *either*

$$Au = f(x,u), \qquad ||u||_{2m,p} \le R \tag{2.20}$$

has a solution in $D(A)$ *or there is a* λ *such that* $0 < \lambda < 1$ *and*

$$Au = \lambda f(x,u), \qquad ||u||_{2m,p} = R \tag{2.21}$$

has a solution in $D(A)$. *For any* λ *sufficiently small there is an* $R \ge 0$ *such that* (2.21) *has a solution. If* (2.19) *holds, then for each* $R > 0$ *satisfying* (2.15) *there is a* $\lambda > 0$ *such that* (2.21) *has a solution.*

We note that any constant coefficient elliptic operator of order $2m$

$$A(D) = \sum_{|\mu| \le 2m} a_\mu D^\mu$$

such that

$$A(\xi) = \sum_{|\mu| \le 2m} a_\mu \xi^\mu \ne 0, \qquad \xi \in \mathbf{R}^n \tag{2.22}$$

will satisfy the hypotheses of Theorem 2.7 for the Dirichlet boundary conditions

$$B_j(x,D) = \frac{\partial^{j-1}}{\partial n^{j-1}}, \qquad 1 \le j \le m$$

where $\frac{\partial}{\partial n}$ denotes the normal derivative to $\partial\Omega$, and $\partial\Omega$ is sufficiently smooth. To see this we recall the estimates of Agmon-Douglis-Nirenberg [**1**]

$$||u||_{2m,p} \le C(||Au||_p + ||u||_p), \qquad u \in D(A) \tag{2.23}$$

holding in general situations. Moreover,in the present case

$$||u||_{m,p} \le C||Au||_{m,p}, \qquad u \in H_0^{m,p}(\Omega) \tag{2.24}$$

where $H_0^{m,p}(\Omega)$ denotes the closure in $H^{m,p}(\Omega)$ of $C_0^\infty(\Omega)$ (cf. [9 p. 55]). If $\partial\Omega$ is sufficiently regular, $D(A) \subset H_0^{m,p}(\Omega)$. Thus

$$||u||_p \le ||u||_{m,p} \le C||Au||_{m,p} \le C||Au||_p, \qquad u \in D(A).$$

This combined with (2.23) gives

$$||u||_{2m,p} \leq C||Au||_p, \qquad u \in D(A).$$

Next we note that we can reduce the assumptions on $f(x, u)$ when $n < sp$. In this case the Sobolev inequality tells us that there is a constant C_1 such that

$$||u||_\infty \leq C_1||u||_{s,p}. \tag{2.26}$$

We have

THEOREM 2.8. *Assume that* $n < sp$ *and that*

$$|f(x, u)| \leq V(x) \exp\{C_2|u|\} \tag{2.27}$$

where $V(x) \in L^q(\Omega)$. *Then all of the conclusions of Theorems* 2.1, 2.4-2.6 *hold if we replace* (2.18) *by*

$$C_0||V||_q \exp\{C_1C_2R\} \leq R. \tag{2.28}$$

Note that (2.28) will hold for some R if

$$eC_0C_1C_2||V||_q < 1. \tag{2.29}$$

A variation of Theorem 2.4 can be obtained as follows.

THEOREM 2.9. *Suppose there exists a function* $u_0(x)$ *in* $D(A)$ *such that*

$$|f(x, u) - f_0(x)| \leq \sum_{k=1}^{\infty} V_k(x)|u - u_0(x)|^{b_k} \tag{2.30}$$

where $f_0(x) = Au_0$ *and* (2.4)-(2.6), (2.15) *hold. If* $C_0W(R) \le R$ *then* (2.8) *has a solution.*

As a special case of a boundary value problem (1.1,2) satisfying our hypotheses, we can mention the Dirichlet problem for a second order elliptic operator of the form

$$A(x,D) = \sum_{i,j=1}^{n} a_{i,j}(x)D_iD_j + \sum_{i=1}^{n} b_i(x)D_i + c(x) \tag{2.31}$$

on a bounded domain Ω with smooth boundary. Assume that the coefficients of (2.31) are continuous in $\overline{\Omega}$ and that $c(x) \ge 0$. If $n \le p$, then the operator (2.31) is a bijective map of $D(A) = H^{2,p}(\Omega) \cap H^{1,p}(\Omega)$ onto $L^p(\Omega)$ (cf. [**15**]). Moreover, the operator A^{-1} is positive, i.e., $Au \ge 0$ implies $u \ge 0$. For such cases we can improve Theorem 2.6 in the following way.

THEOREM 2.10. *Assume that* (2.4)-(2.6), (2.14), (2.15) *hold, and let* A *be a positive continuous linear bijective map of* $D(A) \subset H^{s,p}(\Omega)$ *to* $L^q(\Omega)$. *Assume also that there is a nondecreasing function* $\alpha(t)$ *defined for* $t \ge 0$ *such that* $t/\alpha(t)$ *is bounded on any bounded interval and such that*

$$\alpha(t) \le f(x,t), \qquad t \ge 0. \tag{2.32}$$

If A *has a nonnegative bounded eigenfunction, then there is a* $\lambda > 0$ *such that* (2.9) *has a solution* $u \ge 0$.

COROLLARY 2.11. *Let* Ω *be a smooth bounded domain and let* A *be the operator* (2.31) *acting on* $D(A) = H^{2,p}(\Omega) \cap H_0^{1,p}(\Omega)$ *under the assumptions described above. Let* $\alpha(t)$ *be as described, and assume that* (2.4)-(2.6), (2.14), (2.15), (2.32) *hold with* $q = p$ *and* $s = 2$. *Then there is a* $\lambda > 0$ *such that* (2.9) *has a solution* u *positive in* Ω. *If* $n < 2p$, *we can replace* (2.14) *with* (2.27).

3. Compactness criteria.

In this section we show that certain operators are compact.

LEMMA 3.1. *Let* $B_N(x,u)$ *denote the right hand side of* (2.3). *If* (2.4)-(2.6) *hold, it is a compact and continuous operator from*

$H^{s,p}(\Omega) to L^q(\Omega)$.

PROOF. Suppose

$$||u_j||_{s,p}^{\Omega} \le R. \tag{3.1}$$

Since $H^{s,p}$ is reflexive and continuously embedded in L^p, there is a subsequence (also denoted by$\{u_j\}$) which converges weakly to some $u \in H^{s,p}$ and such that

$$u_j \to u \text{ a.e.} \tag{3.2}$$

By Theorem 2.2, each $V_k^{\frac{1}{b_k}} u_j$ converges to $V_k^{\frac{1}{b_k}} u$ in L^{qb_k}. In particular, we have

$$||V_k|u_j|^{b_k}||_q \to ||V_k|u|^{b_k}||_q.$$

Since

$$|V_k(x)(|u_k|^{b_k} - |u|^{b_k})| \le V_k(x)(|u_j|^{b_k} + |u|^{b_k})$$

and the left hand side approaches 0 as $j \to \infty$, we have

$$\sum_1^N ||V_k(|u_j|^{b_k} - |u|^{b_k})||_q \to 0 \tag{3.3}$$

which implies

$$||B_N(\cdot, u_j) - B_N(\cdot, u)||_q \to 0. \tag{3.4}$$

LEMMA 3.2. *Let $B(x,u)$ denote the right hand side of (2.14). If (2.15) holds as well, then $B(x,u)$ is a compact and continuous operator from the set*

$$||u||_{s,p}^{\Omega} \le R \tag{3.5}$$

to $L^q(\Omega)$.

PROOF. Let $\varepsilon > 0$ be given, and take N so large that

$$\sum_{k=N}^{\infty} C_k M_{\alpha_k, r_k, t_k}(V_k) R^{b_k} < \varepsilon.$$

Put $B^N(x,u) = B(x,u) - B_N(x,u)$. Then by (2.12)

$$||B^N(\cdot,u)||_q \leq \sum_N^{\infty} ||V_k|u|^{b_k}||_q \leq \sum_N^{\infty} C_k M_{\alpha_k, r_k, t_k}(V_k) R^{b_k} < \epsilon$$

whenever u satisfies (3.5). Thus if (3.1) and (3.2) hold, then

$$||B(\cdot,u_k) - B(\cdot,u)||_q \leq ||B_N(\cdot,u_j) - B_N(\cdot,u)||_q + 2\varepsilon.$$

Hence $B(x,u_j)$ converges to $B(x,u)$ in L^q.

THEOREM 3.3. *Under hypotheses* (2.4)-(2.6), (2.14), (2.15), $f(x,u)$ *is a compact and continuous map from the set* (3.5) *to* $L^q(\Omega)$.

PROOF. Suppose (3.1), (3.2), hold. Then

$$|f(x,u_j) - f(x,u)| \leq B(x,u_j) + B(x,u)$$

The right hand side converges to $2B(x,u)$ in L^q by Lemma 3.2. The left hand side converges to 0 a.e. Hence $f(x,u_j)$ converges to $f(x,u)$ in L^q.

In proving Theorems 2.1 and 2.3, we shall make use of a simple consequence of the Schauder fixed point theorem (cf. Schaefer [**14**]).

THEOREM 3.4. *Let X be a normed vector space and let S be a closed bounded convex subset of X containing* 0 *as an interior point. Let T be a continuous compact map of S into X. Then either*

(a) *There is a $u \in S$ such that $u = Tu$ or*

(b) *There are $u \in \partial S$ and real λ such that $0 < \lambda < 1$ and $u = \lambda Tu$.*

PROOF. For each $u \in X$, let $g(u) = \inf\{c > 0 | \frac{u}{c} \in S\}$. Clearly $g(u) \leq 1$ for $u \in S, g(u) > 1$ for $u \notin S$ and $u/g(u) \in \partial S$. Define the mapping

$$rw = \begin{cases} w & \text{if } w \in S \\ w/g(w) & \text{if } w \notin S \end{cases}$$

The mapping rT is continuous and compact from S to S. Hence we may apply the Schauder fixed point theorem to conclude that there is a $u \in S$ such that $u = rTu$. If $Tu \in S$, then $rTu = Tu$ and (a) is true. If Tu is not in S, then $r(Tu) = \lambda Tu \in \partial S$, where $\lambda = 1/g(Tu) < 1$, and (b) holds.

Now we can give the Proof of Theorems 2.1 and 2.3. Let S be the set (3.5), and put $Tu = A^{-1}f(x,u)$. By Theorem 3.3 and the hypotheses on A, T is a compact continuous map from S to $H^{s,p}$. The results now follow from Theorem 3.4.

PROOF OF THEOREM 2.4. By Theorem 2.2

(3.6)

$$||f(\cdot,u)||_q \leq ||V_0||_q + \sum_1^\infty ||V_k|u|^{b_k}||_q \leq ||V_0||_q + \sum_1^\infty C_k M_{\alpha_k,r_k,t_k}(V_k)||u||_{s,p}^{b_k}.$$

If u satisfies (3.5), then by (2.15) and (2.17)

$$||A^{-1}f(\cdot,u)||_{s,p} \leq C_0(||V_0||q + W(R)) \tag{3.7}$$

By (2.18), $Tu = A^{-1}f(x,u)$ also satisfies (3.5). Since T is a compact operator on the set of those $u \in D(A)$ satisfying (3.5) (Theorem 3.3), we can apply the Schauder fixed point theorem to obtain the desired conclusion.

PROOF OF THEOREM 2.5. If u satisfies (3.5), then by (3.15) and (3.7) there exist $R > 0, \lambda > 0$ such that

$$\lambda ||A^{-1}f(\cdot,u)||_{s,p} \leq R.$$

If we apply the Schauder fixed point theorem to $\lambda T = \lambda A^{-1}f(x,u)$, we see that there is a u satisfying (3.5) such that $u = \lambda Tu$. Note that we have not excluded $u = 0$.

PROOF OF THEOREM 2.6. By (2.19)

$$\inf||Tu||_{s,p} > 0, \qquad ||u||_{s,p} = R$$

where $Tu = A^{-1}f(x,u)$ is a compact operator on this set. We can now apply a theorem of Krasnoseľskiĭ [18, p. 161] to obtain the desired conclusion.

Theorem 2.7 is n immediate consequence of Theorems 2.3-2.6.

PROOF OF THEOREM 2.8. In this case we have

$$B(x,u) = V(x)e^{C_2|u|} = V(x)\sum_{k=1}^{\infty} C_2^k|u|^k/k!$$

Moreover, by (2.26)

$$||V|u|^k||_q \leq ||V||_q C_2^k ||u||_{s,p}^k$$

and consequently

$$||B(\cdot,u)||_q \leq ||V||_q \sum_{k-1}^{\infty} C_1^k C_2^k ||u||_{s,p}^k/k! = ||V||_q \exp\{C_1C_2||u||_{s,p}\}.$$

This expression is finite for all $u \in H^{s,p}$. All of the proofs go through as before.

PROOF OF THEOREM 2.9. We follow the proof of Theorem 2.4. By Theorem 2.2

$$(3.8) \qquad ||f(\cdot,u) - f_0||q \leq \sum_{1}^{\infty} ||V_k|u - u_0|^{b_k}||q \leq W(||u - u_0||_{s,p}^{\Omega}).$$

If $u \in D(A) satisfies$

$$(3.9) \qquad ||u - u_0||_{s,p}^{\Omega} \leq R$$

then by (2.15) and (2.17)

$$||A^{-1}f(\cdot,u) - u_0||_{s,p} \leq C_0W(R) \leq R.$$

Thus the mapping $Tu = A^{-1}f(x,u)$ maps the set (3.9) into itself. Again the conclusion follows from the Schauder fixed point theorem.

In proving Theorem 2.10 we make use of the following theorem due to Kransnoselśkiĭ [18, p. 178].

THEOREM 3.4. *Let B, N be operators defined on a cone K of a Banach space X such that*

$$0 \le Bu \le Nu, \qquad u \ge 0, \tag{3.10}$$

N is compact on K and

$$0 \le Bu \le Bv \text{ when } \qquad 0 \le u \le v. \tag{3.11}$$

Assume that there is an element $u_0 \ge 0$ such that $u_0 \ne 0$ and

$$\gamma = \inf\{\tau | tu_0 \le v, ||v|| \le R \text{ implies } t \le \tau\} \tag{3.12}$$

is finite. Assume finally that there is a constant $\alpha > 0$ such that

$$\alpha t u_0 \le B(tu_0), \qquad 0 \le t \le \gamma. \tag{3.13}$$

Then there exist $\lambda > 0, u \ge 0$ such that $||u|| = R$ and $Nu = \lambda u$.

PROOF. For $\delta > 0$ put $N_\delta u = Nu + \delta u_0$. Then N_δ is compact on K and

$$N_\delta u \ge Bu + \delta u_0 \ge \delta u_0.$$

Thus

$$\inf ||N_\delta u|| > 0, \qquad u \ge 0, ||u|| = R.$$

By the theorem of Krasnoselśkiĭ used in the proof of Theorem 2.6 ([**18**, p. 161]), there is a $\lambda_\delta > 0$ and a $u_\delta \ge 0$ such that

$$N_\delta u_\delta = \lambda_\delta u_\delta, \qquad ||u_\delta|| = R \tag{3.14}$$

Thus

$$Bu_\delta \le \lambda_\delta u_\delta, \qquad \delta u_0 \le \lambda_\delta u_\delta \tag{3.15}$$

This implies the existence of a number t_δ such that

$$0 < t_\delta \leq \gamma, \qquad t_\delta u_0 \leq u_\delta \tag{3.16}$$

and

$$t \leq t_\delta \text{ when } t u_0 \leq u_\delta. \tag{3.17}$$

Hence by (3.11), (3.13), (3.15) and (3.16)

$$\alpha t_\delta u_0 \leq B(t_\delta u_0) \leq B(u_\delta) \leq \lambda_\delta u_\delta.$$

In view of (3.17), this implies $\alpha \leq \lambda_\delta$. By the compactness of N_δ and (3.14), there is a sequence of $\{\delta_n\}$ converging to 0 such that $N_\delta u_\delta \to y$ in X. Thus $\lambda_\delta = ||N_\delta u_\delta||/R$ also converges to some number $\lambda \geq \alpha$. Hence $u_\delta = N_\delta u_\delta/\lambda_\delta \to y/\lambda = u$. Then $u \geq 0, ||u|| = R$ and $Nu = y = \lambda u$.

PROOF OF THEOREM 2.10. Put $Nu = A^{-1}f(x,u), Bu = A^{-1}\alpha(u)$. We show that the hypotheses of Theorem 3.5 are satisfied. Clearly (3.10), (3.11) hold and N is compact. Let u be a positive eigenfunction of A^{-1} with positive eigenvalue μ. If $tu_0 \leq v$, then $t||u_0||_p \leq ||v||_p \leq ||v||_{s,p}$. Thus γ given by (3.12) is finite. It remains to verify (3.13). By hypotheses, there is a constant $\beta > 0$ such that

$$\beta \leq \alpha(t)/t, \qquad 0 \leq t \leq \gamma M$$

where $M = \max u_0$. Thus

$$\beta\mu t u_0 = \beta t A^{-1} u_0 \leq A^{-1}\alpha(tu_0) = B(tu_0)$$

and (3.13) is verified.

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF CALIFORNIA, IRVINE, CA 92717