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BOUNDED AND ALMOST PERIODIC SOLUTIONS OF SEMI-LINEAR PARABOLIC EQUATIONS

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0. Introduction. Let Ω be a bounded domain in $\mathbf{R}^N, N \geq 1$, of class $C^{2+\alpha}, 0 < \alpha < 1$, and let $F: \mathbf{R} \times \Omega \times \mathbf{R}^2 \rightarrow \mathbf{R}$ be a continuous function. The purpose of this paper is to discuss the existence of bounded and almost periodic (in time) solutions of nonlinear parabolic boundary problems with a possible time delay. In particular semi-linear problems of the form (0.1), (.02) or (0.1), (0.3) will be studied.

$$u_t - \Delta u = F(t, x, u(t,x), u(t-r, x)) \text{ in } \mathbf{R} \times \Omega \tag{0.1}$$

$$u(t,x) = 0 \text{ on } \mathbf{R} \times \partial\Omega \tag{0.2}$$

$$\frac{\partial u}{\partial \nu}(t,x) = 0 \text{ on } \mathbf{R} \times \partial\Omega. \tag{0.3}$$

Here $u = u(t,x)$ is a real valued function on $\mathbf{R} \times \overline{\Omega}, \Delta := \sum_{i=1}^N \frac{\partial^2}{\partial x_i^2}$ is the N-dimensional Laplacian, $x = (x_1, \ldots, x_N), r \geq 0$, and $\partial/\partial\nu$ indicates the outward normal derivative on $\partial\Omega$, the boundary of Ω.

Suppose $G: \mathbf{R} \times \mathbf{R}^n \rightarrow \mathbf{R}^n, n \geq 1, (t,x) \mapsto G(t,x)$, is continuous and uniformly almost periodic in t. It is a well known result of Amerio that if there is a compact set $K \subset \mathbf{R}^n$ such that for every $G^* \in \operatorname{Hull}(G)$ the ordinary differential equation

$$\dot{u} = G^*(t,u)$$

has a unique solution on $\mathbf{R}$ with range in K then each such solution is an almost periodic function; see [**4**] for a proof and further references. Amerio's result has been generalized in several ways for both ordinary differential equations and for abstract evolution equations. In the latter case an extension has been made by Dafermos [**3**] by using the concept of an almost periodic process, a two parameter family of maps related to the usual evolution operator; his results depend upon the existence of such a process on a complete metric space. Haraux [**6**] has also

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extended Amerio's result to classes of evolution equations in Hilbert space governed by monotone operators; he studies the equation directly.

Here we limit our attention to semi-linear parabolic equations of the form (0.1) and obtain another extension of Amerio's result. We study the equation directly and do not need the existence of a process associated with (0.1), nor do we need monotonicity in our main result or a restriction to Hilbert space. We obtain the existence of classical solutions, however, and must make some smoothness assumptions on $\partial\Omega$ and on F. Our approach is useful, e.g., if there is blow up in finite time of some solutions, if time delays occur, if the nonlinearities preclude a Hilbert space approach or if monotonicity is missing. We are thus able to obtain the existence of classical almost periodic solutions by, e.g., the method of upper and lower solutions to obtain a bounded solution; a uniqueness argument then implies almost periodicity. To illustrate our main result we apply it to obtain an existence result of non-resonance type for (0.1), (0.2) or (0.1), (0.3). This will be done in §2.

1. An Extension of Amerio's result. Let $\Omega \subset \mathbf{R}^N, N \geq 1$, be a bounded domain with boundary of class $C^{2+\alpha}$ for some $0 < \alpha < 1$. Let $F : \mathbf{R} \times \overline{\Omega} \times \mathbf{R}^2 \to \mathbf{R}, (t, x, u, y) \mapsto F(t, x, u, y), t, u, y \in \mathbf{R}, x \in \overline{\Omega}$, be Hölder continuous in t and x, uniformly for u, y in bounded sets, so that $F(\cdot, \cdot, u, y)$ is of class $C^{\alpha/2,\alpha}([a, b] \times \overline{\Omega})$ for all real $a < b$. We assume F is continuously differentiable in u and y. Here and in the following, let $\mathbf{R} = (-\infty, \infty)$ and $\mathbf{R}^+ = [0, \infty)$.

We assume that F is bounded in (t, x). Explicitly, we assume

(H1) *For each $m \geq 0$ there is aconstant $C(m) > 0$ such that $| F(t, x, u, y) | \leq C(m)$ for all $(t, x, u, y) \in Dom(F)$ with $|u| \leq m$ and $|y| \leq m$.*

We wish under certain condition to be able to conclude that a bounded solution of (0.1), (0.2) or (0.3) must be almost periodic if it is unique; it is thus convenient that bounded solutions are equicontinuous on $\mathbf{R} \times \overline{\Omega}$, which is perhaps already known, but we know of no reference; we thus establish it in the following lemmas.

Let $f \in C(\mathbf{R} \times \overline{\Omega}, \mathbf{R}) \cap C^{\alpha/2,\alpha}([a, b] \times \overline{\Omega}, \mathbf{R})$, for all $a < b$ (where $0 < \alpha < 1$) be bounded on $\mathbf{R} \times \overline{\Omega}$, i.e., $\sup\{|f(t, x)| : t \in \mathbf{R}, x \in \overline{\Omega}\} < \infty$, and consider the linear problems

$$u_t - \Delta u = f(t,x) \text{ in } \mathbf{R} \times \Omega \tag{1.1}$$
$$u = 0 \text{ on } \mathbf{R} \times \partial\Omega \tag{1.2}$$

and

$$u_t - \Delta u + cu = f(t,x) \text{ in } \mathbf{R} \times \Omega \tag{1.3}$$
$$\frac{\partial u}{\partial \nu} = 0 \text{ on } \mathbf{R} \times \partial\Omega \tag{1.4}$$

where $c > 0$ is a real number.

LEMMA 1.1. *Let f be as above; then if $u \in C^{1+\alpha/2,2+\alpha}$ is the unique bounded solution of either of the above linear problems then $u = u(t,x)$ is uniformly continuous in t on $\overline{\Omega}$; i.e., for each $\varepsilon > 0$ there is a $\delta > 0$ such that $|t-s| < \delta$ implies $|u(t,x) - u(s,x)| < \varepsilon$ for all $x \in \overline{\Omega}$.*

PROOF. We consider only (1.1), (1.2); the Neumann case is similar. Let $p > N/2$ and $D(A) = W^{2,p}(\Omega) \cap W_0^{1,p}(\overline{\Omega})$. Define $A : D(A) \to L^p(\Omega)$ by $Au = \Delta u$; then A is a sectorial operator and generates a semigroup $T(t), t \geq 0$ on $L^p(\Omega)$; (see [**5**]). For $t > 0$ this semigroup maps $L^p(\Omega)$ into $W^{2,p}(\Omega) \subset C^{\beta}(\overline{\Omega})$ for $0 \leq \beta < 2 - N/p$. Thus by choosing $p > N/2$ we insure that $T(t)$for $t > 0$ maps $C(\overline{\Omega})$ into itself compactly. Also (see [**5**], p. 31) there are constants $M > 0, 0 < \beta < 1$, and $\gamma > 0$ such that for $u \in L^p(\Omega)$ and $t > 0$ we have

$$|T(t)u|_{C(\overline{\Omega})} \leq M||u||_{L^p} t^{-\beta} e^{-\gamma t}. \tag{1.5}$$

Let $\tilde{u}(t) = u(t,\cdot) \in C(\overline{\Omega})$ and use similar notation for f. We have the representation

$$\tilde{u}(t) = \int_{-\infty}^{t} T(t-s)\tilde{f}(s)ds. \tag{1.6}$$

Let $\eta > 0$ be fixed and let $t, \tau \in \mathbf{R}$ with $t < \tau$ and pick $0 < \varepsilon < \tau - t$; we have using (1.6)

$$\begin{aligned}\tilde{u}(\tau) - \tilde{u}(t) &= \int_t^\tau T(\tau - s)\tilde{f}(s)ds \\ &+ \int_{-\infty}^{\tau-\varepsilon} [T(\tau - s) - T(t - s)]\tilde{f}(s)ds \\ &+ \int_{t-\varepsilon}^{t} [T(\tau - s) - T(t - s)]\tilde{f}(s)ds \\ &\equiv I_1 + I_2 + I_3.\end{aligned}$$

Let $|\cdot|_0$ denote the norm in $C(\overline{\Omega})$ and $|\cdot|_p$ that in $L^p(\Omega)$. Now the boundedness of $f(t,x)$ implies that $||\tilde{f}(s)||_p \leq C_1$ for some $C_1 > 0$ and all $s \in \mathbf{R}$. Using (1.5) we have

$$\begin{aligned}|I_1|_0 &\leq \int_t^\tau M(\tau - s)^{-\beta} e^{-\gamma(\tau - s)} |\tilde{f}(s)|_p ds \\ &\leq MC_1 \int_0^{\tau - t} s^{-\beta} e^{-\gamma s} ds.\end{aligned}$$

Hence there exists a $\delta_1 > 0$ such that

$$|I_1|_0 < \eta \text{ for } |t - \tau| < \delta_1. \tag{1.7}$$

We may estimate I_3 similarly to get

$$|I_3|_0 \leq \int_0^\varepsilon 2MC_1 s^{-\beta} e^{-\gamma s} ds;$$

thus by choosing $\varepsilon > 0$ sufficiently small we get

$$|I_3|_0 < \eta. \tag{1.8}$$

As for I_2, we use the fact that $T(t)$ is compact for $t > 0$ on $L^p(\Omega)$ which implies that $T(t)y$ is uniformly continuous in t for y in bounded $L^p(\Omega)$ sets, as long as t is bounded away from 0, say $t \geq \varepsilon$. Thus there is a number $\delta_2 > 0$ such that

$$|[T(\tau - t + \varepsilon) - T(\varepsilon)]f(s)|_p < \eta \tag{1.9}$$

whenever $0 \le \tau - t < \delta_2$, and for all $s \in \mathbf{R}$. Writing $T(\tau - s) - T(t-s) = T(t-s-\varepsilon)[T(\tau - t+\varepsilon) - T(\varepsilon)]$ we have using (1.5) and (1.9)
(1.10)

$$|I_2|_0 \le \int_{-\infty}^{t-\varepsilon} M(t-s-\varepsilon)^{-\beta} \exp(-\gamma(t-s-\varepsilon).\ |[T(\tau - s+\varepsilon) - T(\varepsilon)]f(s)|_p ds$$
$$\le \eta \int_0^\infty M s^{-\beta} e^{-\gamma s} ds = \eta C_3.$$

Combining (1.7), (1.8), and (1.10) we get

$$|I|_0 \le (2 + C_3)\eta,$$

whenever $0 < \tau - t < \min(\delta_1, \delta_2)$. Since C_3 is fixed and η is arbitrary this proves the lemma.

REMARK. If we already had bounds on $\Delta u(t,x)$ on $\mathbf{R} \times \overline{\Omega}$ in (1.1), then Lemma 1.1 would follow immediately from the equation (1.1).

LEMMA 1.2. *Under the same hypotheses as Lemma* 1.1 *the unique bounded solution* $u = u(t,x)$ *of* (1.1), (1.2) (*or* (1.3), (1.4)) *is continuous in* x *uniformly in* $t \in \mathbf{R}$, *i.e., for each* $\varepsilon > 0$ *there exists a* $\delta > 0$ *such that* $|x - y| < \delta$ *and* $t \in \mathbf{R}$ *imply* $|u(t,x) - u(t,y)| < \varepsilon$.

PROOF. Using the same notation as in Lemma 1.1, fix $p > 2N$ and let $B = -A$. Then for $0 < \alpha \le 1$ the fractional powers B^α are defined on X^α, with $X^\alpha \subset C^\mu(\overline{\Omega})$ when $0 \le \mu < 2\alpha - N/p$; choose $0 < \alpha < 1$ so close to one that $X^\alpha \subset C^1(\overline{\Omega})$. Using the fact ([**5**], p. 26) that $||B^\alpha T(t)||_p \le C_\alpha t^{-\alpha} e^{-\delta t}$ for some constants $C_\alpha > 0, \delta > 0$, and all $t > 0$, we obtain, for each $t \in \mathbf{R}$

$$||\tilde{u}(t)||_{C^1(\overline{\Omega})} \le ||B^\alpha \tilde{u}(t)||_p \le \int_{-\infty}^{t} ||B^\alpha T(t-s)\tilde{f}(s)||_p ds$$
$$\le K_1 \int_0^\infty s^{-\alpha} e^{-\delta s} ds \equiv K_2.$$

Hence $|\nabla u(t,x)| \le K_2$ for all $(t,x) \varepsilon\ \mathbf{R} \times \overline{\Omega}$.

Suppose that the lemma is false. Then there exist sequences (t_n, x_n) and $(t_n, y_n)(n = 1, 2, \dots)$ and a number $\varepsilon_0 > 0$ such that $|x_n - y_n| \leq n^{-1}$ and $|u(t_n, x_n) - u(t_n, y_n)| \geq \varepsilon_0$. We may assume that the sequences (x_n) and (y_n) converge to some $z \in \overline{\Omega}$. If $z \in \Omega$ then there is a $\delta > 0$ such that the neighborhood $N = \{x : |x - z| < \delta\}$ is contained in Ω. Moreover there is an integer m such that x_n, $y_n \in N$ for all $n \geq m$. Thus the line segment joining x_n and y_n lies in Ω and by the mean value theorem there is for each n a point c_n on that line segment such that

$$\begin{aligned}\varepsilon_0 &\leq |u(t_n, x_n) - u(t_n, y_n)| = \nabla u(t_n, c_n) \cdot (x_n - y_n)| \\ &\leq |\nabla u(t_n, c_n)| \cdot |x_n - y_n| \leq |\nabla u(t_n, c_n)| n^{-1}.\end{aligned}$$

Thus $|\nabla u(t_n, c_n)| \to \infty$ as $n \to \infty$, contrary to $|\nabla u(t, x)| \leq K_2$ for all $(t, x) \in \mathbf{R} \times \overline{\Omega}$. Thus if there are such sequences (t_n, x_n) and (t_n, y_n) we must have $z \in \partial\Omega$. This will also lead to a contradiction, as follows. If $z \in \partial\Omega$, there is a number $\delta_0 > 0$ such that for all $0 < \delta \leq \delta_0$ if $N = N(z, \delta) = \{x : |x - z| \leq \delta\}$ then (i) $N \cap \partial\Omega$ is simply connected and (ii) $N - \partial\Omega$ has two components N_1 and N_2 with $N_1 \subset \Omega$ and $N_2 \cap \Omega = \emptyset$. Now $\partial\Omega$ is $C^{2,\alpha}$ (and hence C^1, which is all we need here). There is therefore for each $0 < \delta \leq \delta_0$ a number $\eta_0 = \eta_0(\delta), 0 < \eta_0 < \delta$, such that if $0 < \eta \leq \eta_0$ and $x \in \partial\Omega \cap N(z, \eta)$ there is a vector $v = v(x)$ with the properties: (i) $|v| \geq \delta/2$ (ii) $v = v(x)$ is perpendicular to the hyperplane at x tangent to $\partial\Omega$ (iii) $x + v(x) \in N_1$. Moreover we may choose η_0 so small that if $x, y \in \partial\Omega \cap N(z, \eta), 0 < \eta \leq \eta_0$, then the line segment joining $x + v(x)$ to $y + v(y)$ lies in $N_1 = N_1(\delta)$.

Now since $x_n, y_n \to z$ we may choose a sequence $(\delta_n), 0 < \delta_n \leq \delta_0$, and corresponding $(\eta_n), 0 < \eta_n < \delta_n$, with $\delta_n \to 0$ and $x_n, y_n \in N(z, \eta_n)$. Let $\overline{x}_n$ denote a point on $\partial\Omega \cap N(z, \eta_n)$ of minimal distance to x_n; at least one such exists; similarly define $\overline{y}_n$. If $x_n \neq \overline{x}_n$ and $y_n \neq \overline{y}_n$ the directed line segments joining $\overline{x}_n$ to x_n and $\overline{y}_n$ to y_n lie in $\overline{\Omega}$. We see that x_n and y_n may be joined by a polygonal line lying in the closure of $N_1(z, \delta_n)$ and having at most five segments: a segment from x_n to $\overline{x}_n$, thence to $\overline{x}_n + v(\overline{x}_n)$, thence to $\overline{y}_n + v(\overline{y}_n)$; thence to $\overline{y}_n$ and finally to y_n. By the mean value theorem applied to each of these segments (note, however, there maybe fewer than five segments). We find $a_n, b_n, c_n, d_n, e_n \varepsilon N(z, \delta_n)$ with (letting $u_n = \overline{x}_n + v(\overline{x}_n), v_n =$

$\overline{y}_n + v(\overline{y}_n))$:

$$\begin{aligned}\varepsilon_0 \le |u(t_n, x_n) - u(t_n, y_n)| \le \\ |\nabla u(t_n, a_n) \cdot (x_n - \overline{x}_n)| + |\nabla u(t_n, b_n) \cdot (\overline{x}_n - u_n)| \\ + |\nabla u(t_n, c_n) \cdot (u_n - v_n)| + \\ |\nabla u(t_n, d_n) \cdot (v_n - \overline{y}_n)| + |\nabla u(t_n, e_n) \cdot (\overline{y}_n - y_n)| \le 10K_2\delta_n.\end{aligned}$$

Since $\delta_n \to 0$ we have a contradiction. This proves the lemma.

COROLLARY 1.3. *If the hypotheses of Lemma* 1.1 *hold then the unique bounded solutions of* (1.1), (1.2) *and of* (1.3), (1.4) *are uniformly continuous on* $\mathbf{R} \times \overline{\Omega}$.

PROOF. Since $|u(t,x) - u(s,y)| \le |u(t,x) - u(s,x)| + |u(s,x) - u(s,y)|$ we have only to apply each of Lemmas 1.1 and 1.2.

We now recall the concept of a uniformly almost periodic function. Let $S \subset \mathbf{R}^n$ be a non-empty set. A continuous function $f : \mathbf{R} \times S \to \mathbf{R}$, $(t,x) \mapsto f(t,x)$, is said to be uniformly almost periodic (in t) on S (UAP on S) if for any $\varepsilon > 0$ and compact set $K \subseteq S$ there exists a number $L = L(\varepsilon, K)$ such that any interval of length L contains a number τ such that $|f(t+\tau, x) - f(t,x)| < \varepsilon$ for all $t \in \mathbf{R}$ and $x \in K$. For the convenience of the reader we quote three useful theorems; proofs maybe found in [**4**] and [**8**].

THEOREM A. *Let* $D \subseteq \mathbf{R}^n$ *and* $f \in C(\mathbf{R} \times D, \mathbf{R})$ *be UAP on* D. *Then for any sequence* $\alpha' = (\alpha'_n) \subseteq \mathbf{R}$ *there is a subsequence* $\alpha = (\alpha_n)$ *and a continuous function* $g(t,x)$ *on* $\mathbf{R} \times D$ *such that*

$$\lim_{n \to \infty} f(t + \alpha_n, x) = g(t,x)$$

uniformly on every $\mathbf{R} \times K, K \subseteq D$ *and compact. Moreover any such* g *is also UAP on* D.

The set of all such functions g obtainable from a UAP function f via some sequence as in Theorem A is called the hull of f, which we will write as Hull (f). Let $\alpha = (\alpha_n)$ be a real sequence and $f, g : \mathbf{R} \times D \to \mathbf{R}$ where $D \subseteq \mathbf{R}^n$. We will write $T_\alpha f = g$ to mean $f(t + \alpha_n, x) \to f(t, x)$ as $n \to \infty$, where the convergence will be pointwise unless otherwise indicated. The following is due to Bochner.

THEOREM B. *Let $f \in C(\mathbf{R}, \mathbf{R}), g \in C(\mathbf{R} \times D, \mathbf{R}), D \subseteq \mathbf{R}^n$, and let $\alpha' = (\alpha'_n), \beta' = (\beta'_n)$ be real sequences. Then*

(1) *f is almost periodic if and only if there are common subsequences $\alpha \subset \alpha', \beta \subset \beta'$ such that $T_{\alpha+\beta} f = T_\alpha T_\beta f$ (pointwise).*

(2) *g is uniformly almost periodic on D if and only if there are common subsequences $\alpha \subset \alpha', \beta \subset \beta'$ such that $T_{\alpha+\beta} g = T_\alpha T_\beta g$ uniformly on each $\mathbf{R} \times K$, K any compact subset of D.*

THEOREM C. *A function $f \in C(\mathbf{R} \times D, \mathbf{R}), D \subseteq \mathbf{R}^n$, is uniformly almost periodic on D if and only if:*

(1) *For each fixed $x \in D$ the mapping $t \mapsto f(t, x)$ is AP.*

(2) *For every compact $K \subseteq D, f(t, x)$ is continuous in $x \in K$ uniformly in t, i.e., for every $x \in K$ and $\varepsilon > 0$ there exists a $\delta > 0$ such that $y \in K$ and $|x - y| < \delta$ implies $|f(t, x) - f(t, y)| < \varepsilon$ for all $t \in \mathbf{R}$.*

We now need the following:

LEMMA 1.3. *Let $u \in C^{1,2}(\mathbf{R} \times \overline{\Omega})$ be a bounded solution of* (1.1), (1.2) *or* (1.3), (1.4) *where $f \in C^{\alpha/2, \alpha}(\mathbf{R} \times \Omega, \mathbf{R})$ is a bounded function. If for each $x \in \overline{\Omega}$ and each pair α', β' of sequences there are common subsequences $\alpha \subseteq \alpha', \beta \subseteq \beta'$ (which may depend on x) such that $T_{\alpha+\beta} u(\cdot, x) = T_\alpha T_\beta u(\cdot, x)$ (pointwise), then $u = u(t, x)$ is UAP in t on $\overline{\Omega}$.*

PROOF. By Lemma 1.2, $u(t, x)$ is continuous in x, uniformly in t. By Theorem B, $u(t, x)$ is AP in t for each fixed $x \in \overline{\Omega}$. Theorem C now implies the conclusion.

Our extension of Amerio's result may now be given.

THEOREM 1.4. *Let* $F : \mathbf{R} \times \overline{\Omega} \times \mathbf{R}^2 \to \mathbf{R}, (t,x,u,y) \to F(t,x,u,y)$, *satisfy* (H1) *and be uniformly almost periodic on* $\overline{\Omega} \times \mathbf{R}^2$ *and suppose:*

(1) *Each* $F^* \in \mathrm{Hull}\,(F)$ *is Hölder continuous with exponents* $\alpha/2, \alpha, \alpha$, *and* $\alpha, 0 < \alpha < 1$, *in the variables* t, x, u, y, *respectively* (α *may depend on* F^*).

(2) *There are function* c, d *on* $\overline{\Omega}$ *with* $-\infty \le c(x) \le d(x) \le \infty$ *for all* $x \in \overline{\Omega}$ *such that for each* $F^* \in \mathrm{Hull}\,(F)$ *problem* (0.1), (0.2) *(or* (0.3)*) (with* F^* *in place of* F*) has a unique bounded solution* $u^* \in C^{1,2}(\mathbf{R}\times\overline{\Omega})$ *satisfying* $c(x) \le u^*(t,x) \le d(x)$*for all* $(t,x) \in \mathbf{R}\times\overline{\Omega}$. *Then* (0.1), (0.2) *(or* (0.3)*) has a uniformly almost periodic solution* $\tilde{u} \in C^{1,2}(\mathbf{R} \times \overline{\Omega})$ *with* $c(x) \le \tilde{u}(t,x) \le d(x)$ *for all* $(t,x) \in \mathbf{R} \times \overline{\Omega}$.

REMARK. If $F(t,x,u,y) = g(x,u,y) + h(t,x)$ then the regularity condition are easy to check.

PROOF. Let $u = u(t,x)$ be the bounded solution of (0.1), (0.2) (if u satisfies (0.3) the proof is the same) with $c(x) \le u(t,x) \le d(x)$ for all $(t,x) \in \mathbf{R} \times \overline{\Omega}$. Since $F(t,x,u(t,x),u(t-r,x))$ will be locally Hölder continuous of exponent $0 < \alpha < 1$, we have that actually $u \in C^{1+\alpha/2,2+\alpha}(\mathbf{R} \times \overline{\Omega})$.

Let α, β be real sequences. By Theorem B there are common subsequences, which we call again α and β, such that $T_{\alpha+\beta}F = T_\alpha T_\beta F \in \mathrm{Hull}\,(F)$ uniformly on compact sets in $\overline{\Omega} \times \mathbf{R}^2$. Let, for $n = 1, 2, \ldots, u_n(t,x) := u(t+\beta_n, x)$. Then u_n satisfies

$$\frac{\partial u_n}{\partial t} - \Delta u_n = F(t+\beta_n, x, u_n(t,x), u_n(t-r,x)) \text{ on } \mathbf{R} \times \Omega$$
$$u_n = 0 \text{ on } \mathbf{R} \times \partial\Omega$$

and $c(x) \le u_n(t,x) \le d(x)$ on $\mathbf{R} \times \overline{\Omega}$.

Now for each $T > r/2$ there is a $C_T > 0$ such that

$$|u_n|_{\alpha,T} \le C_T$$

where the subscripts α, T denote the norm in $C^{1+\alpha/2,2+\alpha}([-T,T]\times\overline{\Omega})$. There is thus a subsequence of (u_n), converging in $C^{1+\overline{\alpha}/2,2+\overline{\alpha}}([-T,T]\times$

$\overline{\Omega}$) for some $0 < \overline{\alpha} < \alpha$ to a solution of the problem

$$\frac{\partial u}{\partial t} - \Delta u = F_{\tilde{\beta}}, \text{ on } [-T + r, T] \times \overline{\Omega}$$

where $F_{\tilde{\beta}} = T_{\tilde{\beta}} F(t, x, u, y), u = 0$ on $[-T + R, T] \times \partial\Omega$. Here $\tilde{\beta}$ denotes the corresponding subsequence of β. Letting $T \to \infty$ and using the usual diagonalization process gives us a function $z \in C^{1+\hat{\alpha}/2, 2+\hat{\alpha}}, 0 < \hat{\alpha} < \overline{\alpha} < 1$, and a subsequence β' of β such that $z = T_{\beta'}(u)$ on $\mathbf{R} \times \overline{\Omega}$. In addition, $c(x) \leq z(t, x) \leq d(x)$ on $\mathbf{R} \times \overline{\Omega}$, and

$$\begin{aligned} z_t - \Delta z &= T_{\beta'} F \text{ on } \mathbf{R} \times \Omega \\ z &= 0 \quad \text{ on } \mathbf{R} \times \partial\Omega. \end{aligned}$$

If we now let $z_n(t, x) := z(t + \alpha_n, x)$ a similar argument gives us $y = T_{\alpha'} z = T_{\alpha'} T_{\beta'} u$ which solves

$$\begin{aligned} y_t - \Delta y &= T_{\alpha'} T_{\beta'} F \quad \text{in } \mathbf{R} \times \Omega \\ y &= 0 \qquad \text{on } \mathbf{R} \times \partial\Omega. \end{aligned} \tag{1.11}$$

Moreover

$$c(x) \leq y(t, x) \leq d(x).$$

It is clear that we may again repeat the above procedure with subsequences α'', β'' of α', β', respectively, obtaining a function $v = T_{\alpha''+\beta''} u, v \in C^{1,2}(\mathbf{R} \times \overline{\Omega}), c(x) \leq v(t, x) \leq d(x)$ on $\mathbf{R} \times \overline{\Omega}$, and solving

$$\begin{aligned} v_t - \Delta &= T_{\alpha''+\beta''} F \quad \text{in } \mathbf{R} \times \Omega \\ v &= 0 \qquad \text{on } \mathbf{R} \times \partial\Omega. \end{aligned} \tag{1.12}$$

However $T_{\alpha''+\beta''} F = T_{\alpha'+\beta'} F = T_{\alpha'} T_{\beta'} F$ so that (1.11) and (1.12) are the same equation. Our uniqueness hypothesis implies $y = v$, that is, $T_{\alpha''+\beta''} u = T_{\alpha''} T_{\beta''} u$. By Lemma 1.3, $u = u(t, x)$ is uniformly almost periodic in t on $\overline{\Omega}$. This completes the proof.

Bounded solutions of almost periodic equation are not as hard to obtain as one might expect as the following shows.

THEOREM 1.5. *Let* $F : \mathbf{R} \times \overline{\Omega} \times \mathbf{R}^2 \to \mathbf{R}, (t, x, u, y) \to F(t, x, u, y)$ *be locally Hölder continuous with exponents* $\alpha/2, \alpha, \alpha, \alpha$ *in* t, x, u, y, *respectively. Suppose also that* F *is UAP in* t *on* $\overline{\Omega} \times \mathbf{R}^2$. *If there is a* $\tilde{u} \in C^{\alpha/2,\alpha}((t_0, \infty) \times \overline{\Omega}) \cap C^{1+\alpha/2,2+\alpha}((t_0+r, \infty) \times \overline{\Omega})$ *which is bounded and solves* (0.1), (0.2) *(or* (0.3)*) on* $(t_0 + r, \infty) \times \overline{\Omega}$ *then the same problem has a* $C^{1,2}$ *solution defined and bounded on* $\mathbf{R} \times \overline{\Omega}$.

PROOF. Since F is UAP on $\overline{\Omega} \times \mathbf{R}^2$ there exists a sequence $\alpha = (\alpha_n), \alpha_n \to \infty$ as $n \to \infty$, such that

$$F(t + \alpha_n, x, u, y) \to F(t, x, u, y)$$

uniformly on compact subsets of $\overline{\Omega} \times \mathbf{R}^2$. If we let $u_n(t, x) = \tilde{u}(t+\alpha_n, x)$ we see by arguments similar to those of Lemmas 1.1-1.3 and Theorem 1.4 that a subsequence of (u_n) converges in $C^{1+\eta,2+\eta} 0 < \eta < \alpha$, to a bounded function which solves (0.1), (0.2) (or (0.3)) on $\mathbf{R} \times \overline{\Omega}$.

2. An application to a non-resonance problem. It is our aim in this section to give just one application of Theorem 1.4, to an equation which satisfies a "non-resonance" condition. We thus illustrate Theorem 1.4 by obtaining more verifiable conditions for an almost periodic solution of (0.1), (0.2).

We assume $F = F(t, x, u, y)$ is uniformly almost periodic on $\overline{\Omega} \times \mathbf{R}^2$, each $F^* \in \operatorname{Hull}(F)$ is Hölder continuous in t and x with exponents $\alpha/2$ and α, respectively, $0 < \alpha < 1$, uniformly on bounded subsets on $\mathbf{R}^2$, and continuously differentiable in u and y. We also assume Ω is a bounded domain in $\mathbf{R}^N$ with boundary of class $C^{2+\alpha}$.

Let $\lambda_1 > 0$ be the first eigenvalue for the Dirichlet problem

$$-\Delta u = \lambda u \quad \text{in} \quad \Omega \tag{2.1}$$

$$u = 0 \quad \text{on} \quad \partial\Omega. \tag{2.2}$$

We let $\theta \in C^{2+\alpha}(\overline{\Omega})$ be a positive eigenfunction for (2.1),(2.2) with $\lambda = \lambda_1$, so that $\theta(x) > 0$ for $x \in \Omega$. We recall that there exists $c_0 > 0$ such that

$$\frac{\partial \theta}{\partial \nu}(x) \leq -c_0, \quad x \in \partial\Omega, \tag{2.3}$$

where $\partial/\partial\nu$ denotes the outer normal derivative on $\partial\Omega$.

THEOREM 2.1. *Let F and Ω be as described above. In addition suppose that for each $F^* \in \mathrm{Hull}(F)$:*

(i) *There is a number $\mu_1 < \lambda_1$ such that $\partial F^*/\partial u \leq \mu_1$ for all $(t,x,u,y) \in \mathbf{R} \times \overline{\Omega} \times \mathbf{R}^2$.*

(ii) $0 \leq \partial F^*/\partial y \leq \delta_0$ *for some $\delta_0 \geq 0$ and all $(t,x,u,y) \in \mathbf{R}\times\overline{\Omega}\times\mathbf{R}^2$.*

(iii) $\delta_0 < \lambda_1 - \mu_1$.

Then (0.1), (0.2) *has a unique $C^{1.2}$ uniformly almost periodic solution on $\mathbf{R} \times \overline{\Omega}$.*

PROOF. We first show there can be no more than one bounded solution; we will afterwards sketch an existence proof.

Suppose $u, v \in C^{1,2}(\mathbf{R} \times \overline{\Omega})$ are bounded solutions of (0.1), (0.2) and let $z = u - v$. Define $S \in C^1(\mathbf{R}, \mathbf{R})$ by

$$S(t) = \frac{1}{2}\int_\Omega z^2(t,x)dx \tag{2.4}$$

and let

$$M = \sup_{t\in\mathbf{R}} S(t). \tag{2.5}$$

We will show that $M = 0$. All the following integrals are over Ω.

Now

$$S'(t) = \int zz_t = \int z(\Delta z + \tilde{F}(u) - \tilde{F}(v))$$

where by $\tilde{F}(u)$ we mean $F(t,x,u(t,x),u(t-r,x))$, etc. An integration by parts and the mean value theorem yield

$$\begin{aligned} S'(t) &= \int [-|\nabla z|^2 + F_u \cdot z^2 + F_y z(t,x)z(t-r,x)]dx \\ &\leq (\mu_1 - \lambda_1)\int z^2 dx + \delta_0 \int |z(t,x)z(t-r,x)|dx. \end{aligned}$$

Applying the Cauchy-Schwartz inequality to the last integral and using (2.4),(2.5) we obtain

$$S'(t) \leq 2(\mu_1 - \lambda_1)S(t) + \delta_0 M. \tag{2.6}$$

Fixing $t_0 \in \mathbf{R}$ and integrating (2.6) on $[t_0, t]$ we have

$$S(t) \leq S(t_0)e^{K(t_0-t)} + \delta_0 M(K^{-1})(1 - e^{K(t_0-t)}), \tag{2.7}$$

where $K = 2(\lambda_1 - \mu_1) > 0$. Using $|S(t_0)| \leq M$ and letting $t_0 \to -\infty$ in (2.7) we see that

$$S(t) \leq \delta K^{-1} M$$

for all $t \in R$. Since $\delta_0(\lambda_1 - \mu_1)^{-1} < 1$, this contradicts the definition of M unless $M = 0$. We conclude that there is at most one $C^{1,2}$ bounded solution of (0.1), (0.2).

We sketch the proof for existence.

Let $h(t,x) = F(t,x,0,0)$ and choose μ so that $\delta_0 + \mu_1 < \mu < \lambda_1$. Let $q = q(t,x)$ be the unique bounded solution of the linear problem.

$$\begin{aligned} q_t - \Delta q - \mu q &= h(t,x) \quad \text{in } \mathbf{R} \times \Omega \\ q &= 0 \qquad\quad \text{on } \mathbf{R} \times \partial\Omega. \end{aligned} \tag{2.8}$$

Notice q will be UAP since h is. By using arguments like those of Lemma 1.2 can show that there must be a constant $C_1 > 0$ such that

$$|\partial q/\partial \nu(t,x)| \leq C_1 \tag{2.9}$$

for all $(t,x) \in \mathbf{R} \times \partial\Omega$.

Thus there exists a number $\alpha_0 > 0$ such that for all $\alpha_1 \geq \alpha_0$

$$-\alpha_1\theta(x) + q(t,x) \leq 0 \leq \alpha_1\theta(x) + q(t,x) \tag{2.10}$$

on $\mathbf{R} \times \overline{\Omega}$.

Now let

$$\begin{aligned} \mathcal{B} = \{ & v \in C^{\alpha/2,\alpha}(\mathbf{R} \times \overline{\Omega}) : v \text{ is UAP on } \overline{\Omega} \\ & \text{and } -\alpha_1\theta(x) \leq v(t,x) \leq \alpha_1\theta(x) \text{ on } \mathbf{R} \times \overline{\Omega}\}, \end{aligned}$$

and define $F_1(t,x,u,y) = F(t,x,u,y) - F(t,x,0,0) - \mu\mu$ and for each $v \in \mathcal{B}$,

$$G_v(t,x,z) := F_1(t,x,z+q(t,x),v(t-r,x)+q(t-r,x)).$$

We first find a bounded solution in $\mathbf{R} \times \overline{\Omega}$ of

$$z_t - \Delta z - \mu z = G_v(t,x,z) \tag{2.11}$$

$$z = 0 \text{ on } \mathbf{R} \times \partial\Omega. \tag{2.12}$$

Let $z_1 = -\alpha_1\theta(x)$ and $z_2 = \alpha_1\theta(x), \alpha_1 \geq \alpha_0$. We claim z_1, z_2 are respectively lower and upper solution of (2.11), (2.12).

To begin with,

$$z_{1t} - \Delta z_1 - \mu z_1 = (\lambda_1 - \mu)z_1 = -\alpha_1(\lambda_1 - \mu)\theta \leq 0. \tag{2.13}$$

Next, $\frac{\partial}{\partial u}F_1(t,x,u,y) \leq \mu_1 - \mu < 0, 0 \leq \frac{\partial F_1}{\partial y}(t,x,u,y) \leq \delta_0$, and $F_1(t,x,0,0) = 0$. Since $\delta_0 + \mu_1 - \mu < 0$ an application of these facts and the mean value theorem (in the variables u and y) yields (using $z_1 \leq v(t-r,x)$):

$$\begin{aligned} G_v(t,x,z_1) &= F_1(t,x,z_1+q(t,x),v(t-r,x)+q(t-r,x)) \\ &\geq F_1(t,x,z_1+q(t,x),z_1+q(t-r,x)) \geq 0. \end{aligned} \tag{2.14}$$

Since $z_1 \leq 0$, inequalities (2.13) and (2.14) show $z_1 = -\alpha_1\theta$ to be a lower solution of (2.11), (2.12).

Much the same argument shows that $z_2 = \alpha_1\theta$ is an upper solution. Well known results now imply that the initial value problem $z(0,x) \equiv 0$ for (2.11), (2.12) has a solution $\tilde{z} \in C^{1+\alpha/2,2+\alpha}((0,\infty) \times \overline{\Omega})$ continuous on $[0,\infty) \times \overline{\Omega}$ with values between $-\alpha_1\theta(x)$ and $\alpha_1\theta(x)$. Since $G_v(t,x,z)$ is UAP on$\mathbf{R} \times \overline{\Omega}$, by Theorem 1.5 a bounded solution of (2.11),(2.12) exists on $\mathbf{R} \times \overline{\Omega}$. That solution must lie between $-\alpha_1\theta(x)$ and $\alpha_1\theta(x)$. An argument like that for uniqueness given above shows that for each $v \in \mathcal{B}$ the solution so obtained is unique. Thus the obtained solution is UAP by Theorem 1.4 and we have a well defined mapping $\Phi : \mathcal{B} \to \mathcal{B}$. By Lemma 1.3 each solution is uniformly continuous on $\mathbf{R} \times \overline{\Omega}$, but in fact the arguments of Lemmas 1.1-1.3 maybe used to show that the entire image, $\Phi(\mathcal{B})$, is a uniformly continuous family on $\mathbf{R} \times \overline{\Omega}$. From this it follows that there is a constant $C_1 > 0$ such that $|\partial z/\partial t|_0 \le C_1$ for all $z \in \Phi(\mathcal{B})$, and hence also $|\Delta z|_0 \le C_2$ for some $C_2 > 0$ and all $z \in \Phi(\mathcal{B})$. We thus have that $\Phi(\mathcal{B})$ is bounded in the $C_b^{1,2}(\mathbf{R} \times \overline{\Omega})$ norm. That is, there exists $C_3 > 0$ such that

$$|z|_0 + |z_t|_0 + |\nabla z|_0 + \sum_{i,j=1}^{N} \left|\frac{\partial^2 z}{\partial x_i, \partial x_j}\right|_0 \le C_3. \tag{2.15}$$

Thus it follows that in particular there is a number $M > 0$ such that $z \in \Phi(\mathcal{B})$ implies $|z|_{\alpha,T} \le M$ for all $T > 0$. (Recall $|\ |_{\alpha,T}$ is the norm in $C^\alpha([-T,T] \times \overline{\Omega})$).

Let $\mathcal{B}^* = \{v \in \mathcal{B} : |v|_{\alpha,T} \le M \text{ for all } T > 0\}$ and let $\mathcal{B}_1$ be the closure of $\mathcal{B}^*$ in the topology of norm convergence in $C^\alpha([-T,T] \times \overline{\Omega})$ for each $T > 0$ (i.e., we use the compact-open topology of $C^\alpha(\mathbf{R} \times \overline{\Omega})$). $\mathcal{B}_1$ is just all $u \in C^\alpha(\mathbf{R} \times \overline{\Omega})$ with $-\alpha_1\theta \le u \le \alpha_1\theta$. We may extend Φ as a mapping of $\mathcal{B}_1$ into itself as follows. Let $v \in \mathcal{B}_1$ and$(v_n) \subset \mathcal{B}^*$ with $v_n \to \overline{v}$ (in $C^\alpha(\mathbf{R} \times \overline{\Omega})$). Let (z_{n_K}) be any subsequence of $\Phi(v_n)$. It has a subsequence converging in $C^{1,2}(\mathbf{R} \times \overline{\Omega})$ to some $z_0 \in \mathcal{B}_1$ which must be a solution of (2.11), (2.12) with right hand side $G_v(t,x,z)$. If we take any other subsequence of $\Phi(v_n)$ we get a solution of the same problem; since solutions are unique we can define $\Phi(\overline{v}) = z_0$. Thus Φ may be extended to be a self map of $\mathcal{B}_1$. Arguments like those just given for the extension show that $\Phi(\mathcal{B}_1)$ is relatively compact and that Φ is continuous on $\mathcal{B}_1$. The generalized version of the Schauder-Tychonov Theorem ([**7**], p. 25) now implies that Φ has a fixed point $\tilde{z}$ in $\mathcal{B}_1$. This fixed point is $C^{1+\alpha/2,2+\alpha}$ and is bounded (since $-\alpha_1\theta \le v \le \alpha_1\theta$ for all $v \in \mathcal{B}_1$). In addition $\tilde{u} = \tilde{z} + q$ is a bounded $C^{1,2}$ solution of (0.1), (0.2). We have shown such solutions are unique. This existence argument maybe made for each $F^* \in \text{Hull}\,(F)$. It follows by Theorem 1.4 that $\tilde{u}$ is uniformly almost periodic.

REMARKS. Theorem 2.1 maybe compared with a result of Corduneanu [**1**] who has shown that if the non-delay problem ((0.1), (0.2) with F independent of y) satisfies $\partial F/\partial u \le \mu_1 < \lambda_1$ and (0.1), (0.2) has an L^2-bounded solution then it has an L^2-almost periodic solution (see also [**2**])

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