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INSTABILITIES IN STEADY FLOWS OF TWO FLUIDS

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ABSTRACT. Steady shearing flows and convection, involving two immiscible liquids separated by an interface, will be discussed with particular emphasis on the case when the fluids have similar densities but different viscosities. Many interface positions are theoretically allowed but only a few are observed experimentally, thus motivating a study of their stability. Numerical computations and asymptotic analyses for the stability of various arrangements will be discussed.

1. Introduction. This paper is based on a lecture given at the Conference on Nonlinear PDE, and is a review of the author's involvement in the study of stability of flows of two immiscible fluids.

Examples of two-fluid flows arise in a variety of contexts. In the pipeline transport of very viscous oils (which, in the simplest case, may be modelled by Hagen-Poiseuille flow), it has been observed that the addition of a small amount of water greatly reduces the pressure drop required for transportation [**1**]. The resistance to the flow is expected to arise mainly from friction at the pipe wall, so that replacing the viscous fluid by a less viscous immiscible one just along the wall would lower the work required to transport the viscous oil. For horizontal pipelining, such an arrangement was thought to be possible if the densities of the fluids are similar. In fact, experiments showed that the water migrates to the pipe wall, thus shielding the oil from intense shearing [**2**]. Moreover, by using additives in the less viscous liquid, the pipe wall could be protected from corroding. A related example is the extrusion of two molten polymers vertically out of a pipe and cooling in air [**3, 4, 5, 6**]. Experimental data are variable for flows with equal volumes and for flows where there is a small amount of the less viscous fluid. Again, the less viscous fluid eventually encapsulates the more viscous fluid, when the fluids are otherwise similar (see Figure 1): this appeared to be independent of the initial arrangement. Phenomena of this type are of immediate application to the fiber industry, in the spinning of bicomponent fibers, which are important for their self-crimping characteristics. Another area of industrial importance is the extrusion of multi-layer polymer melts through a slit die to form a film having the desired optical and mechanical properties [**7**].

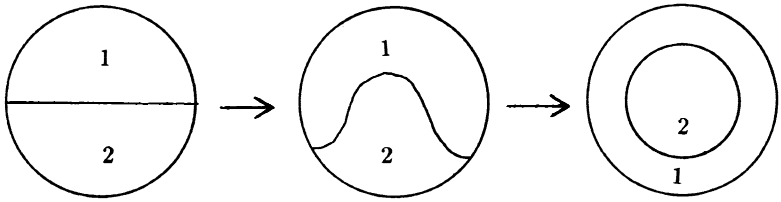

Figure 1. Evolution of flat interface into the encapsulated arrangement Fluid 2 is the more viscous.

The above pipe flows have been modelled by Hagen-Poiseuille flow with fluids of equal density. Mathematically, anti-plane shear flow (exclusively axial flow with only one non-zero component of velocity which depends on the coordinates perpendicular to the axial coordinate) of two fluids at low Reynolds number in a cylindrical pipe of arbitrary cross-section has a continuum of solutions. Here, there is no flow within the cross-section perpendicular to the axis of the pipe, so the pressure must be a constant in each of the fluids. At the interface, the jump in the normal stress is the jump in the pressure, hence a constant, and this must equal the product of the surface tension with the principle radius of curvature. Therefore, if the surface tension is zero, any interface curvature is allowed. If the surface tension is not zero, the curvature must be a constant so the interface is a circle or a circular arc terminating at the pipe wall (see Figure 2). In either case, there is an infinite number of possible interface positions. What has to be done here is to reconcile the theoretical non-uniqueness with the unique observed encapsulated arrangement. This dilemma turns out to be typical of steady density-matched two-component flows.

In the commercial area of ink-jet printing, one of the conventional methods is to force an electrically charged fluid out of a small nozzle under high pressure and to direct the flow with an electric field. Among some restrictions of this method is the clogging of the nozzle by pig-

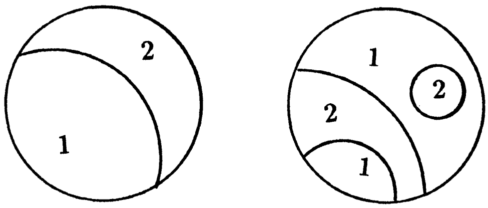

Figure 2. Some possible arrangements with surface tension.

ments in the ink. To overcome this, a compound jet has been suggested [**8**]: a primary fluid jet emerges from a nozzle below the surface of a stationary secondary fluid (see Figure 3). The jet in air then consists of a core of the primary jet surrounded by a layer of the secondary fluid, having a parabolic velocity profile on leaving the nozzle. Experiments [**8**] detailing the instabilities of the compound jet show that it tends to be more stable if the outer thin layer is the less viscous of the fluids.

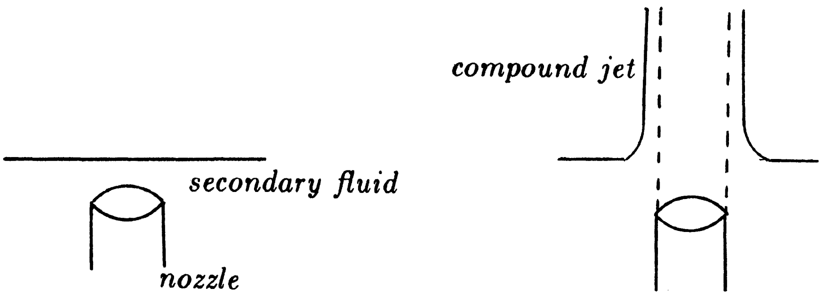

Figure 3.

The Earth's crust and mantle have been modeled [**9**] as a two-layer fluid system: heat sources in the crust induce convection in the entire mantle which is coupled back to the motion in the crust. Models of the Earth's mantle are sometimes based on the assumption that convection takes place in two chemically uniform layers[**10, 11**], the upper and the lower mantles. Earthquakes originate in the upper mantle: the scale of convection cells in the lower mantle may determine the scale of flow in the upper mantle via viscous coupling.

In rheological experiments on extensional flow, a lubricated planar-die rheometer [**12**] has been used to measure planar extensional viscosity. The boundaries of the rheometer are shaped so that a viscoelastic fluid can be forced into it from two opposing directions in the plane, meet in the center and leave through two exists perpendicular to the entrances (see Figure 4). Hyperbolic streamlines are desired, and the shape of the boundaries reflects this, but the no-slip condition at the walls prevents the desired velocity field. A small amount of a second less viscous fluid is introduced to the flow and this migrates to the walls, where most of the shearing takes place, and the more viscous test fluid then produces the desired flow field. Why the less viscous fluid stays next to the wall is not completely understood.

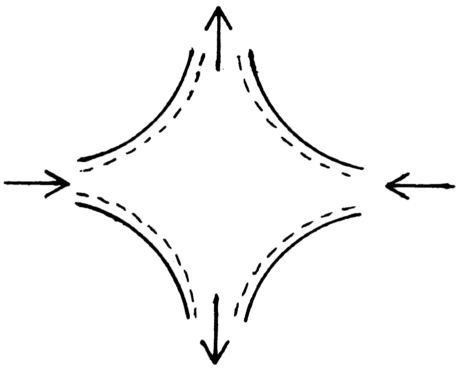

Figure 4. Sketch of boundaries of a rheometer. Arrows show flow direction of a viscous test fluid. A less viscous lubricant flows next to the walls.

The above examples cover a wide range of Reynolds numbers, from low to high, and the fluids may be highly elastic. In the shearing flows, the viscosity difference appears to play a key role in determining how the two fluids position themselves.

2. Shearing flows.

2.1 Formulation of equations. We consider shearing flows composed of two immiscible fluids with different viscosities μ_i and densities ρ_i, for which the general formulation is presented in detail by Joseph Nguyen and Beavers [**13**]. We concentrate on the effect of viscosity stratification. Each fluid is governed by the Navier-Stokes equations and assumed to be incompressible. In dimensionless form, these equations read

$$\mathfrak{R}_i\left[\frac{\partial u}{\partial t} + (\underline{\mathrm{u}} \cdot \nabla)\underline{\mathrm{u}}\right] = -\nabla(p + F^{-2}z) + \nabla \cdot \mathbf{T}, \tag{1}$$

$$\nabla \cdot \underline{\mathrm{u}} = 0, \tag{2}$$

where $\mathfrak{R}_i$, $(i = 1, 2)$, is the Reynolds number of fluid i, $\underline{\mathrm{u}}$ is the velocity, F is a Froude number, and $\mathbf{T}$ is the dimensionless extra stress tensor,

$$\mathbf{T} = \nabla\underline{\mathrm{u}} + (\nabla\underline{\mathrm{u}})^T. \tag{3}$$

If the fluids are viscoelastic, more complicated expressions for the extra stress tensor are applied.

The interface between the two fluids is an unknown, at which the normal is denoted by $\underline{\mathrm{n}}$ and the orthonormal tangential vectors are $\underline{\tau}_1$ and $\underline{\tau}_2$. The following interfacial conditions are expressed in dimensional form for the sake of compactness. Across the interface, the velocity and shear stresses are continuous:

$$[\![\mathrm{u}]\!] = 0 \tag{4}$$

$$\underline{\tau}_\ell \cdot [\![\tilde{T}]\!] \cdot \underline{\mathrm{n}} = 0, l = 1, 2, \tag{5}$$

where $\tilde{T}$ is the dimensional extra stress tensor. The jump in the normal stress across the interface is balanced by surface tension S^*:

$$\underline{\mathrm{n}} \cdot [\![\tilde{T}]\!] \cdot \underline{\mathrm{n}} - [\![\tilde{p}]\!] + 2HS^* = 0, \tag{6}$$

where $\tilde{p}$ is the dimensional pressure and H is the sum of principal curvatures. The kinematic free-surface condition holds at the interface, i.e., if the interface is described by $f(\underline{\mathrm{x}}(t), t) = 0$, then

$$\frac{\partial f}{\partial t} + \underline{\mathrm{u}} \cdot \nabla f = 0 \tag{7}$$

The volume ratio of the two fluids is given. Appropriate boundary conditions are imposed to complete the formulation, e.g., the no-slip condition at solid walls, periodicity in the unbounded direction. There are at least five dimensionless parameters: a Reynolds number, a surface tension parameter, volume ratio, the ratio of viscosities and the ratio of densities. A possible additional parameter is a Froude number for flows with gravity.

The problem is to find out how the two fluids would arrange themselves, and what physical parameters govern the arrangement and selection of interface shapes. A major problem in the theory of bicomponent flows lies in their nonuniqueness: the position of the interface is one of the unknowns, but the equations of motion may permit an infinite number of different interface configurations. In experiments, on the other hand, only certain preferred interface positions have been observed. The question thus arises: which interface positions can be observed?

2.2 Couette flow.

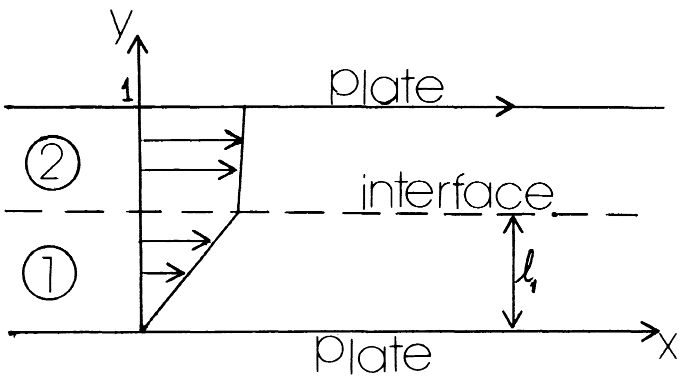

Figure 5. Undisturbed two-layer flow. Fluid 1 occupies $0 \le y \le \ell_1$. Fluid 2 occupies $\ell_1 \le y \le 1$. The fluids have viscosities $\mu_i (i = 1, 2)$ and densities ρ_i. Upper plate speed is 1.

In two-layer Couette flow (Figure 5), there are six dimensionless parameters: a viscosity ratio, $m = \mu_1/\mu_2$, density ration $r = \rho_1/\rho_2$, dimensionless depth of lower fluid $l_1 = l_1^*/l^*$, a Froude number F given by $F^2 = U^{*2}/gl^*$ where g denotes the gravitational acceleration constant and U^* denotes dimensional upper plate speed, a surface tension parameter $S = \frac{s^*}{(\mu_1 U^*)}$, where S^* is the surface tension coefficient, and a Reynolds number $\mathfrak{R} = U^* l^*/\nu_1^*$ where ν_1 is the kinematic viscosity of fluid 1. We denote $l_2 = 1 - l_1$. We make distance, velocity, time and pressure dimensionless with respect to $l^*, U^*, l^*/U^*$ and $\rho_1 U^{*2}$ respectively. A steady shearing flow solution to the two-layer problem is given by a velocity $(U_1(z), 0)$, and a flat interface at $z = l_1$, where

$$
\begin{aligned}
U_1(z) &= \frac{z}{l_1 + ml_2} \text{ for } 0 \le z \le l_1, \\
&= \frac{m(z-1)}{\ell_+ m\ell_2} + 1 \text{ for } \ell_1 \le z \le 1.
\end{aligned}
\tag{8}
$$

We examine the linear stability of this solution by adding small disturbances, $\underline{\mathrm{u}} = (u, v)$ to the velocity and h to the interface position, that are taken to be proportional to $\exp(i\alpha x + \sigma t)$. The resulting equations governing linear stability are: in Fluid i,

$$\frac{1}{\mathfrak{R}_i}\Delta u - \frac{\rho_1}{\rho_i}\frac{\partial p}{\partial x} - vU_1z(z) - U_1(z)i\alpha u = \sigma u, \tag{9}$$

$$\frac{1}{\mathfrak{R}_i}\Delta v - \frac{\rho_1}{\rho_i}\frac{\partial p}{\partial z} - i\alpha v U_1(z) = \sigma v, \tag{10}$$

$$\nabla \cdot \underline{\mathrm{u}} = 0, \tag{11}$$

where $\mathfrak{R}_i = U^* l/\nu_i$.

The interface conditions, linearized at $z = l_1$ yield:

$$\text{kinematic free surface condition: } v - hi\alpha U_1(l_1) = h\sigma, \tag{12}$$

$$\text{continuity of velocity: } [\![v]\!] = 0, \tag{13}$$

$$[\![u]\!] + h[\![U_{1z}(\ell_1)]\!] = 0, \tag{14}$$

$$\text{continuity of shear stress: } [\![\mu\Big(\frac{\partial u}{\partial z} + \frac{\partial v}{\partial x}\Big)]\!] = 0, \tag{15}$$

(16)

balance of normal stress: $$\cdot[\![-p+\frac{\mu}{\mu_1}\frac{2}{\Re}\frac{\partial v}{\partial z}]\!] = h\left(-\frac{\alpha^2 S}{\Re}+\frac{1}{F^2}\left(\frac{1}{r}-1\right)\right).$$

The boundary conditions are:

$$u = 0, \quad v = 0 \quad \text{at} \quad z = 0, 1. \tag{17}$$

Here $[\![\cdot]\!]$ denotes the jump of $\cdot$ across the interface, or $\cdot 1^{-} \cdot 2$. For the one-fluid problem ($r = 1, m = 1, S = 0$) with no interface at $z = l_1$, Romanov [**14**] has proved that the real parts of all eigenvalues σ are negative at any Reynolds number and wave number α. The addition of an interface at $z = l_1$ to the linear stability problem gives rise to an eigenvalue which was referred to as the "interfacial mode" by Yih [**15**]. This follows from the observation that if the two fluids have identical viscosity and density, and if there is no surface tension, then

$$u = 0, \quad v = 0, \quad h = \exp(i\alpha x + \sigma t), \sigma = -i\alpha l_1 \tag{18}$$

satisfies the above equations. In fact, on can think of the one-fluid problem as having an infinite number of neutrally stable eigenvalues, if each streamline is considered as an interface. When the two fluids have different properties, the interfacial eigenvalue can be unstable: this can occur at any Reynolds number.

In [**16**], Hooper and Boyd have furnished a complete solution to the linear stability problem of two-dimensional plane Couette flow when the boundaries are an infinite distance apart and each fluid occupies a half-plane. They also obtained the short-wave limit of the interfacial eigenvalue in closed form both rigorously and with a formal method. A heuristic reason as to why the formal method applies to other two-layer parallel shearing flows is that the eigenfunction for a short-wave disturbance dies out exponentially with distance away from the interface and is not aware of the presence of other boundaries.

What is surprising is that in the absence of surface tension or a density difference, the interfacial mode is unstable for sufficiently short waves at any Reynolds number [**16, 17**]. For the problem in Figure 5, the growth rate $\text{Re}(\sigma)$ of the interfacial mode, in the limit of short disturbance wavelength α is essentially an expression consisting of three terms:

$$\text{Re}(\sigma) \sim \frac{\mathfrak{R}_1 m(1-m)(1-m^2/r)}{2(l_1+ml_2)^2\alpha^2(1+m)^2} - \frac{m\mathfrak{R}_1(1-/r)}{2(1+m)\alpha F^2} - \frac{\alpha S}{2(1+m)} \text{ as } \alpha \to \infty. \tag{19}$$

The term of $O(1/\alpha^2)$ involves a Reynolds number, and viscosities, densities, and depth ratios of the fluids. The term of $O(1/\alpha)$ involves a Froude number, a Reynolds number, and the viscosities and densities of the fluids. The dominant term is $O(\alpha)$, involving surface tension, and is always negative in sign. Intuitively, one might have expected short waves to be damped by viscosity, but instead they are damped by surface tension. Density differences can also stabilize short waves (e.g., due to gravity, centrifugal force) but not as effectively as surface tension.

The long-wave asymptotics has been investigated by Yih [**15**] who was one of the first to look at the question of stability of multi-layer flows. His method of expansion in the parameter $\alpha\mathfrak{R}$ where $\mathfrak{R}$ is a Reynolds number has now been applied to a variety of multi-layer shearing flows [**18, 19, 20**]. Since the effect of the boundaries is important in a long-wave analysis, the calculations can be done in closed form but computing is necessary. Yih showed graphs of results for the case of equal densities. We notice that they show that when the less viscous fluid is in a thin layer, the interfacial eigenvalue is stable to long waves. Recently, Hooper [**21**] considered the long-wave asymptotic when the upper fluid occupies a semi-infinite plane for the case of stable density stratification. She found that at leading order, the flow is linearly stable when the lower fluid is the less viscous, and unstable if it is the more viscous.

We conclude from the above that for the case of equal density, there is a rule of thumb for finding a linearly stable arrangement. We stabilize long waves by putting the less viscous fluid in a thin layer, and stabilize short waves by including surface tension. What happens when there is a density difference is that this arrangement can be linearly stable even when the upper fluid is the heavier! We call this the 'thin-layer effect'. Figure 6 is an example of such a situation for which the eigenvalues were computed numerically. We use a spectral method based on Chebyschev polynomials to discretize the dependent variables [**22**]:

this approximates C^∞ functions with infinite-order accuracy. We have applied this discretization method to the numerical study of linear stability of other flows described below.

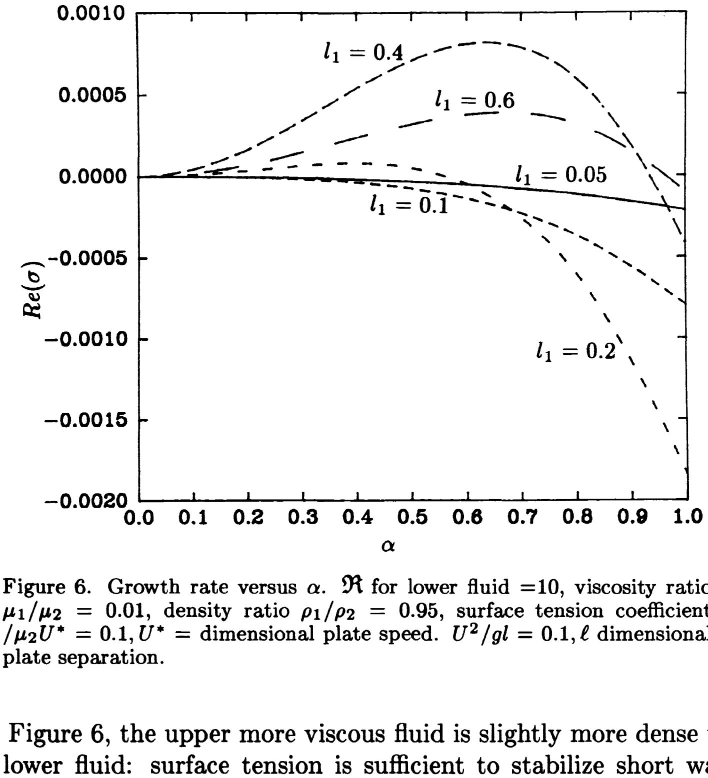

Figure 6. Growth rate versus α. $\mathfrak{R}$ for lower fluid =10, viscosity ratio $\mu_1/\mu_2 = 0.01$, density ratio $\rho_1/\rho_2 = 0.95$, surface tension coefficient $/\mu_2 U^* = 0.1, U^*$ = dimensional plate speed. $U^2/gl = 0.1, \ell$ dimensional plate separation.

In Figure 6, the upper more viscous fluid is slightly more dense than the lower fluid: surface tension is sufficient to stabilize short waves. The Froude number is chosen not too small, otherwise long waves will be destabilized by gravity, and the Reynolds number in each fluid is low. The parameter $l_1 (0 \leq l_1 \leq 1)$ measures the depth of the lower fluid. We note that for small l_1 (e.g., $l_1 = 0.05, 0.1$), the growth rate $\mathrm{Re}(\sigma)$ is negative for all wavelengths of the disturbance α. For larger l_1, instabilities arise from the long wave end, due to gravity.

We remark that it is not always possible in this problem to extrapolate the long and short wave asymptotics to make a conclusion about the linear since there may be instabilities which have nothing to do with either of the asymptotics. In fact, a numerical study [**17**] has revealed such instabilities.

2.3 Pipe flow. For density-matched pipe flow, we investigated numerically the linear stability of the concentric arrangement for various volume ratios [**23**]. In the basic flow, fluid 1 occupies $r \leq R_1$ and fluid 2 occupies $R_1 \leq r \leq R_2$ where R_2 is the radius of the pipe. The axial velocities W_i in fluid i are parabolic:

$$\begin{aligned} W_1 &= \frac{G}{\mu_1}[mR_2^2 + (1-m)R_1^2 - r^2] \text{ for } r \leq R_1 \\ W_2 &= \frac{G}{\mu_2}(R_2^2 - r^2) \text{ for } R_1 \leq r \leq R_2, \end{aligned} \tag{20}$$

where μ_i are the viscosities, $m = \mu_1/\mu_2$, and G is the applied pressure gradient. Long wave asymptotics of the interfacial eigenvalue for the case in which the less viscous fluid is encapsulated by the more viscous fluid [**18**] showed this to be unstable at all Reynolds numbers.

We find that the arrangement with a thin layer of the less viscous fluid next to the wall is linearly stable: this is the arrangement which is observed in experiments. If there is too much of the less viscous fluid or if the viscous fluid is at the wall, then the arrangement is unstable (see Figure 7).

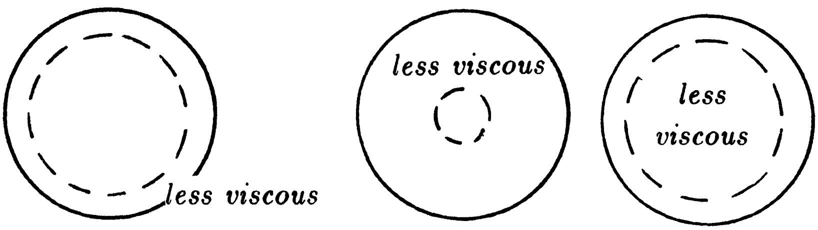

Figure 7.

An open problem for vertical pipe flow is to look at the effect of density stratification in addition to viscosity stratification: one would expect the more dense fluid to move to the center in order to fall faster, but this effect may be countered by the viscosity stratification.

2.4. Taylor flow.

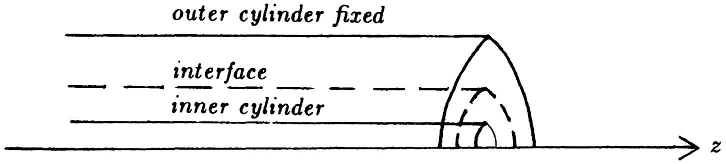

Figure 8.

Two fluids lie concentrically between two cylinders with the outer cylinder fixed and inner cylinder rotating [**24**] (see Figure 8). In the unperturbed flow, fluid 1 occupies $R_1 \leq r \leq D$, fluid 2 occupies $D \leq r \leq R_2$ and the azimuthal velocity field is given by

$$V_i(r) = A_i r + B_i/r, \quad i = 1, 2, \tag{21}$$

$$A_1 = \left(m\left(\frac{R_1^2}{R_2^2} - \frac{R_1^2}{D^2}\right) + \frac{R_1^2}{D^2}\right)/q,$$

$$A_2 = \frac{R_1^2}{qR_2^2},$$

$$B_1 = -1/q, \quad B_2 = -m/q, \quad m = \mu_1/\mu_2,$$

$$q = m\left(\frac{R_1^2}{R_2^2} - \frac{R_1^2}{D^2}\right) + \frac{R_1^2}{D^2} - 1.$$

A linearly stable configuration for fluids of equal density is again the one in which the less viscous fluid occupies a thin layer next to a solid. Contrary to intuition, it has been found [**24**] numerically that it is possible for the more dense fluid to lie inside, despite the centrifugal force, provided that it is the less viscous fluid, is in a sufficiently thin layer, the rotation rate is not too fast, and there is an adequate amount of surface tension. Here, the centrifugal force plays the role of gravity in the thin-layer phenomenon for Couette flow (see §2.3 above).

There is another 'thin-layer' effect in the Taylor problem. In contrast to the Hagen-Poiseuille and Couette flows discussed in §2.2 and §2.3, in which the corresponding one-fluid problem is linearly stable, the one-fluid Taylor flow has a criticality. We therefore examined the effect of the introduction of a second fluid on the onset of that first instability [**24**]. Figure 9 illustrates the effect. The infinitesimal disturbance is proportional to $\exp(i\alpha z + \text{ in } \theta + \sigma t)$ where z is the axial direction and θ is the azimuthal direction. The inner cylinder has radius 1 and the outer cylinder has radius 2. The interface radius on the horizontal axis varies from 1 to 2. The parameters were chosen so that if the outer fluid occupies the entire flow (i.e., interface radius is 1), then the one-fluid problem is at criticality. A more viscous fluid is then introduced next to the inner cylinder, and we see how the growth rate changes as the interface radius increases. Intuitively, one expects that there should be stability as the interface radius increases because then the amount of the more viscous or stabilizing fluid is increased. However, all modes first become unstable as the interface radius is increased. We have also looked at the case when a less viscous fluid is introduced next to the inner cylinder, in which case all modes became stable as the interface radius was increased from 1. We conclude that the onset of Taylor instability is delayed by the addition of thin layer of a less viscous fluid next to the inner cylinder and promoted by the addition of a thin layer of more viscous fluid next to the inner cylinder.

The thin-layer effect (stabilization due to viscosity stratification even in the presence of adverse density stratification) is absent if there is no shearing in the basic flowfield. For example, we have examined the stability of the rigid-body rotation of two fluids where the geometry is the two-layer Taylor flow but with both cylinders rotating at the same rate. The viscosity difference has been found to be irrelevant to the linear marginal-stability criterion [**25**]: criticality depends on the centrifugal terms and surface tension, as would be expected from seeing a centrifuge work. Here, the stability of the interface is governed by a variational principle and we have derived criteria for the concentric interface to be a local or global minimizer.

2.5 Thin-Layer effect. In the study of two-layer shearing flows, of particular interest is the role played by viscosity stratification, which

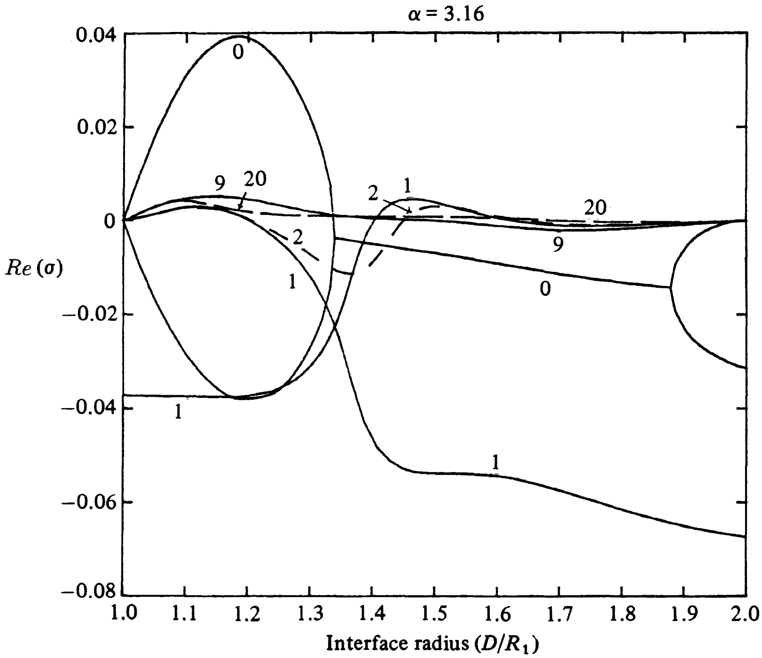

Figure 9. Growth rates when the less viscous fluid lies next to the outer cylinder. $\mu_1/\mu_2 = 1.08, \rho_1 = \rho_2$, zero surface tension. Numbers next to graphs denote azimuthal mode numbers. The amount of the less viscous fluid decreases as D/R_1 increases but various modes are unstable except when most of the gap is occupied by the more viscous fluid. At $D/R_1 = 1$ mode 0 is at the first criticality and mode 1 is slightly below.

has an especially pronounced effect when one of the fluids is in a thin layer. We have seen in §2.2 to §2.4 that it is even possible to have stable arrangements in the presence of an adverse density stratification. In order to look at the thin-layer effect analytically, the two-layer Couette flow has been studied for the particular case when the fluids have similar properties [**26**]. This is a perturbation problem in which the basic unperturbed problem is the two-layer Couette flow where the fluids have identical properties. Here, we know what the eigenvalues are. In the perturbed problem, the viscosities and densities of the fluids differ by an order ε, the surface tension is also of order ε, and we seek an expansion of the interfacial eigenvalue for small ε. This leads to a regular perturbation of simple eigenvalue and we can write down the leading terms in closed form. They involve Airy functions. We then analyze the thin-layer limit of the interfacial eigenvalue in closed form, i.e., let $l_1 (0 \leq l_1 \leq 1)$ measure the depth of one of the fluids, and assume l_1 is small compared with any other parameter which measures length in the problem. We find that the growth rate $\text{Re}(\sigma)$ consists of two terms at leading order: a term proportional to the viscosity difference at $O(\varepsilon l_1^2)$ which is stabilizing if the less viscous fluid is in the thin layer and destabilizing otherwise, and at an order l_1 smaller come the terms with the density difference and surface tension. This shows the dominant effect of viscosity stratification for the stability of thin layers. This result is to be interpreted for wavelengths that are not too short.

2.6 Nonlinear effects. What happens when the interface becomes unstable? The answer depends crucially on surface tension. If there is a sufficient amount of surface tension, then linear instability, if it occurs, involves long or order one waves. If periodic boundary conditions are assumed, then this is the type of instability to which bifurcation theory applies. On the other hand, if there were no surface tension, then waves of arbitrarily short wavelengths are linearly unstable. In this case, standard methods of bifurcation theory are not applicable, and it seems possible that no smooth interface, steady or unsteady, would be stable. This may be a mechanism for the formation and sustenance of emulsions.

Yih had conjectured in 1967 that finite amplitude waves may occur on the interface. Hooper and Grimshaw **[27]** have done a formal analysis of weakly nonlinear interaction of long waves for the two-layer Couette flow in the presence of surface tension. They use multiple scaling to find that the amplitude is governed by the Kuramoto-Sivashinsky equation. From this equation, they deduce that for equal density, the convective nonlinearity causes a sinusoidal disturbance to deform into a traveling wave with a steep front face and a long gradual tail. The final state maybe either apparent chaotic oscillatory motion or a steady state involving only a few harmonics. Ooms et al., **[28]** use an order of magnitude argument to derive an equation for the evolution of finite amplitude long waves. They choose a symmetric sinusoidal wave as an initial value and compute periodic sawtooth shaped waves that travel with a constant velocity in the streamwise direction and appear not to change after a certain time.

A Hopf bifurcation theorem [**29**] has been proved for two-layer Couette flow with surface tension, showing that when one mode becomes unstable, there exist new bifurcating solutions which are traveling interfacial waves. The main part of this work was to derive coercive estimates for the underlying partial differential equations. Also, an algorithm for computing the bifurcating solutions is given. These have yet to be computed and the question of whether they are stable is open.

Bifurcation theory is useful for studying the solution close to the onset of instability. For a more global understanding of the behavior of solutions, work needs to be done to compute the fully nonlinear evolution of interfaces as an initial value problem [**30, 31**].

3. Convection.

3.1 Formulation of equations. We consider the effect of heating in systems composed of two fluids with different viscosities μ_i, densities ρ_i, coefficients of cubical expansion $\hat{\alpha}_i$, thermal diffusivities κ_i, and thermal conductivities k_i.

Temperature differences are assumed to be small. In addition to the formulation of §2. above, the linear heat equation

$$\frac{D\Theta}{Dt} = \kappa\Delta\Theta, \tag{22}$$

where κ denotes thermal diffusivity and Θ the temperature, and the Boussinesq approximation [**32**] are applied. Across the interface, additional jump conditions are the continuity of heat flux and temperature:

$$[\![k\underline{n} \cdot \nabla\Theta]\!] = 0, \tag{23}$$

$$[\![\Theta]\!] = 0, \tag{24}$$

where k denotes the thermal conductivity.

On top of the dimensionless parameters in §2. above, there are at least five more: a Rayleigh number which is a measure of the temperature differences in the system, a Prandtl number which measures the relative effect of momentum diffusivity versus thermal diffusivity for one of the fluids, and ratios of the thermal properties of the fluids.

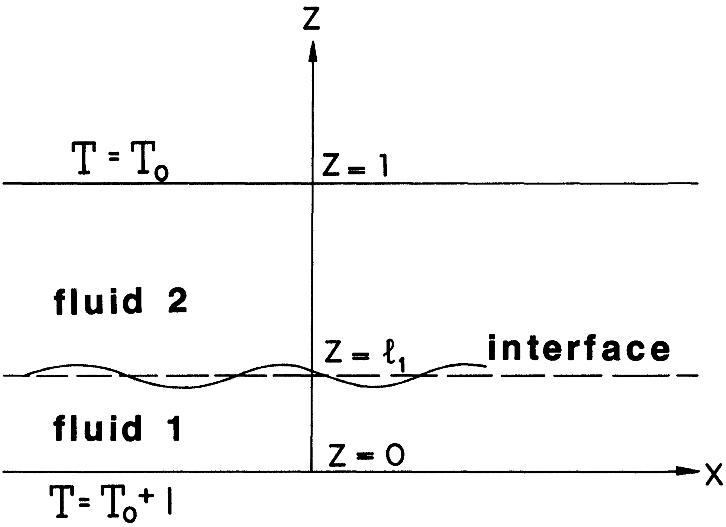

Figure 10.

3.2 Bénard problem.

We consider the two-layer Bénard problem of Figure 10. The lower plate, at $z = 0$, is kept at a higher temperature than the upper plate at $z = 1$. A solution to the equations is the rest state with a linear temperature gradient and a flat interface. To look at the linear stability of this, we add a small disturbance to the velocity and the interface position proportional to $\exp(i\alpha x + \sigma t)$.

In the Bénard problem for one fluid, "exchange of stabilities" (i.e., σ is real) holds for a variety of boundary conditions [**32**], whether the fluid is bounded by walls or by stress-free surfaces, or by a wall below and by a gas above. The consideration of additional effects can, however, introduce "overstability" (i.e., σ is imaginary at criticality). Overstability occurs, e.g., if there are temperature-dependent surface tension gradients [**33**], if there is a temperature-dependent solute gradient [**34, 35**], in mixtures of superfluids [**36**], or in the situation where two fluid layers, each having a gradient of a different solute, are superposed [**37, 38**]. On the other hand, previous to the work of [**39**], the Bénard problem with two fluid layers having different thermal and mechanical properties, but without surface tension gradients or solutes, had not been examined [**40**] for the possibility of over stability.

For the linear stability analysis, only two-dimensional disturbances need to be considered because of the rotational symmetry about the vertical axis. In [**39**], it is shown that the equations of the two-layer Bénard problem are not-self-adjoint, so that complex eigenvalues are possible. A time-periodic instability may occur as follows. There may be a convective instability in one or both of the fluids, counteracted by a stable interface. In fact, we have found a situation where the marginal eigenvalues are a purely imaginary conjugate pair of multiplicity two (the same eigenvalues appear for negative wavenumbers). Marginal eigenvalues of this type are associated with Hopf bifurcation from the motionless state to either a pair of traveling waves or a standing wave [**41**]. According to Ruelle [**41**], both the traveling and standing waves are solutions to the nonlinear problem. The question of which one is stable or whether either is stable is still open.

Figure 11 is an example of a Hopf bifurcation [**39**]. The vertical axis is the growth rate $\mathrm{Re}(\sigma)$ and the horizontal axis is the wavenumber of the disturbance. The densities were chosen so that there is no buoyancy instability at the interface, and this choice also stabilized the short waves. There are 5 branches to the graph. Branches 1, 3, and 5 belong to the interfacial mode. Branches 2, 3 and 4 belong to a mode that comes from the one-fluid problem. The σ is real on branches 1 and 2. They coalesce to form branch 3, on which they are a complex conjugate pair. They reach criticality before $\alpha = 5$, then eventually split to form branches 4 and 5. Branch 4 decays very fast for short waves: this is typical of a stable one-fluid mode. The shortwave behavior of the interfacial mode on branch 5 has been obtained in closed form [**39**]. Differences in density, coefficients of cubical expansion, and surface tension are important in the short-wave limit of the interfacial mode. In the long-wave limit, the volume ratio, and differences in the thermal conductivity, density, and coefficients of cubical expansion are important [**42, 43**]. If the Rayleigh number is raised slightly, then branch 3 comes up and crosses the line $\mathrm{Re}(\sigma) = 0$, yielding a time-periodic instability.

The above example was found numerically. In order to look for instabilities systematically in a problem such as this where there are lots of parameters, it is not feasible to do so numerically. We therefore

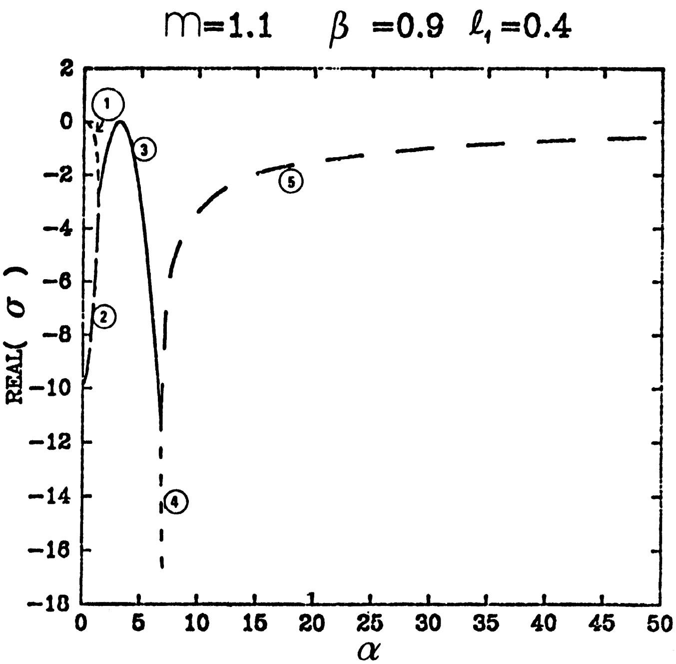

Figure 11. Growth rate versus wavenumber. Rayleigh number of fluid 1 is 1695.7, Prandtl number of fluid is 1; $\mu_1/\mu_2 = 1.1, \ell_1 = 0.4, \hat{\alpha}_1 \Delta T = 0.001, \Delta T$ = dimensional temperature difference between plate, $\hat{\alpha}_1/\hat{\alpha}_2 = 0.9$, surface tension $= 0$.

analyzed the case when the two fluids have only slightly differing mechanical and thermal properties. This is a perturbation of the case when the two fluids are identical. In addition, we replaced the no-slip condition with the stress-free condition. Although the slip condition is physically unrealistic, it has the advantage that the eigenvalue and eigenfunction at the critical Rayleigh number and critical disturbance wavenumber of the unperturbed problem are known in closed form. The unperturbed problem is the linear stability analysis of the two-layer Bénard problem when the fluids are identical.

There are two cases to look at. First, the Rayleigh number is below that of the criticality of the one-fluid problem [**43**]. The equations are real, so that the perturbed interfacial eigenvalue is also real; hence, overstability does not occur. When the interface is unstable, one would expect the fluids to go into a different arrangement. For example, if the instability results from the upper fluid being the heavier, one would expect that it should fall and the fluids should exchange positions. However, there are more subtle cases, such as a "thin-layer" effect described below. In [**43**], the thin-layer asymptotic limit of the interfacial mode is derived.

In the second case, the Rayleigh number and wavenumber are such that if the interface were absent, the one-fluid problem is at the first criticality. The analysis therefore concerns the regular perturbation of a double zero eigenvalue (the interfacial mode and the first critical mode of the one-fluid problem) of Riesz index 2. The leading terms in the perturbation expansion are calculated in closed form. It is found that whether the eigenvalue splits into real values or complex conjugates depends on the coefficients of thermal conductivity, densities, volume ratio and surface tension. When an oscillatory instability occurs, the ensuing flow is expected to look like the convection cells of the one-fluid problem. If the bifurcation results in the interface being a standing wave, the cells would turn slightly clockwise for one period, then anticlockwise the next period (see Figure 12). In the case of bifurcation to a traveling wave solution, the cells would keep turning in one direction and would travel through the fluid.

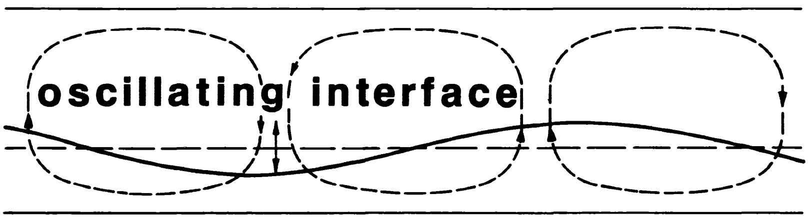

Figure 12.

3.3 Thin-Layer effect. In the equations governing the two-layer Bénard problem, the ratio of the coefficients of thermal conductivity appears in roles similar to those of the viscosity ratio in the equations governing shearing flows. Hence, a "thin-layer" effect also occurs in the Bénard problem [**43**]: depending on the thermal conductivities, and surface tension, it is possible to find linearly stable arrangements with the more dense fluid on top!

3.4 Nonlinear effects. In 3-d, even the one-fluid Bénard problem has lots of possible solutions such as hexagons, rolls and cells. The two-fluid problem is further complicated by the oscillatory nature of the instabilities. This leads to interesting questions of pattern selection which need to be investigated. The bifurcation analysis is related to that of double diffusion [**35**]. Double diffusion involves one fluid and the properties of the fluid vary continuously throughout the domain. In [**35**], there are still some aspects of the stability of the bifurcated solutions that are left unresolved, e.g., the stability with respect to all types of perturbations, not only those with the same symmetry as the bifurcated solutions themselves.

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