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ON THE GEOMETRY OF LEVEL SETS OF POSITIVE SOLUTIONS OF SEMILINEAR ELLIPTIC EQUATIONS

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1. Introduction. In a recent paper [3] the authors showed how certain a priori lower bounds on positive solutions of the quasilinear boundary value problem

$$\begin{aligned} -\Delta &= f(u) \text{ in } \Omega \\ u &= 0 \text{ on } \Omega \end{aligned} \tag{1.1}$$

may be obtained which are similar to bounds for positive solutions of

$$\begin{aligned} -u'' &= f(u) \text{ in } (0,\pi) \\ u(0) &= u(\pi) = 0. \end{aligned} \tag{1.2}$$

The crucial assumption in obtaining these bounds for (1.1) was that the bounded domain $\Omega \subseteq \mathbf{R}^n$ satisfied certain symmetry conditions. In particular, if Ω satisfies the symmetry conditions of Gidas-Ni-Nirenberg [7] (see below for a precise formulation) with respect to the standard basis of $\mathbf{R}^n$, then the level sets $\Omega_c$of u,

$$\Omega_c = \{x \in \Omega : u(x) > c\},$$

are starlike (see [9]). This fact, together with an identity of Rellich (see [1]), then implied the desired a priori lower bound. The question then arose whether these level sets still would be starlike in case the symmetry conditions of Gidas-Ni-Nirenberg are satisfied with respect to a not necessarily orthogonal basis. This paper is addressed to this question and we show, using some group theoretic considerations that these level sets are indeed starlike. We hence obtain extensions of a result of Kawohl [9] on starlike level sets and of our a priori estimates in [3].

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Figure 1, below, indicates a situation to which our considerations apply.

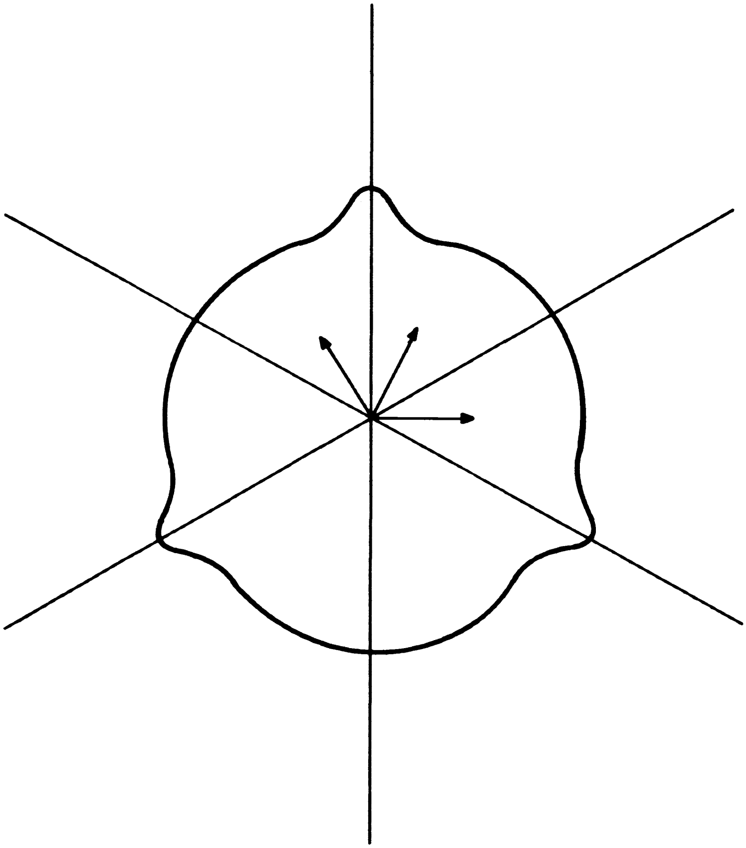

Figure 1.

2. Preliminaries, symmetries. Let $\Omega \subseteq \mathbf{R}^n$ be a smooth, bounded domain with $0 \in \Omega$. Let $f : \mathbf{R} \to \mathbf{R}$ be a function of class C^1. In order to study the geometry of level sets of positive solutions of (1.1) we employ a fundamental result of Gidas-Ni-Nirenberg which we shall state here for the readers' convenience. To introduce this result, and for later purposes, we need some notation.

For γ a unit vector in $\mathbf{R}^n$ and $\delta \in \mathbf{R}$, we denote by $P(\gamma, \delta)$ the hyperplane

$$P(\gamma, \delta) = \{x \in \mathbf{R}^n : \gamma \cdot x = \delta\}.$$

DEFINITION. The domain Ω has GNN *symmetry about* $0 \in \mathbf{R}^n$ *with respect to* γ *if*:

i) Ω is symmetric with respect to reflection through the hyperplane $P(\gamma, 0)$;

ii) if $\delta^* = \sup\{|\delta| : P(\gamma, \delta) \cap \Omega \neq \emptyset\}$, then, for $0 < \delta < \delta^*$, the reflection of the set $\{x \in \Omega : \gamma \cdot x > \delta\}$ through $P(\gamma, \delta)$ lies in Ω, and similarly, for $-\delta^* < \delta \leq 0$, the reflection of the set $\{x \in \Omega : \gamma \cdot x < \delta\}$ through $P(\gamma, \delta)$ lies in Ω;

iii) for $0 < |\delta| < \delta^*, P(\gamma, \delta)$ is not orthogonal to $\partial\Omega$.

THEOREM 1. (GIDAS-NI-NIRENBERG [**7**]) *Let* $u \in C^2(\overline{\Omega})$ *be a solution of* (1.1), $u(x) > 0, x \in \Omega$ *and suppose that* Ω *has* GNN *symmetry about* $0 \in \mathbf{R}^n$ *with respect to a unit vector* γ. *Then, for all* $x \in \Omega$ *with* $0 < \gamma \cdot x < \delta^*$, *one has*

$$\gamma \cdot \nabla u(x) < 0 \tag{2.1}$$

and $u(x)$ *is symmetric with respect to* $P(\gamma, 0)$, *and hence, for all* $x \in \Omega$,

$$(\gamma \cdot x)(\gamma \cdot \nabla u(x)) \leq 0, \tag{2.2}$$

with equality occurring only on $P(\gamma, 0)$.

If u is a solution of (1.1) on a domain Ω containing $0 \in \mathbf{R}^n$, then the level set

$$\Omega_c = \{x \in \Omega : u(x) > x\},$$

is starshaped with respect to $0 \in \mathbf{R}^n$, whenever $x \cdot \nabla u(x) \leq 0, x \in \Omega_c$.

Let$\{e_k : 1 \leq k \leq n\}$ be an orthonormal basis for $\mathbf{R}^n$.

THEOREM 2. (KAWOHL [9]) *Suppose that* Ω *has* GNN *symmetry about the origin with respect to the vectors* $\{e_k : 1 \leq k \leq n\}$. *Let* u *be a solution of* (1.1) *with* $u(x) > 0, x \in \Omega$. *Then, for any* $c, c \geq 0$, *the level sets* Ω_c *of* u *are starshaped with respect to* $0 \in \mathbf{R}^n$.

PROOF. We compute

$$x \cdot \nabla u = \sum_{k=1}^{n} (e_k \cdot x)(e_k \cdot \nabla u).$$

By Theorem 1, $(e_k \cdot x)(e_k \cdot \nabla u) \leq 0, x \in \Omega, 1 \leq k \leq n$. Hence $x \cdot \nabla u \leq 0$ on Ω and hence on any level set Ω_c.

REMARK. Theorem 2 and our results below remain valid if f is no longer C^1 but is nonnegative and can be expressed as the sum of a C^1 function and a monotone increasing function. Also, the results hold for equations more general than (1.1); the key point and limiting factor is the applicability of the results of [**7**].

3. Starshapedness of level sets, reflection groups. One of our main results on the geometric properties of level sets of solutions of (1.1) is

THEOREM 3. *Suppose that* Ω *has* GNN *symmetry about the origin with respect to a linearly independent set of* n *unit vectors. Then the level sets of any solution* u *of* (1.1), $u(x) > 0, x \in \Omega$, *are starshaped with respect to* $0 \in \mathbf{R}^n$.

Suppose that Ω is GNN symmetric with respect to γ_1 and symmetric with respect to the hyperplane $P(\gamma_2, 0)$. Then Ω must also be GNN symmetric with respect to the reflection of γ_1 through $P(\gamma_2, 0)$; the reflected vector is simply $\gamma_1 - 2(\gamma_1 \cdot \gamma_2)\gamma_2$. Also Ω will be GNN symmetric with respect to $-\gamma_1$. It follows that if Ω has GNN symmetry with respect to all the unit vectors in some set $\{\gamma_k\}$, then in fact Ω must have GNN symmetry with respect to all the vectors in the image of the set $\{\pm\gamma_k\}$ under the action of the group of transformations generated by the set of reflections through the hyperplanes $\{P(\gamma_k, 0)\}$. We shall be especially interested in the case where the group of transformations thus generated is finite. The classification of such groups was given in the fundamental paper of Coxeter [**4**], and finite groups generated by reflections are commonly referred to as Coxeter groups and are discussed in detail in [**4,5,6,8**]. We shall utilize directly only the group elements corresponding to reflections; to know what those are, however, we must consider the entire group generated by reflections through our original hyperplanes of symmetry. The groups we consider are subgroups of the orthogonal group, and since they consist of transformations leaving a closed bounded set invariant, they are closed. The significance of this observation is that if the group generated by a set of reflections is infinite,it must contain a closed, connected subgroup of the orthogonal group, which implies that the group must include all rotations around some axis. This fact will be used later.

If $\mathbf{G}$ is a group generated by reflections through a set of hyperplanes $\{P(\gamma_k, 0)\}$, then the set of vectors $\{\pm\gamma_k\}$ together with the images under the action of $\mathbf{G}$ is called a root system for $\mathbf{G}$. If the group $\mathbf{G}$ can be decomposed into a direct product of groups generated by reflections acting on mutually orthogonal subspaces, then $\mathbf{G}$ is called *reducible*. Geometrically we can then decompose the root system for $\mathbf{G}$ into two (or more) mutually orthogonal subsets. We note that Kawohl's [**9**] symmetry condition imposed on Ω is essentially that the symmetry group generated by the GNN symmetries of Ω contains a subgroup which is reducible into a product of n groups of order 2 each consisting of the identity and a single reflection, with mutually orthogonal unit vectors generating the reflections. It is interesting to note that reducibility of $\mathbf{G}$ allows us to decouple different coordinates in our computations; but if $\mathbf{G}$ is irreducible, this is no longer possible. (The idea describing the impossibility of decoupling in terms of irreducibility (in the sense of transformations in $\mathbf{R}^n$ which admit no nontrivial invariant coordinate subspaces) has been used by the second author in a somewhat different context to study positive solutions of coupled systems of elliptic equations, see [**2**].)

Suppose that $\mathbf{G}$ is finite and Γ is a root system for $\mathbf{G}$. If $x \in \mathbf{R}^n$ is such that $x \cdot \gamma \neq 0$ for all $\gamma \in \Gamma$, then we can divide Γ into two subsets, $\Gamma_x^+ = \{\gamma \in \Gamma : x \cdot \gamma > 0\}$ and $\Gamma_x^- = \{\gamma \in \Gamma : x \cdot \gamma < 0\}$. Let Π_x be a subset of Γ_x^+ such that every $\gamma \in \Gamma_x^+$ can be written as a linear combination with nonnegative coefficients of elements of Π_x, and Π_x is minimal with respect to that property. (That is, if Π^* is a proper subset of Π_x, then some $\gamma \in \Gamma_x^+$ cannot be expressed as a linear combination of elements of Π^* with all coefficients nonnegative.) Such a set Π_x is called an x-base for Γ. We have the following result:

LEMMA 4. *Given a finite Coxeter group* $\mathbf{G}$ *acting in* $\mathbf{R}^n$ *with root system* Γ *and* $x \in \mathbf{R}^n$ *such that* $x \cdot \gamma \neq 0$ *for every* $\gamma \in \Gamma$, *there exists a unique x-base for* Γ. *Let* Π_x *denote the x-base. Then* Π_x *is a basis for* $\mathbf{R}^n$; *if* $\Pi_x = \{\gamma_1, \ldots, \gamma_n\}$, *then for* $i \neq j, \gamma_i \cdot \gamma_j \leq 0$. *If* $\Pi_x^* = \{\mu_1, \ldots, \mu_n\}$ *is the dual basis corresponding to* Π_x, *then for* $i \neq j, \mu_i \cdot \mu_j \geq 0$.

Discussion. Lemma 4 summarizes Propositions 4.1.5 and 4.1.8 and Theorems 4.1.7 and 4.2.6 of [8]; these are standard results used in the classification theory of Coxeter groups.

PROOF OF THEOREM 3. (in the case of a finite symmetry group). Suppose that Ω has GNN symmetry with respect to a set of n unit vectors whose associated reflections generate a finite group $\mathbf{G}$. Let u be a positive solution to (1.1). We must show that $x \cdot \nabla u(x) \leq 0$ for all $x \in \Omega$; by continuity, it suffices to prove the inequality for a dense set of values of x. Since $\mathbf{G}$ is finite, $\mathbf{G}$ has finite root system Γ. Hence the set $\Omega_\Gamma \equiv \{x \in \Omega : x \cdot \gamma \neq 0 \text{ for all } \gamma \in \Gamma\}$ is dense in Ω. Suppose $x \in \Omega_\Gamma$, let $\Pi_x = \{\gamma_1, \ldots, \gamma_n\}$ be the x base for Γ and let $\Pi_x^* = \{\mu_1, \ldots, \mu_n\}$ be the corresponding dual basis for $\mathbf{R}^n$. By the definition of $\Pi_x, x \cdot \gamma_i > 0$ for all $\gamma_i \in \Pi_x$. By Theorem 1, $(x \cdot \gamma_i)(\nabla u \cdot \gamma_i) \leq 0$ for all $\gamma_i \in \Pi_x$, so $(\nabla u \cdot \gamma_i) \leq 0$ for $\gamma_i \in \Pi_x$. Since Π_x^* is the dual basis to Π_x, we have

$$x = \sum_{i=1}^{n} (x \cdot \gamma_i)\mu_i \text{ and } \nabla u = \sum_{j=1}^{n} (\nabla u \cdot \gamma_j)\mu_j.$$

Thus, $x \cdot \nabla u = \sum_{i=1}^n \sum_{j=1}^n (x \cdot \gamma_i)(\nabla u \cdot \gamma_j)(\mu_i \cdot \mu_j)$. Since $\mu_i \cdot \mu_j \geq 0$, by Lemma 4, we have $x \cdot \nabla u \leq 0$ as desired.

To extend Theorem 3 to the case of an infinite symmetry group, we need a lemma describing the case of reducible groups.

LEMMA 5. *Suppose $\Omega \subseteq \mathbf{R}^n$ has* GNN *symmetry with respect to a set of n reflections which generate a group $\mathbf{G}$ such that $\mathbf{G}$ has a subgroup $\mathbf{H}$ which can be decomposed into a direct product*

$$\mathbf{H} = \mathbf{H}_1 \times \cdots \times \mathbf{H}_m$$

of groups $\mathbf{H}_k$, with each $\mathbf{H}_k$ generated by reflections acting in a subspace $V^k \subseteq \mathbf{R}^n$, where the subspaces V^k are mutually orthogonal, $\mathbf{R}^m = V^1 + \cdots + V^m, \mathbf{H}_{\underset{\sim}{k}}$ leaves V^j fixed for $k \neq j$ and each group $\mathbf{H}_k$ has a finite subgroup $\tilde{\mathbf{H}}_k$ with a root system that forms a basis for V^k. Then any positive solution of (1.1) *on Ω has starlike level sets.*

PROOF. Since $\mathbf{R}^n = V^1 + \cdots + V^m$, we can write any $x \in \Omega$ as $x = x^1 + \cdots + x^m$ with $x^k \in V^k$; similarly, $\nabla u = p^1 + \cdots + p^m$. Suppose that we restrict our attention to $x^k \in \Omega \cap V^k$. We can recapitulate the proof of Theorem 3 for a finite symmetry group: for a dense subset of $\Omega \cap V^k$ we choose an x^k-base, $\{\gamma_1^k, \dots, \gamma_\ell^k\}$ for $\tilde{\mathbf{H}}_k$, with dual basis $\{\mu_1^k, \dots, \mu_\ell^k\} \subseteq V^k$. As in the case of a finite symmetry group, we have

$$x^k \cdot p^k = x^k \cdot \nabla u = \sum_{i=1}^{\ell} \sum_{j=1}^{\ell} (x^k \cdot \gamma_i^k)(\nabla u \cdot \gamma_j^k)(\mu_i^k \cdot \mu_j^k) \leq 0.$$

By continuity this extends to all of $\Omega \cap V^k$; since $x \cdot \nabla u = \sum_{k=1}^m x^k \cdot p^k$ the conclusion of the lemma follows. The key point is that the orthogonal decomposition of $\mathbf{R}^n$ allows us to work with one subspace V^k at a time.

REMARK. A crucial point is that $\mathbf{H}_k$ need not be finite, but need only have a finite subgroup; also, $\mathbf{G}$ need not be reducible but need only have a reducible subgroup. Kawohl's result, Theorem 2, can be viewed as the case of Lemma 5 in which V^k has one dimension and $\mathbf{H}_k$ in generated by one reflection for each k.

Completion of the proof of Theorem 3: We consider the case of an infinite group $\mathbf{G}$. The roots of $\mathbf{G}$ may be viewed as points on the unit sphere, so the set of roots must have a cluster point γ. Pick a sequence of roots approaching the cluster point; a compactness argument produces a subsequence $\{\gamma_k\}$ so that $(\gamma - \gamma_k)/|\gamma - \gamma_k|$ converges to a fixed vector tangent to the unit sphere. Using the reflections generated by γ and γ_k acting on $\pm\gamma, \pm\gamma_k$ one can construct a set of roots containing a circle in its closure. The circle will have tangent direction $\lim_{k\to\infty}(\gamma - \gamma_k)/|\gamma - \gamma_k|$ at γ. (The set of roots of **G** must be closed; if $\gamma_k \to \gamma$, then, for $x \in \overline{\Omega}, x - 2(x \cdot \gamma_k)\gamma_k \in \overline{\Omega}$ for all k, so $x - 2(x \cdot \gamma)\gamma \in \overline{\Omega}$. So $\overline{\Omega}$ is symmetric with respect to γ, also.) Suppose now that γ_1 is a root for **G** and lies in a circle S^1 consisting of roots of **G**. The circle must lie in a two dimensional subspace V_2 of **G**; since our original set of unit vectors was independent, there must be at least $n-2$ roots of **G** outside of V_2. If all of them are orthogonal to V_2 we may choose V^1 in Lemma 5 to be V_2, take $\tilde{\mathbf{H}}_1$ to be the reflection group generated by γ_1 and a vector in our circle in V_2 which is orthogonal to γ_1, and proceed to decompose the remainder of **G** in the orthogonal complement of V_2. If not all the remaining roots of **G** are orthogonal to V_2, choose one which is not, say γ_2. Let V_3 be the three dimensional subspace of $\mathbf{R}^n$ spanned by our circle in V_2 and γ_2. Since γ_2 varies continuously as γ moves around S^1 and is thus an irrational multiple of π for a dense subset of S^1. However, if the angle between γ and γ_2 is an irrational multiple of π, the set of roots generated by the reflections in $P(\gamma, 0)$ and $P(\gamma_2, 0)$ is infinite and hence contains a circle. Thus, if we consider a subspace $\tilde{V}_2$ of V_3 orthogonal to γ_2, we find that there is a dense subset $\tilde{S}$ of the unit circle in $\tilde{V}$ such that if $\gamma^* \in S$ then the circle containing γ_2 and γ^* lies in the root system for **G** for all γ^* in the unit circle in $\tilde{V}$; hence, V_3 contains a sphere S^2 consisting of roots of **G**. We now proceed as before.

There must be at least $n-3$ remaining roots of G which lie outside V_3. If all are orthogonal to V_3 we take $V' = V_3$ and decompose **G**. If some vector γ_3 is not orthogonal to V_3, let V_4 be the space spanned by γ_3 and V_3. Arguing as above, γ_3 and γ are irrationally related for a dense set of $\gamma \in S^2 \subseteq V_3$; so again if we take $\tilde{V}$ to be the subspace of V_4 orthogonal to γ_3, we see that, for γ^* belonging to a dense subset of a two-sphere $\tilde{S}^2 \subseteq \tilde{V}$ (and hence for all $\gamma^* \in \tilde{S}^2$ by continuity), we have that the circle containing γ^* and γ_3 consists of roots of **G**. Thus, the set of roots of **G** lying in V_4 is a three-sphere S^3. We may continue this line of argument; we find that either **G** has a root system containing an $n-1$ sphere in $\mathbf{R}^n$, in which case we take **H** to be generated by n orthogonal vectors, or that eventually all of the roots outside of V_k are

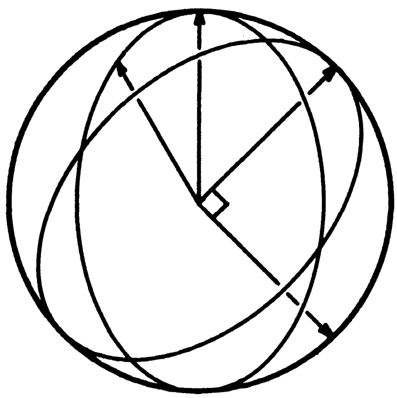

Figure 2a. Circle generated by γ, γ_2

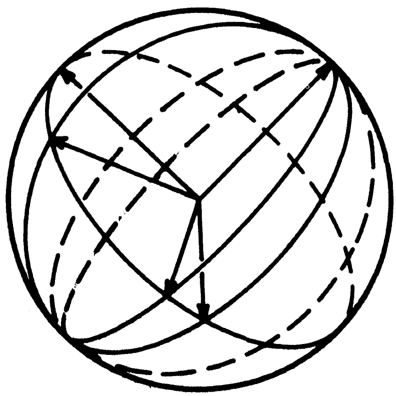

Figure 2b. Since the set of γ^* includes the entire circle $\tilde{S}$, the circles generate a sphere.

orthogonal to V_k. In that case, we take $V^1 = V_k$, the roots of $\mathbf{G}$ in V^1 form a $k-1$ sphere, and we can write $\mathbf{R}^n = V^1 + V^*$; $\mathbf{H} = \mathbf{H}_1 \times \mathbf{G}^*$ where $\mathbf{H}_1$ is generated by any k orthogonal unit vectors in $S^{k-1} \subseteq V^1$.

If $\mathbf{G}^*$is finite, we are done, by Lemma 5. If not, we now decompose V^* and $\mathbf{G}^*$ as above. We can continue this process until we obtain a decomposition of the type required in Lemma 5. This completes the proof of Theorem 3.

REMARK. Theorem 2 of [**3**] remains valid in case Ω satisfies the GNN symmetry condition with respect to any n linearly independent unit vectors $\{\gamma_1, \ldots, \gamma_n\}$. To see this we examine the proof of Theorem 2 of [**3**] and note that it may be carried over verbatim to this situation since the level sets Ω_c are starshaped.

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