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PERIODIC SOLUTIONS OF PERIODICALLY FORCED NON-DEGENERATE SYSTEMS

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1. Introduction. Let $U \subset \mathbf{R}^2$ be open and contain the origin. In this paper we study the ordinary differential equation

$$x' = f(x) + \varepsilon F(t,x) \tag{1.1}$$

where $'$ denotes $d/dt, f : U \to \mathbf{R}^2$ is of class C^2 and such that $f(x) = 0$ if and only if $x = 0, F : \mathbf{R} \times U \to \mathbf{R}$ is continuous, T-periodic in t and of class C^1 in x and ε is a small parameter. (1.1) arises as a perturbation of

$$x' = f(x). \tag{1.2}$$

We assume that the origin is a center of (1.2), that is, we assume there exists a family C of nontrivial periodic solutions of (1.2) whose orbits in $\mathbf{R}^2$ surround the origin. We also assume that C contains a periodic solution u with minimum period T which is nondegenerate. We define degenerate periodic solutions as follows.

Let v be a nontrivial q periodic solution of (1.2). Associated with v we have the linear variational equation

$$y' = f_x(v(t))y \tag{1.3}$$

where $f_x(v(t))$ denotes the Jacobian matrix of f evaluated at $v(t)$.

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DEFINITION 1.1. We say that v, a nontrivial q-periodic solution of (1.2), is degenerate if and only if every solution of (1.3) is q-periodic.

DEFINITION 1.2. We say that (1.2) is degenerate in U, or simply degenerate, if and only if every member of C is degenerate.

The next proposition, which we will state without proof, relates the concept of degeneracy to the periods of the elements of C.

PROPOSITION 1.1 (1.2) *is degenerate in* U *is and only if every element of* C *has the same minumum period.*

See [**2**] and the references therein for a further discussion of degenerate and non-degenerate systems and some of their characterization.

At this point we introduce some notation. The symbol $\cdot$ will denote the scalar product in $\mathbf{R}^2$ and $|\cdot|$ will denote both the absolute value of a real number and the euclidean norm on $\mathbf{R}^2, C^i(T)$, for i a non-negative integer, will denote the Banach space $\{r : \mathbf{R} \to \mathbf{R} | r$ is of class C^i and $r(t+T) = r(t)$, for all $t \in \mathbf{R}\}$. The norm in $C^i(T)$ is given by

$$||r||_i = \sup_{t\in[0,T]} \sum_{j=0}^{i} |r^{(j)}(t)|.$$

We make the following conventions. A vector $x = (x_1, x_2) \in \mathbf{R}^2$ will be identified with its column representation col $(x_1, x_2), x^t$ will then denote the row vector $[x_1, x_2]$. A linear operator $L : \mathbf{R}^2 \to \mathbf{R}^2$ will be identified with its matrix representation with respect to the canonical basis of $\mathbf{R}^2$. Which of these notions is being used in a particular instance will be clear from context. In particular the letter I will denote the identity operator on $\mathbf{R}^2$ or the 2×2 identity matrix. Finally $x(t, x_0, \varepsilon)$ will denote the solution of (1.1) passing through x_0 at $t = 0$, i.e., $x(0, x_0, \varepsilon) = x_0$. We will use x_{x_0} to denote the partial derivative of x with respect to the initial condition coordinate and x_ε to denote the partial derivative of x with respect to the parameter coordinate.

Consider again the non-degenerate T-periodic solution u of (1.2) and its corresponding linear variational equation

$$y' = f_x(u(t))y. \tag{1.4}$$

Since u' is a T-periodic solution of (1.4), it follows from Floquet theory that one characteristic multiplier μ_1 of (1.4) is equal to one. On the other hand since u is a nonisolated periodic solution of (1.2) it is known that the second characteristic multiplier μ_2 of (1.4) is equal to one.

We assume, without loss of generality, that $t = 0$ is chosen such that $u'(0)$ is parallel to the horizontal x_1 axis with positive first component. With this choice of parametrization for u and from $\mu_1 = \mu_2 = 1$, it follows that the principal matrix solution of (1.4) has the following form

$$X(t) = \left(\frac{u'(t)}{|u'(0)|}, p(t) + Kt\frac{u'(t)}{|u'(0)|} \right) \tag{1.5}$$

where K is a constant and $p : \mathbf{R} \to \mathbf{R}^2$ is C^1, T-periodic, and has $p(0) = \mathrm{col}\,(0, 1)$. We note that u is degenerate if and only if $K = 0$.

In the remainder of this paper we study the existence of T-periodic solutions of (1.1) branching from u. A solution $x(t, x_0, \varepsilon)$ of (1.1) will be T-periodic if and only if $x(T, x_0, \varepsilon) = x_0$. The problem, of course, is to find the points x_0 such that this condition holds. To do so, in §2, we will construct a closed curve $\Gamma(\varepsilon) \subset \mathbf{R}^2$ with the following properties:

i) $\Gamma(0) = \{u(s) \in \mathbf{R}^2 | s \in [0, T]\}$;

ii) $\Gamma(\varepsilon)$ is "close" to $\Gamma(0)$ if ε is small; and

iii) if $x_0 \in \Gamma(\varepsilon)$, then $x(T, x_0, \varepsilon)$ lies on a normal line to $\Gamma(0)$ through x_0.

We will see that use of $\Gamma(\varepsilon)$ reduces $x(T, x_0, \varepsilon) = x_0$ from a two dimensional to a one dimensional problem. A similar idea was used by Lazer in **[3]** for a second order scalar equation.

In §3, we will give sufficient conditions to have T-periodic solutions of (1.1) branching continuously from $u(t)$ or from one of its translates. The conditions we obtain generalize some of those given by Loud, in [**4**, Th. 5] and [**5**, Th. 3.12], to systems. Connected with Th. 5 in [**4**] and for a different type of generalization, see also [**3**].

In [**1**] an analysis of the related problem

$$x'' + g(x) = -\lambda x' + \mu f(t) \tag{1.6}$$

is done. The problem is to characterize the number of 2π-periodic solutions of (1.6) which lie in a neighborhood (in the (x, x') plane) of a 2π-periodic non-degenerate solution of the unperturbed equation. Using techniques different from ours they obtain a complete solution of this problem for (λ, μ) in a small neighborhood of (0.0).

Finally in §4 we will give an example which illustrates the theory developed in the previous sections.

2. Construction of $\Gamma(\varepsilon)$. Let u be a non-degenerate T-periodic solution of (1.2) with $t = 0$ chosen so that $u'(0)$ is parallel to the x_1 axis with positive first coordinate. Thus the principal matrix solution of (1.4) has the form (1.5), where $K \neq 0$. We establish a local coordinate system about u:

$$\hat{t}(s) = \frac{u'(s)}{|u'(s)|} \text{ and } \hat{n}(s) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \hat{t}(s). \tag{2.1}$$

$\hat{t}(s)$ and $\hat{n}(s)$ are, respectively, a unit tangent vector and a unit normal vector to $\Gamma(0)$ at $u(s), s \in [0, T]$. Our goal in this section is to find the curve $\Gamma(\varepsilon)$ discussed at the end of §1 and to parametrize it in a convenient form.

Let $\varepsilon_1 > 0$ be such that $|\varepsilon| < \varepsilon_1$, and $|u(s) - x_0| < \varepsilon_1$, for some $s \in [0, T]$ imply that $x(t, x_0, \varepsilon)$ exists at least for all $t \in [0, T]$. We have

THEOREM 2.1. *There exist $\varepsilon_0 > 0, \varepsilon_0 < \varepsilon_1$ and $R : (-\varepsilon_0, \varepsilon_0) \times \mathbf{R} \to \mathbf{R}$ of class C^1 and T periodic in the second variable such that*

$$\begin{aligned} \hat{t}(s) \cdot [x(T, u(s) + R(\varepsilon, s)\hat{n}(s), \varepsilon) \\ - u(s) - R(\varepsilon, s)\hat{n}(s)] = 0 \end{aligned} \tag{2.2}$$

for all $s \in \mathbf{R}$. *Furthermore,* $R(0, s) = 0$, *for all* $S \in \mathbf{R}$.

REMARK. The proof of this theorem is an application of the Implicit Function Theorem (IFT). We prove first the existence of a C^1 mapping $r : (-\varepsilon_0, \varepsilon_0) \to C^0(T)$ such that

$$\hat{t}(s) \cdot [x(T, u(s) + r(\varepsilon)(s)\hat{n}(s), \varepsilon) - u(s) - r(\varepsilon)(s)\hat{n}(s)] = 0 \tag{2.3}$$

for all $s \in \mathbf{R}$ and for $\varepsilon \in (-\varepsilon_0, \varepsilon_0)$. Then we define $R(\varepsilon, s) = r(\varepsilon)(s)$ and by a second application of the IFT we obtain that R is C^1.

PROOF. Let $B(\varepsilon_1) \subset C^0(T)$ denote the open ball with center 0 and radius ε_1. We define $G : (-\varepsilon_1, \varepsilon_1) \times B(\varepsilon_1) \to C^0(T)$ by

$$G(\varepsilon, r)(s) = \hat{t}(s) \cdot [x(T, u(s) + r(s)\hat{n}(s), \varepsilon) - u(s) - r(s)\hat{n}(s)] \tag{2.4}$$

for all $s \in \mathbf{R}$. From the usual theorems on differentiability of solutions with respect to initial condition and parameters it follows that G is a C^1 mapping. Since u is a T-periodic solution of (1.2), we have that $G(0, 0) = 0$. We will solve

$$G(\varepsilon, r) = 0 \in C^0(T) \tag{2.5}$$

using the IFT. Let $G_r(\varepsilon, r)$ denote the partial derivative of G with respect to r evaluated at (ε, r). It can be proved that

$$[G_r(\varepsilon, r)h](s) = \hat{t}(s) \cdot [x_{x_0}(T, u(s) + r(s)\hat{n}(s), \varepsilon) - I]\hat{n}(s)h(s) \tag{2.6}$$

for $h \in C^0(T)$. At $(\varepsilon, r) = (0, 0)$ and using that $\hat{t}(s) \cdot \hat{n}(s) = 0$ for all $s \in \mathbf{R}$, we obtain

$$[G_r(0, 0)h](s) = \hat{t}(s) \cdot x_{x_0}(T, u(s), 0)\hat{n}(s)h(s). \tag{2.7}$$

Thus we must show that

$$\hat{t}(s) \cdot x_{x_0}(T, u(s), 0)\hat{n}(s) \neq 0 \tag{2.8}$$

for all $s \in \mathbf{R}$. Since $x(t, u(s), 0) = u(t+s)$ we have that $x_{x_0}(t, u(s), 0)$ is the principal matrix solution of the linear variational equation

$$y' = f_x(u(t+s))y \tag{2.9}$$

and is given by

$$x_{x_0}(t, u(s), 0) = X(t+s)X^{-1}(s). \tag{2.10}$$

Using the expression for $X(t)$ given in (1.5), we obtain

$$\hat{t}(s) \cdot x_{x_0}(T, u(s), 0)\hat{n}(s) = \frac{KT|u'(s)|^2}{\det X(s)|u'(0)|^2} \tag{2.11}$$

which is evidently non zero for all $s \in \mathbf{R}$. Hence $G_r(0,0)$ is an isomorphism from $C^0(T)$ onto $C^0(T)$. From the IFT for Banach spaces, see [**6**], it follows that there exist an $\varepsilon_0 > 0$ and a C^1 function $r : (-\varepsilon_0, \varepsilon_0) \to C^0(T)$, such that

$$G(\varepsilon, r(\varepsilon)) = 0 \tag{2.12}$$

for all $\varepsilon \in (-\varepsilon_0, \varepsilon_0)$.

Next, we define $R(\varepsilon, s) = r(\varepsilon)(s)$ for all $(\varepsilon, s) \in (-\varepsilon_0, \varepsilon_0) \times \mathbf{R}$. It is clear that R is T-periodic in its second variable and, from the properties of r, continuous. We claim that R is also C^1. The proof of this claim is a second application of the IFT. We define $H : (-\varepsilon_0, \varepsilon_0) \times \mathbf{R} \times (-\varepsilon_1, \varepsilon_1) \to \mathbf{R}$ by

$$\begin{aligned} H(\varepsilon, s, \lambda) = \hat{t}(s) \cdot [&x(T, u(s) + \lambda\hat{n}(s), \varepsilon) \\ &- u(s) - \lambda\hat{n}(s)]. \end{aligned} \tag{2.13}$$

H is C^1 and $H(0, s_0, 0) = 0$ for any fixed $s_0 \in \mathbf{R}$. Also

$$\frac{\partial H}{\partial \lambda}(0, s_0, 0) = \hat{t}(s_0) \cdot x_{x_0}(T, u(s_0), 0)\hat{n}(s_0), \tag{2.14}$$

where we have used that $\hat{t}(s_0) \cdot \hat{n}(s_0) = 0$. From (2.11) and from the IFT it follows that there exist $\bar{\varepsilon}_0(s_0)$, $0 < \varepsilon_0$, $\bar{\delta}(s_0) > 0$, a neighborhood W of 0 in $\mathbf{R}$, and a C^1 function $\Lambda : \bar{\theta}(s_0) \to W \subset \mathbf{R}$ where $\bar{\theta}(s_0) \equiv (-\bar{\varepsilon}(s_0), \bar{\varepsilon}(s_0)) \times (s_0 - \bar{\delta}(s_0), s_0 + \bar{\delta}(s_0))$ such that

$$H(\varepsilon, s, \Lambda(\varepsilon, s)) = 0 \tag{2.15}$$

for all $(\varepsilon, s) \in \overline{\theta}(s_0)$. We observe that Λ is unique in the sense that if $(\varepsilon, s, z) \in \overline{\theta}(s_0) \times W$ and if

$$H(\varepsilon, s, z) = 0 \tag{2.16}$$

then $z = \Lambda(\varepsilon, s)$.

Since $R(0, s) = 0$ for any $s \in \mathbf{R}$, we obtain that $R(0, s_0) \in W$. Moreover, from the continuity of R we obtain the existence of $\tilde{\varepsilon}_0(s_0) > 0$ and $\tilde{\delta}(s_0) > 0$ such that if $|\varepsilon| < \tilde{\varepsilon}(s_0)$ and $|s - s_0| < \tilde{\delta}(s_0)$, then $R(e, s) \in W$. Let us define $\theta(s_0) = (-\varepsilon(s_0), \varepsilon(s_0)) \times (s_0 - \delta(s_0), s_0 + \delta(s_0))$ where $\varepsilon(s_0) = \min\{\overline{\varepsilon}(s_0)\tilde{\varepsilon}(s_0)\}$ and $\delta(s_0) = \min\{\overline{\delta}(s_0), \tilde{\delta}(s_0)\}$. From (2.3) and the definition of H we have that if $(\varepsilon, s) \in \theta(s_0)$, then $(\varepsilon, s, R(\varepsilon, s))$ satisfies (2.16). Hence Λ and R coincide on $\theta(s_0)$. The proof of the claim, and hence of the theorem, follows from the compactness of $[0, T]$ and from redefining ε_0 if necessary.

We now return to the construction of $\Gamma(\varepsilon)$ and define $\gamma : (-\varepsilon_0, \varepsilon_0) \times \mathbf{R} \to \mathbf{R}^2$ by

$$\gamma(\varepsilon, s) = u(s) + R(\varepsilon, s)\hat{n}(s). \tag{2.17}$$

γ is of class C^1 on its domain and (2.2) can now be recast as

$$\hat{t}(s) \cdot [x(T, \gamma(\varepsilon, s), \varepsilon) - \gamma(\varepsilon, s)] = 0 \tag{2.18}$$

for all $\varepsilon \in (-\varepsilon_0, \varepsilon_0)$ and for all $s \in \mathbf{R}$. For $\varepsilon \in (-\varepsilon_0, \varepsilon_0)$ we define $\Gamma(\varepsilon) \subset \mathbf{R}^2$ as the following closed curve

$$\Gamma(\varepsilon) = \{\gamma(\varepsilon, s) | s \in [0, T]\}. \tag{2.19}$$

Clearly $\Gamma(\varepsilon)$ satisfies properties i), ii) and iii) of the introduction.

3. Branching of periodic solutions. In this section we find those (ε, s), such that $\gamma(\varepsilon, s)$ is the initial point of a T-periodic solution of (1.1), i.e., such that

$$x(T, \gamma(\varepsilon, s), \varepsilon) = \gamma(\varepsilon, s). \tag{3.1}$$

To solve (3.1) is equivalent to solving the pair of equations

$$\hat{t}(s) \cdot [x(T, \gamma(\varepsilon, s), \varepsilon) - \gamma(\varepsilon, s)] = 0 \tag{3.2a}$$

$$\hat{n}(s) \cdot [x(T, \gamma(\varepsilon, s), \varepsilon) - \gamma(\varepsilon, s)] = 0. \tag{3.2b}$$

However, in §2, we showed that (3.2a) is satisfied for all $(\varepsilon, s) \in (-\varepsilon_0, \varepsilon_0) \times [0, T]$ and thus solving (3.1) is equivalent to finding the zeroes of $M : (-\varepsilon_0, \varepsilon_0) \times [0, T] \to \mathbf{R}$ defined by

$$M(\varepsilon, s) = \hat{n}(s) \cdot [x(T, \gamma(\varepsilon, s), \varepsilon) - \gamma(\varepsilon, s)]. \tag{3.3}$$

M is of class C^1 on its domain and $M(0, s) = 0$, for all $s \in [0, T]$. (M is indeed C^1 on $(-\varepsilon_0, \varepsilon_0) \times \mathbf{R}$. This fact must be taken into account in some latter smoothness considerations). Hence $(0, s), s \in [0, T]$, produce T-periodic solutions of (1.1) which, in fact, correspond to translates of u. This is, of course, the trivial case. We are interested in pairs (ε, s) with s depending smoothly on ε which produce initial points of T-periodic solutions that tend to translates of u as $\varepsilon \to 0$. Thus we must study (3.3). To do so it is more convenient to work with $M(\varepsilon, s)/\varepsilon$ insted of $M(\varepsilon, s)$. Thus let us define $g : (-\varepsilon_0, \varepsilon_0) \times [0, T] \to \mathbf{R}$ by

$$g(\varepsilon, s) = \begin{cases} \frac{M(\varepsilon, s)}{\varepsilon}, & \varepsilon \neq 0 \\ \hat{n}(s) \cdot x_\varepsilon(T, u(s), 0), & \varepsilon = 0 \end{cases} \tag{3.4}$$

LEMMA 3.1. *g is of class C^1 on it domain.*

The proof of this lemma is lengthy and, in order not to break the continuity of the argument at this point, is deferred to §5.

We are now in a position to prove the main theorem of this section.

THEOREM 3.2. *Assume that there exists an $s_0 \in [0, T]$ such that*

$$g(0, s_0) = \hat{n}(s_0) \cdot x_\varepsilon(T, u(s_0), 0) = 0 \tag{3.5}$$

and

$$\frac{d}{ds}[g(0, s)]_{s_0} = \frac{d}{ds}[\hat{n}(s) \cdot x_\varepsilon(T, u(s), 0)]_{s=s_0} \neq 0. \tag{3.6}$$

Then there exist an $\varepsilon_2 > 0$ *and a* C^1 *function* $\tilde{s} : (-\varepsilon_2, \varepsilon_2) \to \mathbf{R}, s(0) = s_0$*, such that, for any* $\varepsilon \in (-\varepsilon_2, \varepsilon_2)$,

$$\gamma(\varepsilon, \tilde{s}(\varepsilon)) = u(\tilde{s}(\varepsilon)) + R(\varepsilon, \tilde{s}(\varepsilon))\hat{n}(\tilde{s}(\varepsilon))$$

is an initial condition for a T*-periodic solution of* (1.1). *Furthermore, if* $x(t, \gamma(\varepsilon, \tilde{s}(\varepsilon)), \varepsilon)$ *denotes this family of* T*-periodic solutions, then*

$$x(t, \gamma(\varepsilon, \tilde{s}(\varepsilon)), \varepsilon) = u(t + s_0) + \varepsilon\beta(t) + o(\varepsilon), \tag{3.7}$$

where β *is the* T*-periodic solution of* (5.4) *with* $\varepsilon = 0, x_0 = u(s_0)$, *and subject to*

$$\beta(0) = \frac{\partial R}{\partial \varepsilon}(0, s_0)\hat{n}(s_0) + \tilde{s}'(0)|u'(s_0)|\hat{t}(s_0). \tag{3.8}$$

Finally, β *is given explicitly by*

$$\begin{aligned}\beta(t) &= x_\varepsilon(t, u(s_0), 0) \\ &\quad + x_{x_0}(t, u(s_0),)\left[\frac{\partial R}{\partial \varepsilon}(0, s_0)\hat{n}(s_0) + \tilde{s}'(0)|u'(s_0)|\hat{t}(s_0)\right].\end{aligned} \tag{3.9}$$

PROOF. The existence of the C^1 function $\tilde{s}$ is a direct consequence of the fact that g as defined in (3.4) is C^1 and of the IFT. Thus it is only necessary to prove (3.7). Let β be defined by (3.9). Then it is clear that $\beta(0)$ is given by (3.8) and that β is a solution of (5.4) with $\varepsilon = 0, x_0 = u(s_0)$ and initial conditions given by (3.8). The solution β will be T-periodic if and only if $\beta(T) = \beta(0)$, i.e., if and only if

$$x_\varepsilon(T, u(s_0), 0) + x_{x_0}(T, u(s_0), 0)\beta(0) = \beta(0). \tag{3.10}$$

If we replace s by $\tilde{s}(\varepsilon)$ in (3.1), differentiate with respect to ε, and let $\varepsilon \to 0$, we obtain (3.10). Thus β is T-periodic. The rest of the proof comes from the fact that $x(t, \gamma(\varepsilon, \tilde{s}(\varepsilon)), \varepsilon)$ is C^1 in ε and

$$\frac{d}{d\varepsilon}[x(t, \gamma(\varepsilon, \tilde{s}(\varepsilon)), \varepsilon)]_{|\varepsilon=0} = \beta(t).$$

It is interesting to note that, under the hypotheses of Th. 3.2, (1.1) can not have a unique periodic solution, at least for ε small. We have

THEOREM 3.3. *Under the hypotheses of Theorem* 3.2, *there exists* $\varepsilon_3, 0 < \varepsilon_3 \leq \varepsilon_2$, *such that* $|\varepsilon| < \varepsilon_3$ *implies that there exists* $s^*(\varepsilon) \neq \tilde{s}(\varepsilon), s^*(\varepsilon) \in [0, T)$, *such that* $g(\varepsilon, s^*(\varepsilon)) = 0$. *That is,* $x(t, \gamma(\varepsilon, s^*(\varepsilon)), \varepsilon)$ *is a second* T*-periodic solution of* (1.1).

NOTE. We do not assert that s^* is C^1, or even continuous. Neither is necessarily the case.

PROOF. Let $0 < \varepsilon_3 \leq \varepsilon_2$ be such that $\frac{\partial g}{\partial s}(\varepsilon, \tilde{s}(\varepsilon)) \neq 0$ for $|\varepsilon| < \varepsilon_3$. Let $\varepsilon, |\varepsilon| < \varepsilon_3$, be fixed and suppose, for simplicity, that $\frac{\partial g}{\partial s}(\varepsilon, \tilde{s}(\varepsilon)) < 0$. The same is true at $\tilde{s}(\varepsilon)+T$ by periodicity. Thus, slightly to the right of $\tilde{s}(\varepsilon), g < 0$ and slightly to the left of $\tilde{s}(\varepsilon)+T, g > 0$. Thus, by the Intermediate Value Theorem, there exists a point $s^*, \tilde{s}(\varepsilon) < s^* < \tilde{s}(\varepsilon) + T$, such that $g(\varepsilon, s^*) = 0$. If $s^* \in [\tilde{s}(\varepsilon), T)$, then define $s^*(\varepsilon) = s^*$; otherwise let $s^*(\varepsilon) = s^* - T$.

If we have more information about g, it is possible to make more precise statements:

COROLLARY 3.4. *If all the roots of* $g(0, s)$ *are simple, i.e., have nonzero derivative, then there are an even number of* C^1 *functions* $\tilde{s}$ *and hence an even number of branching families of periodic solutions of* (1.1) *given by* (3.7).

4. An illustrative example. As an example of the theory developed in §2 and §3, we consider the system

$$\begin{aligned} x_1' &= x_2(x_1^2 + x_2^2) + \varepsilon F_1(t) \\ x_2' &= -x_1(x_1^2 + x_2^2) + \varepsilon F_2(t), \end{aligned} \tag{4.1}$$

where $F_i : \mathbf{R} \to \mathbf{R}; i = 1, 2$ are continuous and 2π-periodic. The unperturbed system

$$\begin{aligned} x_1' &= x_2(x_1^2 + x_2^2) \\ x_2' &= -x_1(x_1^2 + x_2^2) \end{aligned} \tag{4.2}$$

was studied in [**2**] and was shown to be non-degenerate there. To place

(4.1) and (4.2) into the context of the previous theory, let

$$x = \mathrm{col}\,(x_1, x_2), f(x) = \mathrm{col}\,(x_2(x_1^2 + x_2^2), -x_1(x_2^2)),$$
$$F(t) = \mathrm{col}\,(F_1(t), F_2(t))$$

and let $x(t, x_0, \varepsilon)$ denote the solution of (4.1) passing through x_0 at $t = 0$. Then f, F satisfy the hypothesis detailed in §1.

It is clear that(4.2) possesses $u(t) = \mathrm{col}\,(\sin t, \cos t)$ as a 2π-periodic solution. The linear variational equation for (4.2) associated with this solution is given by

$$\begin{aligned} y_1' &= \sin t \cos t\, y_1 + (1 + 2\cos^2 t)y_2 \\ y_2' &= -(2\sin^2 t + 1)y_1 - 2\sin t \cos t\, y_2. \end{aligned} \tag{4.3}$$

$u'(t) = \mathrm{col}\,(\cos t, -\sin t)$ is a solution of (4.3) and it can be shown that col(sint $+2t$ cost, cost $-2t$ sint) is a second linearly independent solution. Thus

$$X(t) = \begin{bmatrix} \cos t & \sin t\, + 2t\cos t \\ -\sin t & \cos t\, - 2t\sin t \end{bmatrix} \tag{4.4}$$

is the principal matrix solution of (4.3) and, in the notation of (1.5), $p(t)$ =col(sint, cost) and $K = 2$.

The local coordinate system about $u(t)$ is given by

$$\hat{t}(t) = \mathrm{col}\,(\cos t, -\sin t), \hat{n}(t) = \mathrm{col}\,(\sin t, \cos t). \tag{4.5}$$

According to Th. 3.1, 2π-periodic solutions of (4.1) branch from translates of $u, u(t + s_0)$, if

$$\hat{n}(s_0) \cdot x_\varepsilon(2\pi, u(s_0), 0) = 0 \tag{4.6}$$

and

$$\frac{d}{ds}[\hat{n}(s) \cdot x_\varepsilon(2\pi, u(s), 0)]_{s=s_0} \neq 0 \tag{4.7}$$

We recall that $x_\varepsilon(t, x_0, \varepsilon)$ is the solution of (5.4) subject to $y(0) = 0$ and, hence, by the variationof constants formula

$$x_\varepsilon(t, u(s), 0) = X(t + s) \int_0^t X^{-1}(\sigma + s)F(\sigma)d\sigma \tag{4.8}$$

and

$$x_\varepsilon(2\pi, u(s), 0) = X(2\pi + s)\int_0^{2\pi} X^{-1}(\sigma + s)F(\sigma)d\sigma. \tag{4.9}$$

By direct calculation

$$\hat{n}^t(s)X(2\pi + s) = [0, 1]. \tag{4.10}$$

Then from (4.8) and (4.10) we obtain

$$\hat{n}(s) \cdot x_\varepsilon(2\pi, u(s), 0) = \sin s\tilde{F}_1 + \cos s\tilde{F}_2 \tag{4.11}$$

and

$$\frac{d}{ds}[\hat{n}(s) \cdot x_\varepsilon(2\pi, u(s), 0] = \cos s\tilde{F}_1 - \sin s\tilde{F}_2, \tag{4.12}$$

where

$$\tilde{F}_1 = \int_0^{2\pi} [\cos\sigma F_1(\sigma) - \sin\sigma F_2(\sigma)]d\sigma \tag{4.13}$$

$$\tilde{F}_2 = \int_0^{2\pi} [\sin\sigma F_1(\sigma) + \cos\sigma F_2(\sigma)]d\sigma. \tag{4.14}$$

It can be easily checked that in three cases $(\tilde{F}_1 \neq 0, \tilde{F}_2 = 0), (\tilde{F}_1 = 0, \tilde{F}_2 \neq 0)$ and $(\tilde{F}_1 \neq 0, \tilde{F}_2 \neq 0)$, Eq. (4.6) has exactly two solutions separated by π. At these solutions (4.7) is satisfied. If we call these roots s_0 and $s_0 + \pi$, we can conclude that (4.1) has two families of 2π-periodic solutions branching from

$$u(t + s_0) = \mathrm{col}\,(\sin(t + s_0), \cos(t + s_0)),$$
$$u(t + s_0 + \pi) = \mathrm{col}\,(-\sin(t + s_0), -\cos(t + s_0)), \quad \text{respectively.}$$

5. Proof of Lemma 3.1. Recall that we had

$$M(\varepsilon, s) = \hat{n}(s) \cdot [x(T, \gamma(\varepsilon, s), \varepsilon) - \gamma(\varepsilon, s)] \tag{5.1}$$

and

$$g(\varepsilon, s) = \begin{cases} M(\varepsilon, s)/\varepsilon, & \varepsilon \neq 0 \\ n(s) \cdot x_\varepsilon(T, u(s), 0), & \varepsilon = 0. \end{cases} \tag{5.2}$$

In order to apply the Implicit Function Theorem to g, we require

LEMMA 3.1. *g is of class C^1 on its domain.*

PROOF. It is clear that both $\frac{\partial g}{\partial \varepsilon}(\varepsilon, s)$ and $\frac{\partial g}{\partial s}(\varepsilon, s)$ are continuous for all (ε, s) in the domain of g with $\varepsilon \neq 0$. Thus, we need only prove that both partial derivatives exist and match in a C^1 fashion at $(0, s)$.

In proving these facts, the following matrix and vector differential equations will be useful.

$$Y' = f_x(x(t, x_0, \varepsilon))Y + \varepsilon F_x(t, x(t, x_0, \varepsilon))Y \tag{5.3}$$

$$\begin{aligned} y' = f_x(x(t, x_0, \varepsilon))y &+ F(t, x(t, x_0, \varepsilon)) \\ &+ \varepsilon F_x(t, x(t, x_0, \varepsilon))y. \end{aligned} \tag{5.4}$$

We observe that $x_{x_0}(t, x_0, \varepsilon)$ and $x_\varepsilon(t, x_0, \varepsilon)$ satisfy (5.3) subject to $Y(0) = I$ and (5.4) subject to $y(0) = 0$, respectively.

If $\varepsilon \neq 0$, we obtain from (5.1) and (5.2)

$$\begin{aligned} \frac{\partial g}{\partial s}(\varepsilon, s) &= \hat{n}'(s) \cdot [x(T, \gamma(\varepsilon, s), \varepsilon) - \gamma(\varepsilon, s)]/\varepsilon \\ &\quad + \hat{n}(s) \cdot [x_{x_0}(T, \gamma(\varepsilon, s), \varepsilon) - I]\frac{\partial \gamma}{\partial s}(\varepsilon, s)/\varepsilon. \end{aligned} \tag{5.5}$$

From the differentiability properties of solutions of (1.1) it can be proved that

$$\begin{aligned} &\lim_{\varepsilon \to 0}[x(T, \gamma(\varepsilon, s), \varepsilon) - \gamma(\varepsilon, s)]/\varepsilon \\ &= [x_{x_0}(T, u(s), 0) - I]\frac{\partial R}{\partial \varepsilon}(0, s)\hat{n}(s) + x_\varepsilon(T, u(s), 0) \end{aligned} \tag{5.6}$$

uniformly on $s \in [0,T]$. Next we define $Z(t,\varepsilon,s)$ by

$$Z(t,\varepsilon,s) = [x_{x_0}(t,\gamma(\varepsilon,s),\varepsilon) - x_{x_0}(t,u(s),0)]/\varepsilon. \tag{5.7}$$

We observe that

$$\hat{n}^t(s)Z(T,\varepsilon,s) = \hat{n}^t(s)[x_{x_0}(T,\gamma(\varepsilon,s),\varepsilon) - I]/\varepsilon \tag{5.8}$$

where we have used the fact that from (2.10) and (1.5) it follows that

$$\hat{n}^t(s)x_{x_0}(T,u(s),0) = \hat{n}^t(s). \tag{5.9}$$

Using (5.3) first with $x_0 = \gamma(\varepsilon,s), \varepsilon$ and then with $x_0 = u(s), \varepsilon = 0$, it is possible to prove that $Z(t,\varepsilon,s)$ converges uniformly on $(t,s) \in [0,T] \times [0,T]$ as $\varepsilon \to 0$ to a limit which we call $Z(t,0,s)$. $Z(t,0,s)$ satisfies the differential equation

$$\begin{aligned} Y' = \{&f_{xx}(x(t,u(s),0))[x_{x_0}(t,u(s),0)\frac{\partial R}{\partial \varepsilon}(0,s)\hat{n}(s) + x_\varepsilon(t,u(s),0)] \\ &+ F_x(t,x(t,u(s),0))\}x_{x_0}(t,u(s),0) + f_x(x(t,u(s),0)Y, \end{aligned} \tag{5.10}$$

subject to $Y(0) = 0$. Here $f_{xx}(x(t,u(s),0))$ represents the second derivative of f with respect to its variable, a symmetric bilinear mapping from $\mathbf{R}^2 \times \mathbf{R}^2$ into $\mathbf{R}^2$, evaluated at $x(t,u(s),0)$.

Next we note that by using $x_0 = u(s)$ and $\varepsilon = 0$ in (5.3) and (5.4) we can prove that $\frac{\partial}{\partial s}x_{x_0}(t,u(s),0)$ and $\frac{\partial}{\partial s}x_\varepsilon(t,u(s),0)$ satisfy the corresponding linear ordinary differential equations obtained from (5.3) and (5.4) respectively by formally differentiating with respect to s. From these differential equations and from (5.10) it can then be proved that

$$Z(t,0,s)u'(s) = \frac{\partial}{\partial s}x_\varepsilon(t,u(s),0) + \frac{\partial}{\partial s}x_{x_0}(t,u(s),0)\hat{n}(s)\frac{\partial R}{\partial \varepsilon}(0,s). \tag{5.11}$$

Finally, we are ready to show that $\frac{\partial g}{\partial s}$ is continuous at $(0,s)$. It is sufficient to show that

$$\lim_{\varepsilon\to 0}\frac{\partial g}{\partial s}(\varepsilon,s) = \frac{\partial}{\partial s}g(0,s)$$

uniformly in $s \in [0, T]$. From (5.5), (5.6), (5.8), (5.11) and the fact that

$$\lim_{\varepsilon \to 0} \frac{\partial \gamma}{\partial s}(\varepsilon, s) = u'(s),$$

we obtain

$$\begin{aligned}\lim_{\varepsilon \to 0} \frac{\partial g}{\partial s}(\varepsilon, s) &= \hat{n}'(s)[x_{x_0}(T, u(s), 0) - I]\frac{\partial R}{\partial \varepsilon}(0, s)\hat{n}(s) \\ &\quad + \hat{n}'(s) \cdot x_\varepsilon(T, u(s), 0) + \hat{n}(s) \cdot [\frac{\partial}{\partial s} x_\varepsilon(T, u(s), 0) \\ &\quad + \frac{\partial}{\partial s} x_{x_0}(T, u(s), 0)\hat{n}(s)\frac{\partial R}{\partial \varepsilon}(0, s)],\end{aligned} \tag{5.12}$$

uniformly for $s \in [0, T]$. Multiplying (5.9) by $\hat{n}(s)$ and differentiating with respect to s, we obtain

$$\hat{n}'(s) \cdot [x_{x_0}(T, u(s), 0) - I]\hat{n}(s) = -\hat{n}(s) \cdot \frac{\partial}{\partial s} x_{x_0}(T, u(s), 0)\hat{n}(s). \tag{5.13}$$

Substituting (5.13) into (5.12), we obtain

$$\lim_{\varepsilon \to 0} \frac{\partial g}{\partial s}(\varepsilon, s) = \frac{\partial}{\partial s}[\hat{n}(s) \cdot x_\varepsilon(T, u(s), 0)], \tag{5.14}$$

uniformly for $s \in [0, T]$. Thus, $\frac{\partial g}{\partial s}(\varepsilon, s)$ is continuous for $(\varepsilon, s) \in (-\varepsilon_0, \varepsilon_0)x[0, T]$.

Let us now examine the continuity of $\frac{\partial g}{\partial \varepsilon}$. We must show that

$$\lim_{\varepsilon \to 0} \frac{\partial g}{\partial \varepsilon}(\varepsilon, s) = \frac{\partial g}{\partial \varepsilon}(0, s),$$

uniformly in $s \in [0, T]$. From (5.1) and (5.2) with $\varepsilon \neq 0$ we obtain

$$\begin{aligned}\frac{\partial g}{\partial \varepsilon}(\varepsilon, s) &= \hat{n}(s) \cdot [x_{x_0}(T, \gamma(\varepsilon, s), \varepsilon) - I]\frac{\partial \gamma}{\partial \varepsilon}(\varepsilon, s)/\varepsilon \\ &\quad + \hat{n}(s) \cdot [x_\varepsilon(T, \gamma(\varepsilon, s), \varepsilon) - x_\varepsilon(T, u(s), 0)]/\varepsilon \\ &\quad - \hat{n}(s) \cdot [x(T, \gamma(\varepsilon, s), \varepsilon) - \gamma(\varepsilon, s) - \varepsilon x_\varepsilon(T, u(s), 0)]/\varepsilon^2\end{aligned} \tag{5.15}$$

where we have added and subtracted $x_\varepsilon(T, u(s), 0)/\varepsilon$. From previous results, the first term on the right hand side of (5.15) tends to

$$\hat{n}(s) \cdot Z(T, 0, s)\hat{n}(s)\frac{\partial R}{\partial \varepsilon}(0, s)$$

as ε tends to 0, uniformly in s. To handle the second term, let us define $z(t,\varepsilon,s)$ by

$$z(t,\varepsilon,s) = [x_\varepsilon(t,\gamma(\varepsilon,s),\varepsilon) - x_\varepsilon(t,u(s),0)]/\varepsilon. \tag{5.16}$$

From (5.4) with $x_0 = \gamma(\varepsilon,s), \varepsilon$ first and with $x_0 = u(s), \varepsilon = 0$ second, it can be seen that $z(t,\varepsilon,s)$ satisfies a linear ordinary differential equation. From this differential equation it is possible to prove that $z(t,\varepsilon,s)$ converges uniformly for $(t,s) \in [0,T] \times [0,T]$, as $\varepsilon \to 0$ to a limit which we call $z(t,0,s)$. Thus the second term on the right hand side of (5.15) tends uniformly in s to $\hat{n}(s) \cdot z(T,0,s)$. To prove the uniform convergence of the third term, we begin with the fact that $x(T,\gamma(\varepsilon,s),\ \varepsilon)$ can be written as

$$\begin{aligned} x(T,\gamma(\varepsilon,s),\varepsilon) =& u(s) + \varepsilon \int_0^1 [x_{x_0}(T,\gamma(\lambda\varepsilon,s),\lambda\varepsilon)\frac{\partial R}{\partial\varepsilon}(\lambda\varepsilon,s)\hat{n}(s) \\ & + x_\varepsilon(T,\gamma(\lambda\varepsilon,s),\lambda\varepsilon]d\lambda. \end{aligned} \tag{5.17}$$

Also

$$R(\varepsilon,s) = \varepsilon \int_0^1 \frac{\partial R}{\partial\varepsilon}(\lambda\varepsilon,s)d\lambda, \tag{5.18}$$

since $R(0,s) = 0$. From (5.17) and (5.18) we obtain that

$$\begin{aligned} &\hat{n}(s) \cdot [x(T,\gamma(\varepsilon,s),\varepsilon) - \gamma(\varepsilon,s) - \varepsilon x_\varepsilon(T,u(s),0)]/\varepsilon^2 \\ &= \hat{n}(s) \cdot \int_0^1 [x_{x_0}(T,\gamma(\lambda\varepsilon,s),\lambda\varepsilon) - I](\frac{\partial R}{\partial\varepsilon}(\lambda\varepsilon,s)/\varepsilon)\hat{n}(s)d\lambda \\ &+ \hat{n}(s) \cdot \int_0^1 \{[x_\varepsilon(T,\gamma(\lambda\varepsilon,s),\lambda\varepsilon) - x_\varepsilon(T,u(s),0)]/\varepsilon\}d\lambda. \end{aligned} \tag{5.19}$$

From previous results it can be proved that the right hand side of (5.19) tends, as $\varepsilon \to 0$, to

$$\frac{1}{2}\hat{n}(s) \cdot Z(T,0,s)\hat{n}(s)\frac{\partial R}{\partial\varepsilon}(0,s) + \frac{1}{2}\hat{n}(s) \cdot z(T,0,s),$$

uniformly on $s \in [0,T]$. Since this implies that the limit as $\varepsilon \to 0$, of the left hand side of (5.19) exists and since according to (5.1) and (5.2) this limit must be $\frac{\partial g}{\partial\varepsilon}(0,s)$, we conclude from (5.19) that

$$\begin{aligned} \frac{\partial g}{\partial\varepsilon}(0,s) =& \frac{1}{2}\hat{n}(s) \cdot Z(T,0,s)\hat{n}(s)\frac{\partial R}{\partial\varepsilon}(0,s) \\ & + \frac{1}{2}\hat{n}(s) \cdot z(T,0,s). \end{aligned} \tag{5.20}$$

Finally, from (5.15) and the results just derived, we obtain

$$\lim_{\varepsilon \to 0} \frac{\partial g}{\partial \varepsilon}(\varepsilon, s) = 2\frac{\partial g}{\partial \varepsilon}(0, s) - \frac{\partial g}{\partial \varepsilon}(0, s) = \frac{\partial g}{\partial \varepsilon}(0, s), \tag{5.21}$$

uniformly on $s \in [0, T]$. It is thus clear that $\frac{\partial g}{\partial \varepsilon}(\varepsilon s)$ is continuous for all $(\varepsilon, s) \in (-\varepsilon_0, \varepsilon_0) \times [0, T]$.

This complete the proof of Lemma 3.1.

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