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GENERALIZED HOMOGENEITY OF FINITE AND OF COUNTABLE TOPOLOGICAL SPACES

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ABSTRACT. Finite and countable topological spaces are investigated which are homogeneous, homogeneous with respect to open mappings or with respect to continuous ones. It is shown that for finite spaces all three concepts of homogeneity coincide, while for countable or for uncountable ones they are distinct. Some characterizations of countable spaces that are homogeneous in either sense are found for the metric setting.

0. Introduction. A topological space X with a topology (i.e., the family of open sets) $T(X)$ is said to be homogeneous with respect to a class M of mappings of X onto itself provided, for every two points $p, q \in X$, there is a mapping $f \in M$ such that $f(p) = q$. If M is the class of all homeomorphisms, we get the well-known concept of homogeneity of a topological space. A larger class of mappings than that of homeomorphisms (but not as large as the class of all continuous mappings) is one of open continuous mappings. Recall that a mapping $f : X \to Y$ between topological spaces is called open if images under f of open subsets of X are open in Y. And, finally, if M denotes the family of all continuous mappings of X onto X we get the concept of homogeneity with respect to continuity, that is due to David P. Bellamy.

Given a cardinal number k, let $D(k)$ and $I(k)$ denote a set of cardinality k equipped with the discrete and with the indiscrete topology, respectively. J. Ginsburg has proved in [**5**] that a finite topological space is homogeneous if and only if it is homeomorphic to the product $D(m) \times I(n)$ for some natural numbers m and n. The present paper has been inspired just by that short and nice result.

The paper is divided into two parts that concern finite and countable spaces, respectively (the term countable means of cardinality of the

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integers). At the beginning of the first part the authors prove that Ginsburg's characterization of finite homogeneous spaces is valid in a much more general case, namely for spaces that are homogeneous with respect to continuity. It is also indicated how deep finiteness of the space is essential in the result by showing some obstacles in obtaining a similar characterization for infinite–in particular countable–spaces. Taking homogeneity with respect to various classes M as a distinctive feature, we consider four classes of topological spaces, each being a subset of the next one: the class A of homogeneous spaces, B–of homogeneous with respect to open continuous mappings, C–of homogeneous with respect to all continuous mappings, and finally the class D of all topological spaces. While for finite spaces the classes A, B and C coincide, for countable ones example are presented at the end of the first part of the paper showing that the differences $B \setminus A, C \setminus B$ and $D \setminus C$ are nonempty, i.e., that the discussed concepts differ from each other if finiteness of the spaces is not assumed.

The second part of the paper is devoted to countable spaces. Analysis of some basic properties of the examples discussed at the very end of the previous part enable us (for the case of countable spaces) to characterize homogeneous spaces and–simultaneously–spaces which are homogeneous with respect to the class of open continuous mappings, under an additional assumption of metrizability: namely such a space is either discrete or dense in itself. Further, examples are constructed showing how far metrizability is an essential assumption in this result. For countable regular T_1–spaces, equivalences are proved between homogeneity with respect to continuous mappings, noncompactness and some other properties, and necessity of all assumed conditions is investigated in detail. At the end of the second part of the paper examples are provided of countable spaces and of locally connected metric curves displaying differences between all three concepts of homogeneity (i.e., classes A, B and C above) for the general setting, i.e., without any additional assumptions. Some open questions are also asked in the paper.

Since the terminology concerning such concepts as regular, normal or compact spaces is different in references [**4**], [**8**] and [**9**], we note that the one used in this paper follows the Kuratowski monograph [**8**] rather than [**4**] or [**9**]. In particular the definitions of regularity and normality do not include the T_1 axiom; similarly, for compactness, we do not assume the T_2 (Hausdorff) axiom.

We denote the sets of natural (i.e., positive integer), of integer and of rational numbers by $\mathbf{N}, \mathbf{Z}$ and $\mathbf{Q}$ respectively. They serve as underlying sets for various topologies. However, if nothing is said about the topology, they are considered as equipped with their usual topologies coming from the Euclidean metric on the real line. Although $\mathbf{N}$ and $\mathbf{Z}$ with the usual topologies are homeomorphic, we use both of them to simplify descriptions of some examples.

1. Finite spaces. The main result of this part says that, for finite topological spaces, there is no difference between the three kinds of homogeneity discussed in the introduction. We have the the following observation.

PROPOSITION 1.1. *Each continuous mapping from a finite topological space onto itself is a homeomorphism.*

Indeed, let a continuous surjection $f : X \to X$ on a finite topological space X be given. Thus f is one-to-one. To prove that f is a homeomorphism it is enough to show its openness. By continuity of f, to each open set $V \subset X$ corresponds, in a one-to-one way, an open set $f^{-1}(V)$. Since the family of all open subsets of X is finite, we conclude that each open set is of the form $f^{-1}(V)$ for some open set V, and therefore its image under f is open.

As an easy consequence of Proposition 1.1 and of Ginsburg's characterization [**5**] of finite homogeneous spaces we get the following result.

THEOREM 1.2. *For finite topological spaces X the following conditions are equivalent:*

(1) *X is homogeneous;*

(2) *X is homogeneous with respect to the class of open continuous mappings;*

(3) *X is homogeneous with respect to the class of continuous mappings; and*

(4) *there are natural numbers m and n such that X is homeomorphic to the product $D(m) \times I(n)$.*

Now we shall try to examine how far finiteness of the space is an essential assumption in Theorem 1.2. This will be done in a sequence of remarks below, where suitable examples are constructed.

REMARK 1.3. The reader can easily verify that finiteness is an essential assumption in Proposition 1.1. Even if the mapping under consideration in Proposition 1.1 is additionally assumed to be one-to-one, the conclusion is not valid if the spaces are not finite. To see this, consider the set $\mathbf{Z}$ of all integers equipped with a topology generated by a base consisting of the set of all negative integers and of all singeltons for nonnegative ones. A mapping $f : \mathbf{Z} \to \mathbf{Z}$ which assigns to each number $n \in \mathbf{Z}$ a number $f(n) = n - 1$ is a one-to-one continuous surjection but it is not open.

REMARK 1.4. Since the spaces $D(m)$ and $I(n)$ are homogeneous for all cardinal numbners m and n, and since the product of homogeneous spaces is homogeneous, finiteness of the space is a superfluous assumption in the implication from (4) to (1) in Theorem 1.2. However, in the opposite implication it is a necessary condition, as one can see by the example of rationals with the usual topology.

REMARK 1.5. Finiteness is also an essential assumption in the equivalence of (1), (2) and (3) of Theorem 1.2. Namely, for countable topological spaces, the concepts of homogeneity with respect to the classes of mappings considered in that theorem are all distinct: if A, B, C, D denote the classes of homogeneous, of homogeneous with respect to open continuous mappings, of homogeneous with respect to continuous mappings and of all countable topological spaces correspondingly, then $A \subset B \subset C \subset D$, and no one of these inclusions can be replaced by the equality. In fact, the inclusions follow directly from the definitions of the considered classes of mappings.

A. The discrete space of all integers is homogeneous, so $A \neq \emptyset$.

B. To see $B \backslash A \neq \emptyset$ take the set $\mathbf{Z}$ of all integers and define a topology $T(\mathbf{Z})$ on $\mathbf{Z}$ by declaring open, all sets of the forms $\{z \in \mathbf{Z} : z \leq 3n\}$ and $\{z \in \mathbf{Z} : z \leq 3n + 2\}$, $n \in \mathbf{Z}$, together with $\emptyset$ and $\mathbf{Z}$. Obviously $(\mathbf{Z}, T(\mathbf{Z}))$ is not homogeneous, i.e., $(\mathbf{Z}, T(\mathbf{Z}))$ is not in A.

Now we show that the discussed space is homogeneous with respect to the class of open continuous mappings. So, given two points $p, q \in \mathbf{Z}$ we shall find an open continuous surjection $f : \mathbf{Z} \to \mathbf{Z}$ with $f(p) = q$. First, note that if $p-q$ is divisible by 3, then the translation f_1 defined by $x \to x - p + q$ transforms $\mathbf{Z}$ onto itself homeomorphically and maps p onto q. Second, observe that if $p = 3m + 1$ and $q = 3m + 2$ or vice versa $m \in \mathbf{Z}$, then a mapping f_2 that interchanges these points and keeps all the other points of $\mathbf{Z}$ fixed is a homeomorphism. Third, note that a mapping $f_3 : \mathbf{Z} \to \mathbf{Z}$, defined for a fixed number $k \in \mathbf{Z}$ by

$$f_3(x) = \begin{cases} x, & \text{if } x < 3k, \\ 3k, & \text{if } 3k \leq x \leq 3k+3, \\ x-3, & \text{if } x > 3k+3, \end{cases}$$

is an open continuous surjection satisfying $f_3(3k+1) = 3k$. Fourth, a mapping $f_4 : \mathbf{Z} \to \mathbf{Z}$ defined for a fixed $k \in \mathbf{Z}$ by

$$f_4(x) = \begin{cases} x+3 & \text{if } x < 3k, \\ 3k+1, & \text{if } x = 3k, \\ x, & \text{if } x > 3k \end{cases}$$

if again an open continuous surjection, now having the property $f_4(3k) = 3k + 1$. Using a suitable composition of some of these four open mappings, the reader can easily find, for any $p, q \in \mathbf{Z}$, an open continuous surjective mapping that takes p to q. Thus $(\mathbf{Z}, T(\mathbf{Z}))$ is in B.

C. An example of a space in $C \backslash B$ is constructed on the product $H \times \mathbf{Z}$, where $H = \{0\} \cup \{1/n : n \in \mathrm{N}\}$ and both H and $\mathbf{Z}$ are equipped with their usual topologies, i.e., the topologies inherited from the Euclidean metric on the real line. Since $H \times \mathbf{Z}$ has isolated points and accumulation points, and since each open mapping of a space into itself maps isolated points to themselves, we conclude $H \times \mathbf{Z}$ is not in B. To see it is in C, note that, given two arbitrary points $p, q \in H \times \mathbf{Z}$, we can take the projection $f : H \times \mathbf{Z} \to \mathbf{Z}$ and an arbitrary surjection $g : \mathbf{Z} \to H \times \mathbf{Z}$ satisfying $g(f(p)) = q$. Then g is continuous as a mapping from a discrete space, hence the composition $gf : H \times \mathbf{Z} \to H \times \mathbf{Z}$ is a continuous surjection that maps p to q. For an indirect argument see Theorem 2.15 below.

D. Lastly, to show $D \backslash C \neq \emptyset$, note that $H = \{0\} \cup \{1/n : n \in N\}$ with its usual topology has exactly one accumulation point, namely 0, and therefore for each continuous surjection f on H we have $f(0) = 0$.

So H is not in C (see also Theorem 2.15 below).

2. Countable spaces. As has been shown in Remarks 1.4 and 1.5 of the previous part of the paper, neither Ginsburg's characterization nor coincidence of the three classes A, B and C of spaces are valid if finiteness of the spaces is replaced by their countability. For countable metrizable spaces we have two other characterizations that are main results of this part of the paper. The first of them says that, for countable spaces that are metrizable (equivalently: regular, T_1 and satisfying the first or the second axiom of countability), the concepts of homogeneity and of homogeneity with respect to open continuous mappings coincide, and each countable metrizable space having this attribute must be either discrete or dense in itself (i.e., homeomorphic either to the space $\mathbf{Z}$ of integers or to the space $\mathbf{Q}$ of rationals with their usual topologies)–see Theorems 2.1 and 2.3. The second main result of this part is a characterization of countable regular T_1-spaces (in particular metrizable ones) which are homogeneous with respect to the class of all continuous mappings simply as noncompact spaces–see Theorem 2.15. Examples are presented to show that these characterizations cannot be extended to the class of all countable topological spaces and that countability is an essential assumption. But no full characterization is obtained of countable topological spaces in the general case, i.e., for the nonmetric setting. For this large area of all countable topological spaces we prove some partial results and ask some questions only. However, the authors hope that the results proved and examples constructed will help attain characterizations of countable topological spaces that are homogeneous with respect to the classes of mappings discussed in the paper.

Note that a presentation of some properties of countable topological spaces from a viewpoint of their homogeneity with respect to various classes of mappings was begun in the end of the previous part of the paper. Of the four countable topological spaces we discussed above in Remark 1.5, the first, third and fourth are metrizable, while the second is not. Indeed, to see nonmetrizability of the space $(\mathbf{Z}, T(\mathbf{Z}))$ described in part B of Remark 1.5, observe that if $x_1 = -1$ and $x_2 = -2$, then there is no set in $T(\mathbf{Z})$ that contains exactly one of the points x_1 and x_2. Thus $(\mathbf{Z}, T(\mathbf{Z}))$ is not T_0 even. Therefore a natural question arises whether it is possible to find a countable metrizable topological space X such that $X \in B \setminus A$ (i.e., homogeneous with respect to open continuous mappings without being homogeneous). The question is answered in the negative by the following theorem.

THEOREM 2.1. *For countable metrizable spaces X the following conditions are equivalent:*

(1) *X is homogeneous;*

(2) *X is homogeneous with respect to the class of open continuous mappings; and*

(5) *X is homeomorphic either to a discrete space ($\mathbf{Z}$ of integers) or to a space which is dense in itself ($\mathbf{Q}$ of rationals).*

In fact, both $\mathbf{Z}$ and $\mathbf{Q}$ (with the Euclidean topologies inherited from the real line) are homogeneous, so (5) implies (1). Obviously, (1) implies (2). To see (2) implies (5), consider two cases. If X has an isolated point, then each point of X is also isolated, since open continuous mappings preserve the property of being an isolated point. Thus all singletons are open sets, and therefore X is discrete, i.e., it is homeomorphic to $\mathbf{Z}$. If X has no isolated point, then it is dense in itself. However, all metrizable countable dense in themselves spaces are homeomorphic (see [**8**; §26, V, p. 287]), and so X is homeomorphic to $\mathbf{Q}$.

REMARK 2.2. Recall that a regular second countable T_1-space is metrizable ([**8**; §21, XVII, Theorem, p. 236 and Remark, p. 239]). On the other hand each countable metrizable space is regular, T_1, and–obviously–separable, thus second countable. Then, for countable spaces, metrizability is equivalent to being regular, second countable and T_1. Furthermore, if a space is countable, then it is second countable if and only if it is first countable.

Thus by the above remark, Theorem 2.1 can be reformulated as

THEOREM 2.3. *For countable regular T_1-spaces that are first (second) countable, conditions* (1), (2) *and* (5) *are equivalent.*

The assumption of metrizability in Theorem 2.1 as well as both the conditions of regularity and of being first (second) countable in Theorem 2.3 are essential. This can be seen by the topological spaces

$(\mathbf{Q}, T'(\mathbf{Q}))$ and $(\mathbf{Q}, T''(\mathbf{Q}))$, where the topologies $T'(\mathbf{Q})$ and $T''(\mathbf{Q})$ are both larger than the Euclidean topology $T(\mathbf{Q})$. In both definitions, we let H_r denote the set $\{r + 1/n : n \in \mathbf{N}\}$ for each point $r \in \mathbf{Q}$.

EXAMPLE 2.4. There exists a countable Hausdorff nonregular first (second) countable homogeneous space.

PROOF. On the set $\mathbf{Q}$ of all rationals, extend the Euclidean topology $T(\mathbf{Q})$ in such a way that each point $r \in \mathbf{Q}$ has as its open neighborhoods all the sets $U \in T(\mathbf{Q})$ and the sets of the form $U \backslash H_r$, where $U \in T(\mathbf{Q})$. Since the Euclidean topology $T(\mathbf{Q})$ is Hausdorff, so is $T'(\mathbf{Q})$. However $T'(\mathbf{Q})$ is not regular, because there are no disjoint sets $V_1, V_2 \in T'(\mathbf{Q})$ with $r \in V_1$ and $H_r \subset V_2$ (note that H_r is closed and r is not in H_r). Further, if $\{U_n : n \in \mathbf{N}\}$ is a countable local base at a point $r \in \mathbf{Q}$ for the Euclidean topology $T(\mathbf{Q})$, then $\{U_n \backslash H_r : n \in \mathbf{N}\}$ is a local base at r for $T'(\mathbf{Q})$. So, $(\mathbf{Q}, T'(\mathbf{Q}))$ is first (and thus second) countable. And finally $(\mathbf{Q}, T'(\mathbf{Q}))$ is homogeneous because, for any two rationals p, q, a translation $r \to r - p + q$ $r \in \mathbf{Q}$ is the needed homeomorphism.

EXAMPLE 2.5. There exists a countable normal homogeneous T_1-space in which no point has a countable local base.

PROOF. On the set $\mathbf{Q}$ of all rationals take, as open neighborhoods of a point $r \in \mathbf{Q}$, all sets of the form $\{r\} \cup U$, with $U \in T(\mathbf{Q})$, provided $H_r \backslash U$ is finite (possibly empty). It is worthy to note, that if $r \in U \in T(\mathbf{Q})$, then U is declared to be an open neighborhood of r. Thus $T(\mathbf{Q}) \subset T''(\mathbf{Q})$, whence we conclude that $(\mathbf{Q}, T''(\mathbf{Q}))$ is Hausdorff, so it is a T_1-space.

For any two numbers $a, b \in \mathbf{Q}$ with $a < b$, we put $(a, b) = \{x \in \mathbf{Q} : a < x < b\}$ and $[a, b] = \{x \in \mathbf{Q} : a \leq x \leq b\}$.

Since the space is countable, to prove its normality, it is enough to show regularity of the space (see [**4**; Theorem 1.5.16, p. 66]). To this end consider a closed set $F \subset \mathbf{Q}$ and a point r out of it. So r is in the set $\mathbf{Q} \backslash F$ which is open, and thus there exists a set $U \in T(\mathbf{Q})$ such that the open neighborhood $\{r\} \cup U$ of r is contained in $\mathbf{Q} \backslash F$. Since the set $H_r \backslash U$ is finite, there is a positive integer m such that $r + 1/n \in U$ for $n \geq m$. Then, for each $n \geq m$, there are four points $a_n, b_n, c_n, d_n \in Q$

such that

(6) $a_n < c_n < r + 1/n < d_n < b_n$,

(7) $[a_n, b_n] \cap [a_{n+1}, b_{n+1}] = \emptyset$,

(8) $[a_n, b_n] \subset U$, and

(9) $H_r \cap [a_n, b_n] = \{r + 1/n\}$.

Thus the union $\{r\} \cup \bigcup\{(c_n, d_n) : n \geq m\}$ is an open neighborhood of the point r. Take the union $W = \{x \in \mathbf{Q} : x < r\} \cup \bigcup\{(b_{n+1}, a_n) : n \geq m\} \cup \bigcup\{x \in \mathbf{Q} : x > b_m\}$. Since all the sets forming this union (i.e., the sets in braces above) are open in the usual topology $T(\mathbf{Q})$ that is contained in $T''(\mathbf{Q})$, we see W is open, i.e., $W \in T''(\mathbf{Q})$. It is evident that $\mathbf{Q} \backslash U \subset W$, whence $F \subset W$. Further, we have $(\{r\} \cup U) \cap W = \emptyset$ simply by the definitions. Thus $(\mathbf{Q}, T''(\mathbf{Q}))$ is regular.

To prove no point of the space has a countable local base we show that, given a countable family $\{B_n : n \in \mathbf{N}\}$ of open sets, each containing a fixed point $r \in \mathbf{Q}$, there exists an open set V with $r \in V$ such that no B_n is contained in V. To this end consider a sequence of distinct natural numbers $\{n_k\}$, where $k \in \mathbf{N}$, such that $r + 1/n_k \in B_k$. Then, for each k, there are four points $a_{n_k}, b_{n_k}, c_{n_k}, d_{n_k}$ in $\mathbf{Q}$ such that conditions (6), (8) and (9) hold with n_k substituted in place of n and with B_k instead of U in (8). Further, for each $n \in \mathbf{N} \backslash \{n_1, n_2, n_3, \cdots\}$, we take a pair of points $c_n, d_n \in \mathbf{Q}$ such that

$$c_n < r + 1/n < d_n,$$

$$H_r \cap [c_n, d_n] = \{r + 1/n\},$$

the sets (c_n, d_n) are pairwise disjoint and, moreover, disjoint from all the sets (c_{n_k}, d_{n_k}) constructed before. Thus, for all $n \in \mathbf{N}$ we have defined the sets (c_n, d_n) which form an open covering of H_r. So $V = \{r\} \cup \bigcup\{(c_n, d_n) : n \in \mathbf{N}\}$ has all the required properties.

To close the proof observe that $(\mathbf{Q}, T''(Q))$ is homogeneous: for each two points $p, q \in Q$, the translation $r \to r - p + q$ is a homeomorphism sending p to q. So the proof is complete.

REMARK 2.6. Concerning Example 2.5 observe that examples are known of countable regular T_1-spaces that do not satisfy the first (and hence the second) axiom of countability: see [**4**], a remark on p. 66 just after Theorem 1.5.16 and Examples 1.6.19, 1.6.20 and 2.3.37 on

p. 79, 80 and 120 respectively. The first two of them are evidently not homogeneous because they contain both isolated and accumulation points. The third, Example 2.3.37 of [**4**], is in fact not a single topological space having the considered properties, but a family of such spaces X, depending on how a dense countable subspace X is chosen from the Cantor cube. So a question arises if it is possible to define a space X satisfying all the conditions of Example 2.3.37 of [**4**], p. 120 and 121, which additionally is homogeneous.

REMARK 2.7. The two examples above show that, for nonmetrizable countable spaces, the equivalences between conditions (1), (2) and (5) are not true: the family of countable homogeneous topological spaces is essentially larger than the corresponding family of metric ones which consists of two topologically distinct elements ($\mathbf{Z}$ and $\mathbf{Q}$ with the Euclidean topologies) only. However, till now the authors neither have any characterization of topological spaces that are homogeneous or homogeneous with respect to the class of open continuous mappings in a general case (i.e., not only for metric spaces but for the nonmetric setting as well), nor are they able to present any example of a countable space satisfying higher separation axioms (note that the space $(\mathbf{Z}, T(\mathbf{Z}))$ in part B of Remark 1.5 is not even T_0) to exhibit the difference between the two notions of homogeneity.

As was observed in Part C of Remark 1.5, homogeneity with respect to continuous mappings is a weaker condition than homogeneity with respect to open ones, even if metrizability of the space is assumed. A sequence of propositions below and Theorem 2.15 discuss conditions related to this topic and give full characterizations of continuously homogeneous regular T_1-spaces as noncompact ones.

Below we use the concept of a compact space in the sense that each open covering contains a finite subcovering (no separation axiom is assumed).

PROPOSITION 2.8. *For countable topological spaces X the following conditions are equivalent:*

(10) *there is a continuous mapping from X onto the set $\mathbf{N}$ of all positive integers with the usual topology, and*

(11) *there is an infinite open covering of X whose elements are mutually disjoint.*

Each of them implies that

(12) *X is noncompact, and*

(3) *X is homogeneous with respect to the class of continuous mappings.*

PROOF. Condition (10) implies (11), since if a mapping $f : X \to \mathbf{N}$ is continuous and surjective, then $\{f^{-1}(n) : n \in \mathbf{N}\}$ is the required covering. Conversely, if C is a covering assumed in (11), then each element of C is closed as the complement of the union of other elements of the covering; hence C is a decomposition of X. Further, the space X being countable, the family C is countable, too. So, since elements of C are simultaneously open and closed subsets of X, the quotient space X/C obtained by shrinking each element of C to a point (distinct elements to distinct points) is countable, and the quotient topology is discrete. Therefore X/C is homeomorphic to $\mathbf{N}$.

Condition (12) is an immediate consequence of (11) and of the definition of compactness. Finally, to see that (10) implies (3), take a continuous mapping $f : X \to \mathbf{N}$ of X onto $\mathbf{N}$, fix an arbitrary one-to-one surjection $g : \mathbf{N} \to X$ and note that it is continuous since the domain space $\mathbf{N}$ is discrete. If p and q are points of X, then let $h : \mathbf{N} \to \mathbf{N}$ be a homeomorphism such that $h(f(p)) = g^{-1}(q)$. Then the composition $ghf : X \to X$ is a continuous surjection that maps p to q. The proof is complete.

REMARK 2.9. Neither of the implications from (10) (or (11)) to (12) and (3) in Proposition 2.8 can be reversed in general. In fact, the space $(\mathbf{Z}, T(\mathbf{Z}))$ defined in part B of Remark 1.5 is obviously noncompact, and is proved there to be homogeneous with respect to the class of open continuous mappings, i.e., conditions (2) is satisfied which is stronger than (3). However, the space evidently does not satisfy condition (11), since each two open sets intersect. In particular, the space is not regular.

REMARK 2.10. Further, neither of (12) and (3) implies the other for countable spaces. Indeed, even condition (1), which is much stronger than (3), does not imply (12), as an example of a countable space with the finite complement topology shows (that is, a nonempty subset of X is declared to be open provided its complement is a finite set, see [**9**; Part II, Example 18, p. 49]). Note that this space is compact but not Hausdorff. For Hausdorff spaces no such example exists (see below, Proposition 2.14). To see a noncompact countable space that does not satisfy (3), define on the set $\mathbf{N}$ of positive integers a topology whose nonempty members have the form $\{n \in \mathbf{N} : n \leq k\}$ for each $k \in \mathbf{N}$. Then, for each continuous surjection $f : \mathbf{N} \to \mathbf{N}$, we have $f(1) = 1$. In fact, since the singelton $\{1\}$ is open in $\mathbf{N}$, the set $f^{-1}(1)$ is open, i.e., for some $k \in \mathbf{N}$ we have $f^{-1}(1) = \{n \in \mathbf{N} : n \leq k\} \supset \{1\}$, so $1 \in f^{-1}(1)$. Therefore the space is not homogeneous with respect to the class of continuous mappings. Again observe that the space just constructed is T_0 only, but not T_1.

If we, however, additionally assume that the space under consideration is regular, then (12) implies (11) (equivalently (10)) and therefore (3). It seems to the authors that the implication (for countable regular T_1-spaces) is probably known, but they did not find its proof in the literature. Thus a proof of this result, preceded by a lemma, is presented below.

LEMMA 2.11. *Each point of a countable regular noncompact T_1-space has an open and closed neighborhood whose complement is noncompact.*

PROOF. Let a countable regular noncompact T_1-space X be given and let $x \in X$. There exists an open neighborhood V of x such that $X\backslash V$ is noncompact. Since the space is T_1 regular and countable, it is normal [**4**; Theorem 1.5.16, p. 66], so completely regular, and hence there is a continuous real-valued mapping $f : X \to [0,1]$ such that $f(x) = 0$ and $f(X\backslash V) = \{1\}$. Since $f(X)$ is a countable subspace of $[0,1]$, there exists an open and closed (with respect to $f(X)$) neighborhood W such that $0 \in W \subset [0,1]\backslash\{1\}$. Then $U = f^{-1}(W)$ is an open and closed set containing the point x and contained in V. Further, $X\backslash U$ is noncompact because otherwise $X\backslash V$ would be compact as a closed subset of a compact space $X\backslash U$. The lemma is proved.

PROPOSITION 2.12. *If a countable regular T_1-space is noncompact then it has a countable open covering with mutually disjoint elements.*

PROOF. Denote the space by X and order its elements in a sequence so that $X = \{x_1, x_2, x_3, \cdots\}$. Put $n_1 = 1$ and let U_1 be a neighborhood of x_{n_1} as in the lemma, i.e., open and closed with noncompact complement. Put $X_1 = X \setminus U_1$ and let x_{n_2} denote the first element of the sequence which is in X_1. Applying Lemma 2.11 to the noncompact space X_1 (which is an open and closed subspace of X) and to the point x_{n_2} we find an open and closed (with respect to X_1, and thus with respect to X) neighborhood U_2 of x_{n_2} contained in X_1 and having the noncompact complement $X_1 \setminus U_2 = X_2$, which is again an open and closed subspace of X. The first point $x_n \in X_2$ with $n > n_2$, is denoted by x_{n_3}. Using Lemma 2.11 once more on X_2 and x_{n_3} we get $U_3 \subset X_2$ and so on. By an inductive procedure we define an infinite sequence of points $x_{n_1}, x_{n_2}, \cdots, x_{n_k}, \cdots$ such that $1 = n_1 < n_2 < \cdots < n_k < \cdots$ and a sequence of open (and closed) sets $U_1, U_2, \cdots, U_k, \cdots$ which are pairwise disjoint by their definitions. Furthermore, since $n_k \geq k$, it follows by the choice of x_{n_k} that $x_k \in \bigcup\{U_i : i \in \{1, 2, \cdots, k\}\}$. Therefore the sets U_k cover the whole space X, and so the proof is finished.

Observe that the assumption of regularity of the space in Proposition 2.12 cannot be weakened to being Hausdorff only. Namely the set $X = \{(x, y) \in \mathbf{Q} \times \mathbf{Q} : y \geq 0\}$ equipped with the irrational slope topology (see [**9**; Part II, Example 75, p. 93]) is a countable Hausdorff noncompact space on which every real-valued continuous function is constant [**9**; Part II, Property 5 of Example 75, p. 94]; i.e., (10) does not hold, which is equivalent by Proposition 2.8 to the conclusion (11) of Proposition 2.12.

It has been mentioned above that Proposition 2.12 leads to the implication from noncompactness (12) to homogeneity with respect to continuous mappings (3) for countable regular T_1-spaces via property (11) of Proposition 2.8. The converse implication also holds, even for Hausdorff spaces, as can be seen by Proposition 2.14 below.

The next proposition, perhaps interesting by itself, plays an auxiliary role and serves as a lemma in a proof of Proposition 2.14 which is a stronger result.

PROPOSITION 2.13. *No countable metrizable compact space is homo-*

geneous with respect to the class of continuous mappings.

PROOF. Let a countable metrizable space X be compact. For a given set $A \subset X$ let A^d denote the set of all accumulation points of A (called the derived set of A). For an ordinal number α, by the derived set of A of order α (see [**8**; §24, IV, p. 261]) we understand a set A^α defined by the conditions

$$A^1 = A^d, A^{\alpha+1} = (A^\alpha)^d \text{ and}$$
$$A^\lambda = \cap\{A^\alpha : \alpha < \lambda\} \text{ if } \lambda \text{ is a limit number.}$$

Recall that the sets X^α are closed and they form a decreasing family. It follows from separability of X that there exists a countable ordinal α with $X^\alpha = X^\beta$ for all $\beta \geq \alpha$ (see [**8**; §24, II, Theorem 2,p. 258]). Moreover, since X is countable and compact, it follows from Corollary 4 of [**8**], §34, IV, p. 415 that such X^α is empty. Denote by δ an ordinal number satisfying $X^\delta \neq \emptyset$ and $X^\beta = \emptyset$ for all $\beta > \delta$. Observe that X^δ is a finite set.

Consider now a continuous surjection $f : X \to X$. We shall prove that, for each $A \subset X$, we have

(13) $A^\alpha \subset f(A^\alpha)$
for all ordinals α. In particular $X^\delta \subset f(X^\delta)$, and, by finiteness of X to a point out of X^δ, we get $X^\delta = f(X^\delta)$; this implies that there is no continuous surjection taking a point from X^δ, and therefore X is not homogeneous with respect to the class of continuous mappings.

So, to finish the proof, we have to show (13). We proceed by transfinite induction. To this aim observe that, for every compact set $C \subset X$, we have $(f(C))^d \subset f(C^d)$. Substituting $C = A$, we get (13) with $\alpha = 1$. Assume (13) holds for some α. Putting $C = A^\alpha$ we have

(14) $(f(A^\alpha))^d \subset f(A^{\alpha+1})$.

Taking the derived sets in both members of (13) (the inductive assumption), we obtain $A^{\alpha+1} \subset (f(A^\alpha))^d$, and by (14) we have (13) for $\alpha+1$ in place of α. If λ is a limit ordinal and (13) holds true for $\alpha < \lambda$, by the assumption, we get $A^\lambda = \cap\{A^\alpha : \alpha < \lambda\} \subset \cap\{f(A^\alpha) : \alpha < \lambda\} = f(\cap\{A^\alpha : \alpha < \lambda\}) = f(A^\lambda)$. Thus (13) holds for all α, and therefore the proof is complete.

PROPOSITION 2.14. *If a countable Hausdorff space is homogeneous with respect to the class of continuous mappings, then it is noncompact.*

PROOF. Let a countable Hausdorff space be compact. Then it is second countable (see [**4**; Theorem 3.1.21, p. 171]) and normal ([**4**; Theorem 3.1.9, p. 168]), whence its metrizability follows by the Urysohn theorem (see [**8**; §22, II, Theorem 1, p. 241]). As a consequence of Proposition 2.13 we conclude that the space is not homogeneous with respect to continuous mappings.

Propositions 2.8, 2.12 and 2.14 imply the following result.

THEOREM 2.15. *For countable regular T_1-spaces X the following conditions are equivalent:*

(3) *X is homogeneous with respect to the class of continuous mappings;*

(10) *there is a continuous mapping from X onto a countable discrete space;*

(11) *there is a countable open covering of X whose elements are mutually disjoint; and*

(12) *X is noncompact.*

REMARK 2.16. We shall verify that countability of the spaces under consideration is an essential assumption in the above discussed results, in particular in Theorems 2.1, 2.3 and 2.15, as has been done for finiteness in Theorem 1.2–see Remark 1.5 above. Namely we shall present a few examples of curves (i.e., compact connected one-dimensional metric spaces) showing that, even for so narrow a class of spaces as locally connected curves, the concepts of homogeneity with respect to the classes of mappings considered above are all distinct. So let A, B, C and D have the same meaning as in Remark 1.5 but with referernce to curves.

A. The Menger universal curve (see, e.g., [**2**; Chapter 15, p. 501-506]) or a circle are known to be locally connected homogeneous curves [**1**, p. 322], so $A \neq \emptyset$.

B. As was observed by L.G. Oversteegen in a letter to the first named author, the one-point union of two Menger universal curves is homogenous with respect to the class of light (i.e., having zero-dimensional point-inverses) open continuous mappings without being homogeneous (see [**3**; Example 5.5]; for an example of a two-dimensional metric continuum that is homogeneous with respect to monotone open continuous mappings but is not homogeneous, see the end of the first part of [**6**]). Thus $B\backslash A \neq \emptyset$.

C. An arc is a locally connected curve in $C\backslash B$ (see [**7**; Theorem 1, p. 347] and [**10**; (1.3), p. 184]).

D. Since each locally connected continuum is homogeneous with respect to continuous mappings [**7**; Theorem 1, p. 347], an example in $D\backslash C$ cannot be locally connected. The $\sin(1/x)$-circle is an example of such a curve (see [**7**; Theorem 4, p. 352]).

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