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LOCAL CONNECTEDNESS OF SUPPORT POINTS

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ABSTRACT. It is shown that the set of support points of a boundedly weakly compact convex subset of a Banach space is locally connected in every dimension. If the convex set is also separable the set of support points is contractible.

The present paper refines, extends and localizes the results of [**2**]. The notation is the same as that in [**2**]; however for convenience we repeat some of it.

In what follows E will be a real Banach space and E^* its continuous dual. If $C \subseteq E$, the set of support points of C (written: $\operatorname{supp} C$) is the collection of points $x \in C$ for which there exists $x^* \in E^* \backslash \{0\}$ such that

$$\langle x, x^* \rangle = \sup\{\langle x, x^* \rangle : x \in C\} \equiv M(x^*; C).$$

The set C is boundedly (weakly) compact if $C \cap B$ is (weakly) compact for each closed ball B in E.

A space Y is said to be k-connected if every map $f : S^k \to Y$ is null-homotopic. We say Y is locally k-connected if, for each $y \in Y$ and every neighborhood U of y, there exists a neighborhood U_0 of y such that every map $f : S^k \to U_0$ is null-homotopic in U.

If Y is (locally) k-connected for each $k = 0, \ldots, n$, then Y is said to be $(\mathrm{LC}^n)\mathrm{C}^n$. A space is said to be $(\mathrm{LC}^\infty)\mathrm{C}^\infty$, if it is $(\mathrm{LC}^n)\mathrm{C}^n$ for every n. We use the notation (x, y) for the open line segment $\{\lambda x + (1-\lambda) y : 0 < \lambda < 1\}$ and the notation $B(x; \delta)$ for the open ball with center at x and radius δ.

If C is a closed convex subset of the Banach space E and $0 \in C \backslash \operatorname{supp} C$, then

$$\operatorname{supp} C = \cup \{F_m : m \geq 1\},$$

where $F_m = \{x \in C : \exists x^* \in E^*, ||x^*|| \leq m, \langle x, x^* \rangle = 1 = M(x^*, C)\}$; it is easy to verify that each F_m is closed, so $\operatorname{supp} C$ is an F_σ.

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Recall that a residual set R is the complement of a first category set, and that in a Baire space E this is equivalent to R containing a dense G_δ in E.

The following lemma is a restatement of Lemma 4 of [**2**]; in [**2**] the proof of Lemma 4 showed that the set $R(S)$ (see below) contained a dense G_δ, although the statement of the Lemma was only that $R(S)$ is nonempty.

LEMMA. *Let C be a closed convex subset of a Banach space E. Suppose that $\operatorname{int} C = \emptyset$ and that $0 \in C \backslash \operatorname{supp} C$. If $S \subseteq C$ is such that $S \cap F_m$ is separable for each $m \geq 1$, then*

$$R(S) \equiv \{x \in E \backslash C : \text{ for each } a \in S \text{ we have } C \cap (x, a) \subseteq \operatorname{supp} C\}$$

is a residual subset of E.

Typical applications of this Lemma are with S a compact subset of $\operatorname{supp} C$ or with $S = \operatorname{supp} C$ when C is separable.

The following is a rather evident property of the boundary of a convex body. The result is used in the proof of Proposition 2.

PROPOSITION. *Let E be an infinite dimensional Banach space. If C is a closed convex subset of E and* $\operatorname{int} C \neq \emptyset$, *then* $\operatorname{supp} C$ *is* LC^∞.

PROOF. Without loss of generality we can take $0 \in \operatorname{int} C$; since $\operatorname{int} C \neq \emptyset$ we have $\operatorname{supp} C = \operatorname{bdry} C$.

Let $q : E \to C$ be defined by

$$q(x) = \frac{x}{\mu(x)}$$

where μ is the Minkowski functional of C. If $z \in \operatorname{bdry} C$ and $\varepsilon > 0$, choose $\delta > 0$ such that

$$||q(x) - q(z)|| < \varepsilon$$

whenever $x \in B(z; \delta)$; this is possible since q is continuous at z.

Suppose $f : S^n \to B(z; \delta) \cap \operatorname{bdry} C$ is continuous. Define

$$H : [0,1] \times S^n \to B(z; \varepsilon) \cap \operatorname{bdry} C$$

by $H(t,s) = q(tz + (1-t)f(s))$. This map deforms f to the constant map $s \mapsto q(z) = z$ and the deformation takes place in $B(z;\varepsilon)$ since $tz + (1-t)f(s) \in B(z;\delta)$ for each $0 \le t \le 1$.

The following Theorem is a local version of Theorem 1 of [**2**].

THEOREM 1. *Let E be an infinite dimensional Banach space. If C is a boundedly weakly compact convex subset of E, then* $\operatorname{supp} C$ *is* LC^∞.

PROOF. If $\operatorname{int} C \neq \emptyset$, the conclusion follows by the Proposition. If $C = \operatorname{supp} C$, then $\operatorname{supp} C$ is convex and the result is obvious. Thus we may assume that $\operatorname{int} C = \emptyset$ and $0 \in C \backslash \operatorname{supp} C$.

Let $z \in \operatorname{supp} C$ and $\varepsilon > 0$; suppose

$$f : S^n \to B(z;\varepsilon) \cap \operatorname{supp} C$$

is continuous; let $S = f(S^n)$ and $\delta = \sup\{||z-a|| : a \in S\}$. Since S is compact the supremum is attained, so clearly $\delta < \varepsilon$.

Let $R(S)$ be as in the Lemma; then

$$B\Big(z;\frac{\varepsilon-\delta}{2}\Big) \cap R(S) \neq \emptyset.$$

Let x be an element of this intersection. We may assume [**6**; proof of Theorem 9] without loss of generality that the metric projection p is single-valued and continuous and that $p(E\backslash C) \subseteq \operatorname{supp} C$. Define the homotopy

$$H : [0,1] \times S^n \to B(z,\varepsilon) \cap \operatorname{supp} C$$

by $H(t,s) = p(tf(s) + (1-t)x)$; the map H deforms f to the constant map $s \mapsto p(x)$.

It only remains to show that the deformation takes place in $B(z;\varepsilon)$, all else being obvious.

We have

$$||x - z|| < \frac{\varepsilon - \delta}{2}$$

and $||z - a|| \le \delta$ for each $a \in S$.

Let $x_t = (1-t)x + ta$; then

$$||x_t - p(x_t)|| \le ||x_t - a|| = (1-t)||x - a||$$

and

$||x_t - z|| = ||(1-t)x + ta - ((1-t)z + tz)|| \leq (1-t)||x - z|| + t||a - z||.$

By combining the above inequalities with $||x - a|| \leq ||x - z|| + ||z - a||$ we obtain $||z - p(x_t)|| < \varepsilon$ which complete the proof.

The following Theorem summarizes our results for boundedly weakly compact sets. Note that in any reflexive Banach space this hypothesis can be replaced "C is closed."

THEOREM 2. *Let C be boundedly weakly compact convex subset of an infinite dimensional Banach space E. Then the following hold:*

(1) *If C contains no hyperplane, then* $\operatorname{supp} C$ *is arcwise connected;*

(2) *If C contains no linear variety of finite codimension, then* $\operatorname{supp} C$ *is C^∞;*

(3) *If C contains no linear variety of finite codimension, then the homotopy groups of* $\operatorname{supp} C$ *are all trivial; and*

(4) $\operatorname{supp} C$ *is* LC^∞.

PROOF. Parts (1) and (2) are proved in [**2**]; part (3)is known to be equivalent to (2) [**1**, p. 50], and part 4 is Theorem 1 of this paper.

The following Theorem generalizes Theorem 3 of [**2**] by replacing the hypothesis of "compactness" with that of "weakly compact and separable."

THEOREM 3. *If E is an infinite dimensional Banach space and C is a boundedly weakly compact convex and separable subset of E which contains no linear variety of finite codimension, then* $\operatorname{supp} C$ *is contractible.*

PROOF. If $\operatorname{int} C \neq \emptyset$, then $\operatorname{supp} C = \operatorname{bdry} C$ and it is known that $\operatorname{bdry} C$ is an AR(metric), if C contains no linear variety of finite codimension [**3**, Corollary 2]. If $C = \operatorname{supp} C$, then the conclusion is obvious, so we may assume that $0 \in C \backslash \operatorname{supp} C$.

Since C is separable metric, $\operatorname{supp} C$ is separable. Thus the Lemma is applicable with $S = \operatorname{supp} C$. Let $x \in R(S)$.

By considering $\operatorname{span} C = \operatorname{span}(C - C)$ we can suppose E is separable (since $C \neq \operatorname{supp} C$, we know $C \cap B[0;1]$ is total).

Since E is separable, we may assume E is locally uniformly convex and hence the metric projection $p : E \to C$ is single-valued and continuous.

Let $H : [0,1] \times \operatorname{supp} C \to \operatorname{supp} C$ be defined by $H(t,y) = p((1-t)y+tx)$. Then H deforms the identity on $\operatorname{supp} C$ to the constant map $y \mapsto p(x) \in \operatorname{supp} C$ (since $x \in E \backslash C$). The deformation takes place in $\operatorname{supp} C$ since $C \cap (y,x) \subseteq \operatorname{supp} C$ for each $y \in \operatorname{supp} C$.

COROLLARY. *If C is a closed convex bounded subset of the sequence space $\ell_p (1 < p < \infty)$, then* $\operatorname{supp} C$ *is contractible.*

A set A is finitely convex [**4**] if, given any finite subset S of A and any $\varepsilon > 0$, there is a continuous mapping ϕ from $\operatorname{conv} S$ into A such that $||\phi(x) - x|| \leq \varepsilon$ for each $x \in \operatorname{conv} S$. It if known [**5**] that if A is finitely convex, then $\operatorname{cl} A$ is convex.

If C is a closed convex subset of a Banach space and $A = \operatorname{supp} C \neq C$ is finitely convex, then $\operatorname{int} C = \emptyset$. Is $\operatorname{supp} C$ finitely convex, if $\operatorname{supp} C \neq C$ and $\operatorname{int} C = \emptyset$?

We state the following partial result in this direction. It can be proved by applying the Lemma inductively to the points of a partition of $[a_0, a_1]$ and using continuity of the metric projection.

PROPOSITION. Let C be boundedly weakly compact convex subset of an infinite dimensional Banach space E, and suppose $\operatorname{int} C = \emptyset$. Let $a_0, a_1 \in \operatorname{supp} C$ and set $a_t = (1-t)a_0 + ta_1$ for $0 \leq t \leq 1$. Then, for each $\varepsilon > 0$, there exists a continuous map $\phi : [0,1] \to \operatorname{supp} C$ such that $||\phi(t) - a_t|| < \varepsilon$.

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