ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 18, Number 1, Winter 1988

THE SUPPORT OF CERTAIN RIESZ PSEUDO-NORMS AND THE ORDER-BOUND TOPOLOGY

G.J.H.M. Buskes

ABSTRACT. In this paper we characterize those Riesz spaces E with weak order unit e for which the topology induced on E by the compact-open topology of $\{\varphi \in \mathbf{R}^E | \varphi(e) = 1$ and φ is a Riesz homomorphism$\}$ coincides with the order-bound topology.

Introduction. Suppose E is a Riesz space. The collection of all Riesz seminorms on E generates a locally solid topology on E, called the order-bound topology of E. The order-bound topology occurs very naturally in problems about Riesz spaces. However, to define the order-bound topology is far from understanding its structure. In case E is a Banach lattice, Goffman proved that the order-bound topology coincides with the norm topology [**7**]. Though there is no doubt that it was known that the order-bound topology of $C(X)$ coincides with the topology of uniform convergence on compact sets of the real compactification of X (see satz 4.10 of [**16**] combined with the results of [**9**] and [**12**]), Goffman's theorem seems to have been the only example of a concrete representation of the order-bound topology in the literature for a long time. It was only recently that the order-bound topology for certain function lattices was given explicitly as the topology of uniform convergence on compact sets of the spectrum [**4**], and in [**3**] a similar theorem was proved for complete ordinary function systems.

In this paper we give a unified approach to these problems. The main technique will be to define the notion of support for Riesz pseudonorms on a Riesz space with weak order unit. Implicitly this notion may be found in [**4**] and [**12**] and, explicitly for a special class of Riesz

1980 *Mathematics subject classifications* (Amer. Math. Soc.): 46A40
Received by the editors on November 10, 1985 and in revised form on April 22, 1986.
While writing this paper the author was supported by the Netherlands Organization for the Advancement of Pure Research (Z.W.O.).
The author wishes to thank P.G. Dodds for many valuable discussions on the subject matter.

seminorms on certain function lattices, in a fruitful paper by A.C.M. van Rooij [**15**]. Our main theorem gives necessary and sufficient conditions for the order-bound topology on a Riesz space with weak order unit to coincide with the compact-open topology on its spectrum.

In §1 we introduce the necessary preliminaries. The definition of support for Riesz pseudo-norms is given in §2 and in §3 we discuss how results in [**3**] and [**4**] may be derived from our main result.

Our general reference for the theory of Riesz spaces will be [**11**].

1. Preliminaries. In this paper E will denote an Archimedean Riesz space with weak order unit e. We define $E^* = \{f \in E \mid$ there exists $n \in \mathbf{N}$ such that $|f| \leq ne\}$ and $\Lambda = \{\varphi \in \mathbf{R}^{E^*} \mid \varphi$ is a Riesz homomorphism and $\varphi(e) = 1\}$. Furthermore, $\mathrm{Sp}(E) = \{\varphi \in \mathbf{R}^E \mid \varphi$ is a Riesz homomorphism and $\varphi(e) = 1\}$. $\mathrm{Sp}(E)$ is called the *spectrum of* E. It is clear that Λ is the spectrum of E^*. For all $f \in E^+$ and all $\varphi \in \Lambda$, define $\hat{f}(\varphi) = \sup_n \varphi(f \vee \mathrm{ne}) \in [0, \infty]$. For all $f \in E^{*+}$ and all $\varphi \in \Lambda, \hat{f}(\varphi) = \varphi(f)$. For any $f \in E^*$ we define $\hat{f}(\varphi) = \varphi(f)$for all $\varphi \in \Lambda$.

Suppose $g \in E^+$. The sequence $(f_n)_{n \in \mathbf{N}}$ of elements of E is said to *converge relatively uniformly to* $f \in E$ *with respect to* g (or g-uniformly), if there exists a sequence of positive numbers $(\varepsilon_n)_{n \in \mathbf{N}}$ with $\varepsilon_n \to 0$ such that $|f_n - f| \leq \varepsilon_n g$ for all $n \in \mathbf{N}$. Sometimes this is denoted with $f_n \to f(g)$. The sequence $(f_n)_{n \in \mathbf{N}}$ of elements of E is said to *converge relatively uniformly* to $f \in E$ if there exists $g \in E^+$ such that $f_n \to f(g)$. Every relatively uniformly convergent sequence has a unique limit [**11**; Theorem 63.2]. In a similar way g-uniform Cauchy sequences and relatively uniform Cauchy sequences are defined. E is said to be *uniformly complete* if, for every $g \in E^+$, every g-uniform Cauchy sequence has a g-uniform limit.

A *pseudo-norm* p on E is a function $p : E \to \mathbf{R}$ with

$$p(f) \geq 0, \quad \text{for all } f \in E, \tag{1}$$

$$p(f+g) \leq p(f) + p(g), \quad \text{for all } f, g \in E, \tag{2}$$

$$\lim_{\lambda \to 0} p(\lambda f) = 0, \quad \text{for all } f \in E. \tag{3}$$

A pseudo-norm p on E is called a *Riesz pseudo-norm* if, in addition, $p(f) \leq p(g)$ whenever $|f| \leq |g|$. A Riesz pseudo-norm p on E is called a *Riesz seminorm* if, for all $f \in E$ and all $\alpha \in \mathbf{R}, p(\alpha f) = |\alpha| p(f)$.

2. Compactly supported Riesz pseudo-norms. E^* has a natural norm, and, since Λ is a closed subset of the dual unit ball, it is weak*-closed and compact by Alaoglu's theorem. $E^{*\hat{}} = \{\hat{f} | f \in E^*\}$ is a norm dense Riesz subspace of $C(\Lambda)$.

LEMMA 2.1. *If A and B are disjoint closed subsets of Λ, then there exists $\hat{f} \in (E^{*\hat{}})^+$ such that $\hat{f} \equiv 0$ on A and $\hat{f} \equiv 1$ on B.*

PROOF. Suppose A and B are disjoint closed subsets of Λ. Then, by Urysohn's lemma, there exists $g \in C(\Lambda)$ such that $g \equiv -1$ on A and $g \equiv 2$ on B. Because $E^{*\hat{}}$ is a norm dense Riesz subspace of $C(\Lambda)$ we can find $\hat{h} \in E^{*\hat{}}$ such that $||\hat{h} - g||_\infty \leq 1$. It follows that $\hat{f} = (\hat{h} \wedge 1_\Lambda) \vee 0$ is as required.

Suppose $p : E \to \mathbf{R}$ is any Riesz pseudo-norm. Define $S \subset \Lambda$ to be a *weak support* for p if, for all $f \in E^{*+}, \hat{f}|_S = 0$ implies $p(f) = 0$. S is said to be a *support* for p if, for all $f \in E^{*+}, \hat{f}|_S = 0$ if and only if $p(f) = 0$. Remark that Lemma 2.1 implies that if p has a compact support, this compact support is unique. In the following, p will be a Riesz pseudo-norm such that $p(e) \neq 0$. Denote $\mathcal{Y}_p = \{S | S$ is a compact subset of Λ and S is a weak support for $p\}$. As $\Lambda \in \mathcal{Y}_p, \mathcal{Y}_p$ is nonempty.

It is not hard to see that if S_1 and S_2 are elements of $\mathcal{Y}_p$ then $S_1 \cap S_2 \neq \emptyset$: If $S_1 \cap S_2 = \emptyset$, choose, by Lemma 2.1, $f \in E^{*+}$ such that $\hat{f} = 0$ on A and $\hat{f} = 1$ on B and $f \leq e$; then $p(e) \leq p(e - f) + p(f) = 0$ which is a contradiction. However, Lemma 2.2 shows rather more, namely that $S_1 \cap S_2 \in \mathcal{Y}_p$ if $S_1 \in \mathcal{Y}_p$ and $S_2 \in \mathcal{Y}_p$.

LEMMA 2.2. *$\mathcal{Y}_p$ is closed under finite intersection.*

PROOF. Suppose $S_1, S_2 \in \mathcal{Y}_p$ and $f \in E^{*+}$ is such that $\hat{f}|_{S_1 \cap S_2} = 0$. Suppose $\varepsilon > 0$. Then $(f - \varepsilon e)^{\hat{}\,+} = 0$ on an open subset U of

Λ containing $S_1 \cap S_2$. Choose, by Lemma 2.1, $\hat{G} \in (E^{*\hat{}})^+$ such that $\hat{G}|_{S_1 \cap U^c} = 0$ and $\hat{G}|_{S_2} \geq \sup\{(f - \varepsilon e)^{+\hat{}}(2)|\lambda \in S_2\}$. Because $\hat{G} \wedge (f - \varepsilon e)^{+\hat{}} = (f - \varepsilon e)^{+\hat{}}$ on S_2, we get $p(G \wedge (f - \varepsilon e)^+) = p((f - \varepsilon e)^+)$ and because $\hat{G} \wedge (f - \varepsilon e)^{\hat{}} = 0$ on S_1, we get $p((f - \varepsilon e)^+) = 0$. Using property (3) of Riesz pseudo-norms (listed above) and the fact that $(f - \frac{1}{n}e)^+ \to f$ relatively uniformly, it is straightforward to show that $p(f) = 0$. Thus, $S_1 \cap S_2$ is a weak support for p.

Since $p(e) \neq 0$, it follows that $\emptyset \notin \mathcal{Y}_p$ and $S_\infty = \cap_{S \in \mathcal{Y}_p} S$ is nonempty and compact by Lemma 2.2. We will show that S_∞ is the compact support for p.

LEMMA 2.3. *S_∞ is the compact support for p.*

PROOF. Suppose $f \in E^{*+}$ is such that $\hat{f}|_{S_\infty} = 0$. Let $\varepsilon > 0$. $(f - \varepsilon e)^+ = 0$ on an open subset U containing S_∞. There exists a finite number of elements of $\mathcal{Y}_p$, say $S_1, \ldots, S_n$, such that $\cap_{k=1}^n S_k \subset U$. Since, by Lemma 2.2, $\cap_{k=1}^n S_k \in \mathcal{Y}_p$, it follows that $p((f - \varepsilon e)^+) = 0$. Using $(f - \frac{1}{n}e)^+ \to f$ relatively uniformly, combined with property (3) of Riesz pseudo-norms, leads to $p(f) = 0$. Thus, S_∞ is a weak support for p. Suppose $f \in E^{*+}, a \in S_\infty$ and $\hat{f}(a) = 1$ while $p(f) = 0$. Define $W = \{x \in \Lambda | \hat{f}(x) > \frac{1}{2}\}$. If $g \in E^{*+}$ and $\hat{g}|_{\Lambda \setminus W} = 0$, then $\hat{g} \leq 2||\hat{g}||_\infty 1_W \leq 2||\hat{g}||_\infty \hat{f}$ and thus $p(g) = 0$. Thus $\Lambda \setminus W$ is a weak support for p which is in contradiction with the fact that $a \in S_\infty$. Thus, S_∞ is the compact support for p.

All of the above results are about E^*. To be able to say more, we need a stronger assumption on p. Suppose that p is a nonzero Riesz pseudo-norm on E such that, for every $f \in E^+$, there exists $n \in \mathbf{N}$ such that $p((f - \mathrm{ne})^+) = 0$. It then follows that, for all $f \in E^+, p(f) = \sup p(f \wedge \mathrm{ne})$. Therefore, $p(e) \neq 0$ and Lemma 2.3 applies. Suppose $f \in E^+$ and $N \in \mathbf{N}$. We remark that, by definition, for all $\varphi \in \Lambda, \hat{f}(\varphi) = \sup_n \varphi(f \wedge \mathrm{ne})$. Hence $(\hat{f} \wedge N1_\Lambda)(\varphi) = (\sup_n \varphi(f \wedge \mathrm{ne})) \wedge N = \sup_n \varphi((f \wedge \mathrm{Ne}) \wedge \mathrm{ne}) = (f \wedge \mathrm{Ne})^{\hat{}}(\varphi)$ for all $\varphi \in \Lambda$. Hence, for all $f \in E^+$ and all $N \in \mathbf{N}, (\hat{f} \wedge N1_\Lambda) = (f \wedge \mathrm{Ne})^{\hat{}}$. This will be used in the following.

LEMMA 2.4. *Suppose p is a Riesz pseudo-norm with the property that, for each $f \in E^+$, there exists $n \in \mathbf{N}$ such that $p((f - ne)^+) = 0$. Let S_∞ be the compact support for p. Then, for all $f \in E^+, \hat{f}$ is uniformly bounded on S_∞.*

PROOF. Suppose $f \in E^+$ and let $p((f - \mathrm{ne})^+) = 0$. Assume $m \geq n$. Then $f - (f \wedge \mathrm{ne}) \geq (f \wedge \mathrm{me}) - (f \wedge \mathrm{ne}) \geq ((f \wedge \mathrm{me}) - \mathrm{ne})^+$. For all $m > n$ we have $((f \wedge \mathrm{me}) - \mathrm{ne})^+ \in E^*$ and $p(((f \wedge \mathrm{me}) - \mathrm{ne})^+) = 0$. Lemma 2.3 yields that $((f \wedge \mathrm{me}) - \mathrm{ne})^{+\hat{}}(x) = 0$ for all $x \in S_\infty$, i.e., $(f \wedge \mathrm{me})\hat{}(x) \leq n$ for all $x \in S_\infty$. With the remark preceding this lemma we see that $\hat{f}$ is uniformly bounded on S_∞.

Define $v(\Lambda) = \{\omega \in \Lambda | \hat{f}(\omega) < \infty$ for all $f \in E^+\}$. We have proved that S_∞ is a subset of $v(\Lambda)$. Of course, in case $E = C(X), v(\Lambda)$ coincides with the realcompactification of X. From [**3**; Lemma 21, p. 97 and Lemma 5, p. 132], we know that, for every $\omega \in v(\Lambda), f \to \hat{f}(\omega)(f \in E^+)$ can be extended to a Riesz homomorphism $E \to \mathbf{R}$. We denote the image of f under this Riesz homomorphism with $\hat{f}(\omega)$. Thus, there is a natural map $i : v(\Lambda) \to \mathrm{Sp}\,(E)$ defined by $i(\omega)(f) = \hat{f}(\omega)$ for all $\omega \in v(\Lambda)$ and all $f \in E$. Certainly i is injective. Moreover we will now show that i is surjective. Suppose $\phi \in \mathrm{Sp}\,(E)$. Since $\phi|_{E^*} \in \Lambda$, there exists $\lambda \in \Lambda$ such that $\varphi(f) = \hat{f}(\lambda)$ for all $f \in E^*$. One easily verifies that $\varphi(f) = \sup_n \varphi(f \wedge \mathrm{ne})$ for all $f \in E^+$, hence $\varphi(f) = \hat{f}(\lambda)$ for all $f \in E$.

Since the weak topology determined by the elements of E on $v(\Lambda)$ coincides with the restriction topology of Λ on $v(\Lambda)$, we see that i is a homeomorphism, if we equip $\mathrm{Sp}\,(E)$ with the weak topology determined by E.

It is well known that the existence of nonzero positive linear functionals on E is equivalent to the existence of nonzero Riesz seminorms. As a corollary of the above we observe the following

COROLLARY 2.5. *There exists a nonzero Riesz pseudo-norm p on E such that, for every $f \in E^+$, there exists $n \in \mathbf{N}$ such that*

$p((f - ne)^+) = 0$ if and only if $\mathrm{Sp}(E) \neq \emptyset$.

PROOF. Suppose $\mathrm{Sp}(E) \neq \emptyset$. Choose $\varphi \in \mathrm{Sp}(E)$ and define $p : E \to \mathbf{R}$ by $p(f) = \varphi(|f|)(f \in E)$. For every $f \in E^+, p((f - \varphi(e)f)^+) = \varphi((f - \varphi(e)f)^+) = 0$. Conversely, suppose p is a nonzero Riesz pseudo-norm on E such that, for every $f \in E^+$, there exists an $n \in \mathbf{N}$ such that $p((f - \mathrm{ne})^+) = 0$. By Lemma 2.3, p has a compact nonempty support, and, by Lemma 2.4, this support is a subset of $v(\Lambda)$. By the remarks made just before this corollary, every element of the compact support for p determines an element of $\mathrm{Sp}(E)$. In particular, $\mathrm{Sp}(E) \neq \emptyset$.

For $f \in E$ and $\varphi \in \mathrm{Sp}(E)$, we define $\check{f}(\varphi) = \hat{f}(i^{-1}(\varphi))(= \varphi(f))$. We come to our main theorem.

THEOREM 2.6. *For every nonzero Riesz pseudo-norm p on E the following are equivalent.*

(1) *For every $f \in E^+$ there exists $n \in \mathbf{N}$ such that $p((f - ne)^+) = 0$.*

(2) *There exists a compact nonempty set $A \subset \mathrm{Sp}(E)$ such that $p(f) = 0$ if and only if $\check{f}|_A = 0$ for all $f \in E^+$.*

PROOF. (2)$\Rightarrow$(1) is obvious. Suppose (1). By Lemma 2.4 and the discussion following Lemma 2.4, we get a compact nonempty set A (namely $i(S_\infty)$, where S_∞ is the compact support for p) such that, for all $f \in E^{*+}, p(f) = 0$ if and only if $\check{f}|_A = 0$. Suppose $f \in E^+$ and $p(f) = 0$. Then, for all $n \in \mathbf{N}, p(f \wedge \mathrm{ne}) = 0$; hence, for all $n \in \mathbf{N}, (f \wedge \mathrm{ne})\check{}|_A = 0$. It follows that $\check{f}|_A = 0$.
Conversely, suppose $\check{f}|_A = 0$. Then $p(f \wedge \mathrm{ne}) = 0$ for all $n \in \mathbf{N}$, and by using (1) once again $p(f) = 0$.

3. The order-bound topology. The collection of all Riesz seminorms on E generates a locally convex and locally solid (see [**1**]) topology on E. This topology is called *the order-bound topology.* Unfortunately, at least four names are in use for this topology (see [**2, 3, 4, 10, 14** and **16**]). Every nonempty compact subset A of $\mathrm{Sp}(E)$ determines a nonzero Riesz seminorm p_A on E defined by

$p_A(f) = \sup_{x\in A} |\check{f}(x)|$. The locally convex and locally solid topology generated by $\{p_A | A \subset \mathrm{Sp}\,(E), A$ is compact and nonempty$\} \cup \{$the zero Riesz seminorm$\}$ is called the compact-open topology on E.

THEOREM 3.1. *The compact-open topology on E coincides with the order bound topology on E if and only if, for every Riesz seminorm p and for every $f \in E^+$, there exists $n \in \mathbf{N}$ such that $p((f - ne)^+) = 0$.*

PROOF. Suppose that, for every Riesz seminorm p and for every $f \in E^+$, there exists $n \in \mathbf{N}$ such that $p((f - \mathrm{ne})^+) = 0$. Of course, the order-bound topology on E is finer than the compact-open topology on E. If there is no nonzero Riesz seminorm, there is nothing to prove. Therefore, suppose p is a nonzero Riesz seminorm on E. Since, by assumption, it has property (1) of Theorem 2.6, there exists a nonempty compact set $A \subset \mathrm{Sp}\,(E)$ such that $p(f) = 0$ if and only if $\check{f}|_A = 0$ for all $f \in E^+$. For every $f \in E^+$, $f \leq p_A(f)e + (f - p_A(f)e)^+$ and $(f - p_A(f)e)^+|_A = 0$. Hence $p(f) \leq p(e)p_A(f)$ for all $f \in E$ and $p \leq p(e)p_A$.

Conversely, suppose the compact-open topology on E coincides with the order-bound topology on E. If there exists no nonzero Riesz seminorm, again there is nothing to prove. Suppose p is a nonzero Riesz seminorm on E. Choose a compact nonempty set $A \subset \mathrm{Sp}\,(E)$ and a number $C \in \mathbf{R}$ such that $p \leq Cp_A$. Suppose $f \in E^+$. There exists $n \in \mathbf{N}$ (for instance the first natural number bigger than $p_A(f)$) such that $p((f - \mathrm{ne})^+) \leq Cp_A((f - \mathrm{ne})^+) = 0$.

In many concrete examples the condition of Theorem 3.1 on all Riesz seminorms is quite easily checked. Lemma 3.2 is a convenient reformulation of this condition. Its proof is left to the reader.

LEMMA 3.2. *The following conditions are equivalent.*

(1) *For all $f \in E^+$ and all Riesz seminorms p on E, there exists $n \in \mathbf{N}$ such that $p((f - ne)^+) = 0$.*

(2) *For all $f \in E^+$ and all sequences $(\lambda_n)_{n\in\mathbf{N}}$ of real numbers $\{\lambda_n(f - ne)^+ | n \in \mathbf{N}\}$ is bounded for the order-bound topology.*

In fact, in §4, we will only use the following corollary.

COROLLARY 3.3. *If, for all $f \in E^+$ and all sequences $(\lambda_n)_{n\in\mathbf{N}}$ of real numbers with $\lambda_n \uparrow \infty, \{\lambda_n(f - ne)^+ | n \in \mathbf{N}\}$ is order-bounded (i.e., is dominated by an element of E^+), then the compact-open topology on E coincides with the order-bound topology on E.*

4. Examples and counterexamples.

EXAMPLE 4.1. Suppose X is a completely regular topological space and $E = C(X)$. For every $f \in E^+$ and every sequence $(\lambda_n)_{n\in\mathbf{N}}$ of real numbers with $\lambda_n \uparrow \infty, \{\lambda_n(f - \mathrm{ne})^+ | n \in \mathbf{N}\}$ is order bounded. Sp(E) equals the realcompactification of X. From Corollary 3.3 it follows that the order-bound topology on E coincides with the topology of uniform convergence on compacta of the real compactification of X.

In [3] the result of Example 4.1 was extended to a more general setting. We will now show the relation with Theorem 3.1.

EXAMPLE 4.2. Suppose X is a set and $E \subset \mathbf{R}^X$. We say that E is 1_X-uniformly closed if, for every sequence $(f_n)_{n\in\mathbf{N}}$ of elements of E and for every $f \in \mathbf{R}^X$ such that $f_n \to f$ 1_X-uniformly, $f \in E$. We say that E is closed under inversion if, for all $f \in E$ with $f(x) > 0$ for all $x \in X$, $1/f \in E$. We assume that E is a Riesz subspace of $\mathbf{R}^X$, contains the constants, is 1_X-uniformly closed and is closed under inversion, (i.e., E is a *complete ordinary function system*). We can then prove the fact: *If $f \in E^+$ and $\omega \in C[0,\infty)^+$ is an increasing continuous function, then $\omega \circ f \in E^+$.*

PROOF OF THE FACT. It suffices to show that $(\omega + 1) \circ f \in E^+$. So we may assume that ω is an increasing continuous function with values in $(1, \infty)$. By the inversion property, it suffices to prove that $1/(\omega \circ f) \in E$. For all $n \in \mathbf{N}$, we have $f \wedge n1_X \in E^*$, and E^* is uniformly complete (and hence E^* is Riesz isomorphic with $C(\Lambda)$). It follows that $1/(\omega \circ (f \wedge n1_X)) \in E^*$. Let $n \in \mathbf{N}$. Then

$$(1/(\omega\circ(f\wedge n1_X))-1/(\omega\circ f))(x) = \begin{cases} 0 & \text{if } f(x) \leq n \\ 1/\omega(n) - 1/(\omega \circ f)(x) & \text{if } f(x) > n, \end{cases}$$

and if $f(x) > n$, then

$$1/\omega(n) - 1/(\omega \circ f)(x) \leq 1/\omega(n) - \inf_{y \geq n} 1/\omega(y).$$

So

$$\sup_{x\in X}(1/(\omega\circ(f\wedge n1_X))-1/(\omega\circ f))(x) \leq (1/\omega(n)-\inf_{y\geq n} 1/\omega(y)) \to 0 \text{ if } n \to \infty$$

and

$$1/(\omega \circ (f \wedge n1_X)) \to 1/(\omega \circ f)$$

1_X-relatively uniformly, thus $1/(\omega \circ f) \in E$.

We remark that it was Hausdorff who called a sublattice E of $\mathbf{R}^X$, with the properties that we have used above, a complete ordinary function system. These systems were extensively studied by Mauldin in [**13**]. Of course, every $C(X)$ is a complete ordinary function system. From Theorem 3.1 of [**13**] we know that, for every Riesz subspace E of $\mathbf{R}^X$ containing the constants, the Baire functions of the first class, $\mathcal{B}^1(E)$, is a complete ordinary function system.

If E is a complete ordinary function system, $f \in E^+$ and $(\lambda_n)_{n\in\mathbf{N}}$ is a sequence of real numbers such that $\lambda_n \uparrow \infty$, it follows that $\{\lambda_n(f - n1_X)^+ | n \in \mathbf{N}\}$ is order bounded by applying the above fact to $\omega : [0,\infty) \to [0,\infty)$, defined by $\omega(t) = \sup_n \lambda_n(t - n)^+ (t \in [0,\infty))$. Using Corollary 3.3 we see that the order-bound topology of a complete ordinary function system coincides with its compact-open topology.

We mention here that it is possible to show that if E is a Riesz subspace of $\mathbf{R}^X$ containing the constants, then the order-bound topology on $\mathcal{B}'(E)$ coincides with the product topology induced by $\mathrm{Sp}\,(\mathcal{B}^1(E))$. From there it is easy to show each positive linear functional on $\mathcal{B}^1(E)$ is a finite linear combination of Riesz homomorphisms. However, this generalization of a theorem by C.T. Tucker [**17**] has already been proved in even greater generality by A.C.M. van Rooij [**15**]. We refer the reader to the latter paper in which many other interesting facts can be found as well.

In [4], the result of Example 4.1 was extended to another context. We will show again that a result in [4] is a consequence of Theorem 3.1.

EXAMPLE 4.3. Suppose E is a Riesz space with weak order unit e. E is said to be 2-universally complete if, for any 2-disjoint subset $\{v_n|n \in \mathbf{N}\}$ (i.e., for each $n, v_n \wedge v_m \neq 0$ for at most two $m \neq n$; see [4]) of E^+ such that, for any $\varphi \in \operatorname{Sp}(E)$, there exists $n \in \mathbf{N}$ with $\varphi(v_n) \neq 0$, $\sup\{v_n|n \in \mathbf{N}\}$ exists. Suppose E is a 2-universally complete Riesz space with weak order unit e such that there exists a separating family of real-valued Riesz homomorphisms on E. To show that the order-bound topology on E coincides with the compact-open topology on E, take $f \in E^+$ and $(\lambda_n)_{n\in\mathbf{N}}$ a sequence of real numbers such that $\lambda_n \uparrow \infty$. Define, for each $n \in \mathbf{N}$ the following subsets of $\Lambda = \operatorname{Sp}(E^*)$. $A_n = \{x \in \Lambda | n-1 \leq (f \wedge (n+1)e)\hat{}(x) \leq n\}, B_n = \{x \in \Lambda | (f \wedge \mathrm{n}e)\hat{}(x) \leq n - 5/4\}$ and $C_n = \{x \in \Lambda | (f \wedge (n+1)e)\hat{}(x) \geq n + 1/4\}$. By Lemma 2.1 there exists, for each $n \in \mathbf{N}, \hat{f}_n \in (E^{*\hat{}})^+$ such that $\hat{f}_n|_{A_n} = \lambda_n(n+1)$ and $\hat{f}_n|_{B_n \cup C_n} = 0$. Because E is 2-universally complete, $\sup\{f_n|n \in \mathbf{N}\}$ exists and, for each $n \in \mathbf{N}, \lambda_n(f - \mathrm{n}e)^+ \leq \sup\{f_n|n \in \mathbf{N}\}$. Now apply Corollary 3.3.

Example 4.4 will show that our result is in fact a proper extension of the results in [3] and [4].

EXAMPLE 4.4. Let $\mathbf{Z}$ be the set of integers. Define E to be the subset of $\mathbf{R}^{\mathbf{Z}}$ consisting of those elements $f \in \mathbf{R}^{\mathbf{Z}}$ for which there exists an $N \in \mathbf{N}$ such that $f(n) = f(-n)$ for all $n \geq N$. E is a Riesz subspace of $\mathbf{R}^{\mathbf{Z}}$ and $1_{\mathbf{Z}} =: e$ is a weak order unit in E. Remark that $\operatorname{Sp}(E) = \mathbf{Z}$. Suppose $f \in E^+$. Then $z \to \sup_n \lambda_n(f - \mathrm{n}e)^+(z)$ is an element of E for all sequences of real positive numbers $(\lambda_n)_{n\in\mathbf{N}}$ with $\lambda_n \uparrow \infty$. It follows that the order-bound topology on E coincides with the product topology. However, E is not 2-universally complete nor a complete ordinary function system.

EXAMPLE 4.5. In Example 3 of [**4**] a function on the reals is called ultimately a polynomial if it is continuous and if it is equal to a polynomial on the complement of $[-n, n]$ for some $n \in \mathbf{N}$. Let E be the solid hull in $C(\mathbf{R})$ of the functions which are ultimately polynomial. It is easy to check that E is a uniformly complete Riesz space. In [**4**] the authors show that E is not 2-universally complete. Considering the Riesz seminorm $p : E \to \mathbf{R}$ defined by $p(f) = \int_{-\infty}^{\infty} |f(t)|(e^{-t} \wedge e^{t})dt$ we see that the order-bound topology does not coincide with the compact-open topology on E.

Finally, we wish to mention that, also, Nachbin's result [**12**] can be seen as a corollary of Theorem 3.1 by using a theorem by Schaefer (satz 4.5 of [**16**]). The details, however, we leave to the interested reader.

REFERENCES

1. C.D. Aliprantis and O. Burkinshaw, *Locally solid Riesz space,* Academic Press, New York, 1978.

2. G. Birkhoff, *Lattice Theory,* Am. Math. Soc. Coll. Publ. 25, New York, 1948.

3. Gerard Buskes, *Extension of Riesz Homomorphisms,* thesis, Nijmegen, 1983.

4. W.A. Feldman and J.F. Porter, *The order topology for function lattices and realcompactness,* Internat. J. Math. and Math. Sci. **4** (1981), 289-304.

5. D.H. Fremlin, *Topological Riesz spaces and measure theory,* Cambr. Univ. Press, London-New York, 1974.

6. L. Gillman and M. Jerison, *Rings of continuous functions,* Springer Verlag, New York, 1976.

7. C. Goffman, *Compatible seminorms in a vector lattice,* Proc. Nat. Acad. Sci. U.S.A., **42** (1956), 536-538.

8. H. Gordon, *Relative uniform convergence,* Math. Annalen, **153**, (1964), 418-427.

9. G. Gould and M. Mahowald, *Measures on completely regular spaces,* J. London Math. Soc. **37** (1962), 103-111.

10. J.L. Kelley, I. Namioka et al., *Linear topological spaces,* Berlin, Springer Verlag, 1976.

11. W.A.J. Luxemburg and A.C. Zaanen, *Riesz spaces* I, North Holland Publishing Company, Amsterdam-London, 1971.

12. L. Nachbin, *Topological vector spaces of continuous functions,* Proc. N.A.S. **40** (1954), 471-474.

13. R.R. Mauldin, *Baire functions, Borel sets and ordinary function systems,* Advances Math. **12**, (1974), 418-450.

14. I. Namioka, *Partially ordered linear topological spaces,* Memoir of the Math. Soc. number 24, 1957.

15. A.C.M. van Rooij, *Riesz duals of certain spaces of functions,* Report KUN no. 8209, (1982).

16. H.H. Schaefer, *Halbgeordnete lokalkonvexe Vektorräume,* Mathematische Annalen, Note I **135** (1958), 115-141.

17. C.T. Tucker, *Positive operators on spaces of Baire functions,* III. J. Math. **69** (1977), 551-556.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF MISSISSIPPI, UNIVERSITY, MS 38677.