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NOTES ON THE ANALYTIC YEH-FEYNMAN INTEGRABLE FUNCTIONALS

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ABSTRACT. In this paper we extend Johnson and Skoug's results involving the analytic Feynman integrable functionals on Wiener space to the analytic Yeh-Feynman integrable functionals on Yeh-Wiener space. To do this we define the analytic Yeh-Feynman integral and find a Banach algebra of some Yeh-Feynman integrable functionals. Also we find formulae for the analytic Yeh-Feynman integral and extend some measurability results involving the Wiener measure to the Yeh-Wiener measure.

1. Introduction. In [**1**], Cameron and Storvick treat a Banach algebra $S(L_2[a,b])$ of functionals on Wiener space which are a kind of stochastic Fourier transform of Borel measures on $L_2[a,b]$. Here $L_2[a,b]$ denotes the space of Lebesgue measurable, square integrable functions on $[a,b]$. For such functionals they show that the analytic Feynman integral, defined by analytic continuation of the Wiener integral, exists, and they give formulae for this Feynman integral. In a recent paper [**7**], Johnson and Skoug extend somewhat and simplify substantially some of Cameron and Storvick's results in [**1**].

The main purpose of this paper is to extend Theorem 1 in [**7**] involving the analytic Feynman integrable functionals on Wiener space to the analytic Yeh-Feynman integrable functionals on Yeh-Wiener space. Let **R** and **C** denote the real and complex numbers respectively. Let $C_2(Q)$ denote the Yeh-Wiener space, that is, the space of **R**-valued continuous functions x on $Q = [a,b] \times [\alpha,\beta]$ for some fixed real numbers a and b, and α and β such that $x(a,v) = x(u,\alpha) = 0$ for all $a \leq u \leq b$ and $\alpha \leq v \leq \beta$. In this paper we shall always denote the above

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rectangle by the symbol Q. Let $(C_2(Q), Y, m_y)$ be the Yeh-Wiener measure space. For a complete discussion of Yeh-Wiener measure space, see [**8**]. To obtain the main theorem we define the analytic Yeh-Feynman integral and find a Banach algebra $S(L_2(Q))$ of Yeh-Feynman integrable functionals, which are a kind of stochastic Fourier transform of Borel measures on $L_2(Q)$ where $L_2(Q)$ denotes the space of Lebesgue measurable, square integrable functions on Q. Also we find formulae for the analytic Yeh-Feynman integral and obtain the measurability lemmas involving the Yeh-Wiener measure which are the extensions of the corresponding results in [**7**].

2. Definitions and some results. A subset B of Yeh-Wiener space is said to be scale-invariant measurable if ρB is Yeh-Wiener measurable for every $\rho > 0$. A scale-invariant measurable set N is said to be scale-invariant null if $m_y(\rho N) = 0$ for every $\rho > 0$. A property which holds except on a scale-invariant null set is said to hold scale-invariant almost everywhere ($s - a.e.$). The class of scale-invariant measurable sets form a σ-algebra [**3**; Proposition 3.2]. A function F on $C_2(Q)$ is said to be scale-invariant measurable if it is measurable with respect to this σ-algebra. In this paper we shall use a definition of the analytic Yeh-Feynman integral which is similar to that used in [**1**].

DEFINITION 2.1. Let F be a functional which is scale-invariant measurable and $s - a.e.$ defined and which is such that the Yeh-Wiener integral

$$J(\lambda) \equiv \int_{C_2(Q)} F(\lambda^{1/2}x)dm_y(x) \tag{2.1}$$

exists for all $\lambda > 0$. If there exists a function $J^*(\lambda)$ analytic in $\mathbf{C}^+ = \{\lambda \text{ in } \mathbf{C} : \operatorname{Re} \lambda > 0\}$ such that $J^*(\lambda) = J(\lambda)$ for all $\lambda > 0$, then $J^*(\lambda)$ is defined to be analytic Yeh-Wiener integral of F over $C_2(Q)$ with parameter λ, and, for λ in $\mathbf{C}^+$, we write

$$\int_{C_2(Q)}^{\text{any } w_\lambda} F(x)dm_y(x) \equiv J^*(\lambda). \tag{2.2}$$

DEFINITION 2.2. Let q be a nonzero real parameter and let F be a functional whose analytic Yeh-Wiener integral exists for λ in $\mathbf{C}^+$. If the following limit exists, we call it the analytic Yeh-Feynman integral of F over $C_2(Q)$ with parameter q and we write

$$\int_{C_2(Q)}^{\text{any } F_q} F(x)dm_y(x) \equiv \lim_{\substack{\lambda \to -iq \\ \lambda \in C^+}} \int_{C_2(Q)}^{\text{any } w_\lambda} F(x)dm_y(x). \tag{2.3}$$

We shall say that two functions $F(x)$ and $G(x)$ are equal $s-a.e.$, denoted by $F \sim G$, if, for each $\rho > 0$, the equation $F(\rho x) = G(\rho x)$ holds for $m_y - a.e.$ $x \in C_2(Q)$. Equality $s - a.e.$ is an equivalence relation. It is the appropriate relation for the analytic Yeh-Feynman integral. For example, let $F \equiv 1$ and $G(x) = \chi_{C_1}(x)$ for $x \in C_2(Q)$, where, for $\lambda > 0$, C_λ are disjoint Borel sets in $C_2(Q)$ such that $m_y(C_1) = 1$ and $\nu C_\lambda = C_{\nu\lambda}$ for $\nu > 0$ [**3**]. Then F and G are Borel measurable on $C_2(Q)$ and equal $m_y - a.e.$ Here the analytic Yeh-Feynman integral of F exists but the analytic Yeh-Feynman integral of G does not exist.

We now define the Paley-Wiener-Zygmund (P.W.Z.) integral for functions of two variables which is a simple type of stochastic integral.

DEFINITION 2.3. Let $\{\phi_j\}$ be a complete orthonormal (C.O.N.) set of real valued functions of bounded variation on Q. For v in $L_2(Q)$, let

$$v_n(s,t) = \sum_{j=1}^{n} \left(\int_Q v(p,q)\phi_j(p,q)dpdq \right) \phi_j(s,t). \tag{2.4}$$

The P.W.Z. integral with two parameters is defined by the formula

$$\int_Q v(s,t)\tilde{d}x(s,t) \equiv \lim_{n\to\infty} \int_Q v_n(s,t)dx(s,t) \tag{2.5}$$

for all x in $C_2(Q)$ for which the limit exists.

Some useful facts about the P.W.Z. integral with two parameters are listed in [**4**].

Let ϕ be the map from $Q \times \mathbf{R}$ into $L_2(Q)$ defined by

$$\phi((s,t),v)(c,e) = \begin{cases} v & \text{for } (c,e) \in [a,s) \times [\alpha,t) \\ 0 & \text{otherwise.} \end{cases} \tag{2.6}$$

Then it is easy to show that ϕ is continuous and so is Borel measurable.

We obtain the following two measurability Lemmas 2.1 and 2.2 as the extensions of Lemmas 1 and 2 in [**7**], respectively. Here we prove the first lemma and omit the proof of the second lemma which is much like that of Lemma 2 in [**7**].

LEMMA 2.1. *$\int_\phi((s,t),v)(c,e)\tilde{d}x(c,e)$ is a Borel measurable function of $((s,t),v,x)$ on $Q \times \mathbf{R} \times C_2(Q)$. Moreover, for any Borel measure μ on $Q \times \mathbf{R}$, $\int_Q \phi((s,t),v)(c,e)\tilde{d}x(c,e)$ is defined except on a $\mu \times m_y$-null Borel set.*

PROOF. Let $\{\phi_n\}$ be a C.O.N. set of real valued functions of bounded variation on Q. Then, by "integration by parts formula with respect to two variables" [**6**], we have

(2.7)

$$\begin{aligned}
&\left(\int_Q \phi((s,t),v)(p,q)\phi_n(p,q)dpdq\right)\int_Q \phi_n(c,e)dx(c,e) \\
&= (v\int_\alpha^t\int_a^s \phi_n(p,q)dpdq)(-\phi_n(b,\beta)x(b,\beta) + \phi_n(a,\alpha)x(a,\alpha) \\
&+ \phi_n(b,\alpha)x(b,\alpha) - \phi_n(a,\alpha)x(a,\alpha) - \int_\alpha^\beta \phi_n(a,t)dx(a,t) \\
&+ \int_\alpha^\beta \phi_n(b,t)dx(b,t) - \int_a^b \phi_n(s,\alpha)dx(s,\alpha) + \int_a^b \phi_n(s,\beta)dx(s,\beta) \\
&+ \int_Q x(s,t)d\phi_n(s,t))
\end{aligned}$$

since x is continuous and ϕ_n is of bounded variation on Q. Thus the left hand side of (2.7) is continuous and hence Borel measurable function of $((s,t),v,x)$. Since $\int_Q \phi((s,t),v)(c,e)\tilde{d}x(c,e)$ is defined as

$$\lim_{k\to\infty}\sum_{n=1}^{k}(\int_Q \phi((s,t),v)(p,q)\phi_n(p,q)dpdq)\int_Q \phi_n(c,e)dx(c,e),$$

for all x in $C_2(Q)$ for which the limit exists, we see that $\int_Q \phi((s,t),v)(c,e)$ $\tilde{d}x(c,e)$ is a Borel measurable function of $((s,t),v,x)$.

For every ψ in $L_2(Q)$, $\int_Q \psi(s,t)\tilde{d}x(s,t)$ exists for $m_y-a.e.x$. Thus, for each $((s,t),v)$ in $Q \times \mathbf{R}$, $\int_Q \phi((s,t),v)(c,e)\tilde{d}x(c,e)$ exists for $m_y - a.e.x$. Let μ be any Borel measure on $Q \times \mathbf{R}$. Then, as in the proof of Lemma 2.1 in [**1**], we have that $\int_Q \phi((s,t),v)(c,e)\tilde{d}x(c,e)$ is defined except on a $\mu \times m_y$-null Borel set in $Q \times \mathbf{R} \times C_2(Q)$.

LEMMA 2.2. *Let μ be any Borel measure on $Q \times \mathbf{R}$. Then the P.W.Z. integral* $\int_Q \phi((s,t),v)(c,e)\tilde{d}x(c,e)$ *and the Riemann-Stieltjes integral* $\int_Q \phi((s,t),v)(c,e)dx(c,e)$ *are equal except on a* $\mu \times m_y$*-null Borel set in* $Q \times \mathbf{R} \times C_2(Q)$. *Thus for* m_y *– a.e.* x *they are equal except on a* μ*-null Borel set in* $Q \times \mathbf{R}$.

3. Analytic Yeh-Feynman integrable functionals. Cameron and Storvick introduce a Banach algebra $S(L_2[a,b])$ of functionals on Wiener space and prove the existence of the analytic Feynman integral for every element of $S(L_2[a,b])$ and also evaluate this Feynman integral in terms of formulae that do not involve analytic continuation [**1**]. To obtain Theorem 3.1 which is an extension of [**7**; Theorem 1] we first extend some of Cameron and Storvick's results in [**1**].

Let $\mathcal{M}(L_2(Q))$ be the collection of complex measures defined on $\mathcal{B}(L_2(Q))$, the Borel class of $L_2(Q)$. For $\mu \in \mathcal{M}(L_2(Q))$, we set $||\mu|| = \operatorname{var} \mu$ over $L_2(Q)$.

DEFINITION 3.1. Let $S(L_2(Q))$ be the space of functionals F expressible in the form

$$F(x) \approx \int_{L_2(Q)} \exp\{i \int_Q v(s,t)\tilde{d}x(s,t))d\mu(v) \tag{3.1}$$

where μ is an element of $\mathcal{M}(L_2(Q))$.

LEMMA 3.1. *If* $F \in S(L_2(Q))$, *then the measure* μ *is uniquely determined by equation* (3.1).

The above lemma is a modification of [**1**; Theorem 2.1] which can be easily obtained using the basic Yeh-Wiener integration formula [**4**], the Fubini theorem, and the dominated convergence theorem.

DEFINITION 3.2. If $F \in S(L_2(Q))$, we define the norm of F by $||F|| = ||\mu||$ where μ is associated with F by (3.1).

It follows from Lemma 3.1 that $||F||$ is uniquely determined by F. As in the proofs of Theorems 2.2 and 2.3 of [**1**], we have that $S(L_2(Q))$ is a Banach algebra. For a complete discussion of the proof above, see [**2**].

The following two Propositions 3.1 and 3.2 are the corresponding modifications of Theorems 5.1 and 5.4 in [**1**], respectively. Since the proofs in our setting remain the same, we can easily obtain them [**2**].

PROPOSITION 3.1. *Let* $\mu \in \mathcal{M}(L_2(Q))$ *and let* $F \in S(L_2(Q))$ *be the stochastic Fourier transformation of* μ, *thus*

$$F(x) = \int_{L_2(Q)} \exp\{i \int_Q v(s,t)\tilde{d}x(s,t)\} d\mu(v). \tag{3.2}$$

Then F *is analytic Yeh-Feynman integrable on* $C_2(Q)$, *and if* q *is a nonzero real number,*

$$\int_{C_2(Q)}^{\text{any } f_q} F(x) dm_y(x) = \int_{L_2(Q)} \exp\{\frac{1}{2qi} \int_Q (v(s,t))^2 dsdt\} d\mu(v). \tag{3.3}$$

PROPOSITION 3.2. *Let* $F_n \in S(L_2(Q))$ *for* $n = 1, 2, \cdots$, *and*

$$\sum_{n=1}^{\infty} ||F_n|| < \infty. \tag{3.4}$$

Then $F \in S(L_2(Q))$ *where, for* $m_y - a.e.$ $x \in C_2(Q)$,

$$F(x) = \sum_{n=1}^{\infty} F_n(x), \tag{3.5}$$

and

$$\int_{C_2(Q)}^{\text{any } f_q} F(x) dm_y(x) = \sum_{n=1}^{\infty} \int_{C_2(Q)}^{\text{any } f_q} F_n(x) dm_y(x). \tag{3.6}$$

The following proposition which is a modification of Lemma 3 in [**7**] plays a key role in proving Theorem 3.1. It can be easily obtained using Lemma 2.2, the linearity of the P.W.Z. integral, and the change of variables theorem [**5**; P. 163].

PROPOSITION 3.3. *Let* μ *be a Borel measure on* $Q \times \mathbf{R}$. *Define* G *on* $C_2(Q)$ *by*

$$G(x) = \int_{Q \times R} \exp\{ivx(s,t)\} d\mu((s,t), v). \tag{3.7}$$

Then G is in $S(L_2(Q))$.

Finally we give the main result.

THEOREM 3.1. *Let θ be a complex-valued function on $Q \times \mathbf{R}$ defined by*

$$\theta((s,t),u) = \int_R \exp\{iuv\} d\sigma_{(s,t)}(v), \tag{3.8}$$

where $\{\sigma_{(s,t)} : a \leq s \leq b, \alpha \leq t \leq \beta\}$ is a complex measure of finite variation defined on $\mathcal{B}(\mathbf{R})$satisfying the following two conditions:

(3.9a) For each Borel set E in $Q \times \mathbf{R}$, $\sigma_{(s,t)}(E^{(s,t)})$ is a Borel measurable function of (s,t) where $E^{(s,t)}$ denotes the (s,t) – section of E.

$$||\sigma(s,t)|| \in L_1(Q). \tag{3.9b}$$

Then the function F on $C_2(Q)$ defined by

$$F(x) = \exp\{\int_Q \theta((s,t), x(s,t))dsdt\} \tag{3.10}$$

is in $S(L_2(Q))$, and thus F is analytic Yeh-Feynman integrable on $C_2(Q)$.

REMARK. In [**7**; Theorem 1], the condition that $||\sigma_t||$ is dominated by a function $h(t)$ in $L_1[a,b]$ can be modified as follows:

$$||\sigma_{(s,t)}|| \leq h(s,t) \in L_1(Q). \tag{3.9b)'}$$

Actually the condition (3.9b) is formally weaker than, but equivalent to the condition (3.9b)′

PROOF OF THEOREM 3.1. Since $S(L_2(Q))$ is a Banach algebra, it suffices to show that the function

$$f(x) = \int_Q \theta((s,t), x(s,t))dsdt \tag{3.11}$$

is in $S(L_2(Q))$. For each Borel set E in $Q \times \mathbf{R}$, let

$$\mu(E) = \int_Q \sigma_{(s,t)}(E^{(s,t)})dsdt. \tag{3.12}$$

Then we easily have that μ is a Borel measure on $Q \times \mathbf{R}$ with $||\mu|| < \infty$, by (3.9b), (3.12) and the dominated convergence theorem. By Proposition 3.3, (3.11) and (3.8), it suffices to show that

$$\begin{aligned}&\int_Q \left(\int_R \exp\{ivx(s,t)\}d\sigma_{(s,t)}(v)\right)dsdt \\ &= \int_{Q\times R} \exp\{ivx(s,t)\}d\mu((s,t),v)\end{aligned} \tag{3.13}$$

Now, to prove (3.13), we show, for any bounded Borel measurable function ϕ on $Q \times \mathbf{R}$, $\int_R \phi((s,t),v)d\sigma_{(s,t)}(v)$ is a measurable function of (s,t) and

$$\begin{aligned}&\int_Q \left(\int_R \phi((s,t),v)d\sigma_{(s,t)}(v)\right)dsdt \\ &= \int_{Q\times R} \phi((s,t),v)d\mu((s,t),v).\end{aligned} \tag{3.14}$$

Let us consider the case where $\phi = \chi_E$ for some Borel set E in $Q \times \mathbf{R}$. Then

$$\int_R \chi_E((s,t),v)d\sigma_{(s,t)}(v) = \sigma_{(s,t)}(E^{(s,t)}) \tag{3.15}$$

which is measurable as a function of (s,t) by (3.9a). Also

$$\begin{aligned}&\int_Q \left(\int_R \chi_E((s,t),v)d\sigma_{(s,t)}(v)\right)dsdt \\ &= \mu(E) = \int_{Q\times R} \chi_E((s,t),v)d\mu((s,t),v)\end{aligned} \tag{3.16}$$

by (3.15) and (3.12) so that (3.14) holds. Following the standard procedure in integration theory we proceed from this particular case to simple functions on $Q \times \mathbf{R}$ and bounded measurable functions on $Q\times\mathbf{R}$, using the dominated convergence theorem,to complete the proof of (3.14). Since $\exp\{ivx(s,t)\}$ is a bounded Borel measurable function of $((s,t),v)$ for every x in $C_2(Q)$, we have the desired result (3.13) by (3.14).

Since F belongs to $S(L_2(Q))$, we have by Proposition 3.1 that F is analytic Yeh-Feynman integrable on $C_2(Q)$.

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