ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 18, Number 1, Winter 1988

CONES IN THE GROUP ALGEBRA RELATED TO SCHUR'S DETERMINANTAL INEQUALITY

ROBERT GRONE*, RUSSELL MERRIS *
AND WILLIAM WATKINS

ABSTRACT. Let $c : S_n \to \mathbf{C}$ be a complex valued function on the symmetric group. For $A = (a_{ij})$, an n-by-n matrix, define

$$d_c(A) = \sum_{\sigma \in S_n} c(\sigma) \prod_{t=1}^{n} a_{t\sigma(t)}.$$

Suppose C is the cone of all functions c such that $d_c(A) \geq 0$ for all positive semidefinite A (written $A \geq 0$). We show that $d_c(A) \geq c(e)\det(A)$ for all $c \in C$ and all $A \geq 0$, and then investigate the structure of C.

1. Introduction. Denote by H_n the cone of positive semidefinite hermitian n-by-n matrices. In 1893, J. Hadamard proved that $h(A) \geq \det(A)$ for all $A \in H_n$, where $h(A)$ is the product of the main diagonal entries of A. In 1918, I. Schur published a dramatic improvement of the Hadamard Determinant Theorem: Let G be a subgroup of the symmetric permutation group S_n. Suppose χ is an irreducible, complex character of G. If $A = (a_{ij})$ is an n-by-n matrix,define

$$d_\chi(A) = \sum_{\sigma \in G} \chi(\sigma) \prod_{t=1}^{n} a_{t\sigma(t)}. \tag{1}$$

In the recent literature, it has been customary to state Schur's Inequality as

$$d_\chi(A) \geq \chi(e)\det(A), \tag{2}$$

$A \in H_n$. As pointed out in [**1**], this inequality does not do justice to the full power of Schur's result. We will have more to say about this presently.

The work of these authors was supported in part by ONR contract no. 85-K-0335.
Received by the editors on October 24, 1985.
* AMS *Subject Classification:* 15A15, 15A45, 15A57, 15A69, 20C30, 52A20.

Consider, now, the symmetric group algebra $\mathcal{A} = \mathbf{C}S_n$ consisting of all (formal) complex linear combinations of the $n!$ elements of S_n. Then we may identify $\mathcal{A}$ with the algebra of complex valued functions of S_n under "pointwise" additions and scalar multiplication, and convolution multiplication. An element $c \in \mathcal{A}$ is said to be positive semidefinite (write $c \geq 0$) if

$$\sum_{\sigma,\tau \in S_n} \overline{x(\sigma)} c(\sigma\tau^{-1}) x(\tau) \geq 0$$

for all $x \in \mathcal{A}$. Another way to say this is the following: Let $\sigma \to L(\sigma)$ be the left regular (matrix) representation of S_n. Then $c \geq 0$ if and only if $\sum_{\sigma \in S_n} c(\sigma)L(\sigma)$ is a positive semidefinite hermitian matrix. Consider the cone $\mathcal{C}^+ = \{c \in \mathcal{A} | c \geq 0\}$. Since the field is the complex numbers, $\mathcal{C}^+$ is contained in the cone $\mathcal{H} = \{c \in \mathcal{A} | c(\sigma^{-1}) = \overline{c(\sigma)}, \sigma \in S_n\}$, corresponding to the hermitian matrices in the left regular representation. If we define

$$d_c(A) = \sum_{\sigma \in S_n} c(\sigma) \prod_{t=1}^{n} a_{t\sigma(t)}, \tag{3}$$

then (see e.g., [**1**])

$$d_c(A) \geq c(e) \det(A), \tag{4}$$

for all $c \in \mathcal{C}^+$ and for all $A \in H_n$. (This result simplifies earlier work done in [**6**].) It is the main purpose of the present paper to show that (4) continues to hold for a larger cone $\mathcal{C}$ and then to study the structure of $\mathcal{C}$.

One way to produce examples of elements of $\mathcal{C}^+$ is this: Let $\{B(\sigma) = (b_{ij}(\sigma)) | \sigma \in S_n\}$ be an irreducible, unitary representation of S_n of degree m. Then $b_{ii} \in \mathcal{C}^+, 1 \leq i \leq m$. Indeed, it is an exercise in group representation theory to show that the cone $\mathcal{C}^+$ is generated by these main diagonal entry functions from irreducible, unitary representations of S_n. (See the discussion following Theorem 4 below.) Consequently (as mentioned in [**1**]), inequality (4) is implicit in Schur's 1918 paper.

We now define the cone

$$\mathcal{C} = \{c \in \mathcal{A} | d_c(A) \geq 0 \text{ for all } A \in H_n\}.$$

It is a consequence of (4) that $\mathcal{C}^+ \subseteq \mathcal{C}$. To see that the two are not equal, let $p \in \mathcal{C}^+$ be the function which is 1 on every $\sigma \in S_n$ (so that

$d_p = \text{per}$, the permanent function). If ε denotes the signum function, then $p - \varepsilon \in \mathcal{C}$ by (2). But, for $x = \varepsilon$,

$$\begin{aligned}&\sum_{\sigma,\tau\in S_n} \overline{x(\sigma)}(p-\varepsilon)(\sigma\tau^{-1})x(\tau)\\ &= \sum_{\sigma,\tau\in S_n} \varepsilon(\sigma)(1-\varepsilon(\sigma)\varepsilon(\tau))\varepsilon(\tau)\\ &= \sum_{\sigma,\tau\in S_n} (\varepsilon(\sigma\tau)-1) = -(n!)^2.\end{aligned}$$

So, $p - \varepsilon \notin \mathcal{C}^+$.

We now come to our first main result which shows that a Schur type inequality is available for all $c \in \mathcal{C}$.

THEOREM 1. *If $c \in \mathcal{C}$, then $d_c(A) \geq c(e)\,det(A)$, for all $A \in H_n$. In other words, $d_c(A) \geq 0, A \in H_n$, implies $d_c(A) \geq c(e)\,det(A), A \in H_n$.*

PROOF. Observe first that if A is singular, there is nothing to prove. If $A = (a_{ij})$ is positive definite, let $\alpha = \det A/\det A(1)$, where $A(1)$is the principal submatrix of A obtained by deleting row 1 and column 1. Denote by E the n-by-n matrix with a 1 in the $(1,1)$ position and zeros elsewhere. Then $A_0 = A - \alpha E \in H_n$ is singular. Define $A_x = A_0 + xE$, so that $A_\alpha = A$. Define a (linear) function $f(x) = d_c(A_x) - c(e)\det(A_x)$. Then $f'(x) = d_c(1 \oplus A(1)) - c(e)\det(A(1))$. It follows by induction that $f'(x) \geq 0$ (for all x). Since $f(0) = d_c(A_0) \geq 0$, we may conclude that $f(x) \geq 0$ for all $x \geq 0$. In particular, $f(\alpha) \geq 0$. Since $A = A_\alpha$ the proof is complete.

EXAMPLE 1. Let $Q = (q_{ij}) \in H_n$ be fixed but arbitrary. For any $c \in \mathcal{C}$ define $c_Q(\sigma) = c(\sigma)\prod_\sigma(Q)$, where

$$\prod_\sigma(Q) = \prod_{t=1}^{n} q_{t\sigma(t)}.$$

Then $d_{c_Q}(A) = d_c(Q \circ A)$, where $Q \circ A$ is the Hadmard-Schur product of Q and A. Since $Q \circ A \in H_n$ for all $A \in H_n$, it follows that $c_Q \in \mathcal{C}$. By Theorem 1,

$$\begin{aligned}d_c(Q\circ A) &\geq c_Q(e)\det(A)\\ &= c(e)h(Q)\det(A).\end{aligned} \tag{5}$$

Taking, for example, $c = \varepsilon$, (5) becomes Oppenheim's inequality [**8**]

$$\det(Q \circ A) \geq h(Q)\det(A).$$

One of the most significant outstanding problems involving the permanent is the Lieb-Marcus-Minc conjecture,

$$\chi(e)\operatorname{per}(A) \geq d_\chi(A), \tag{6}$$

$A \in H_n$, a sort of dual to (2). As above, denote by p the principal (identically 1) character of S_n. For $Q \in H_n, p_Q(\sigma) = \prod_\sigma(Q)$, and

$$d_{p_Q}(A) = \operatorname{per}(Q \circ A).$$

It seems plausible to extend conjecture (6) to the p_Q functions. This would result in

$$p_Q(e)\operatorname{per}(A) \geq d_{p_Q}(A)$$

or

$$h(Q)\operatorname{per}(A) \geq \operatorname{per}(Q \circ A).$$

This is a sort of dual to Oppenheim's Inequality which was suggested recently by R.B. Bapat and V.S. Sunder [**1**; Conjecture 2]. (Also see [**2**].)

2. Cones in $\mathcal{A}$. Stimulated by Example 1, we consider the cone $\mathcal{B}$ in the symmetric-group algebra $\mathcal{A}$ generated by $\{p_Q | Q \in H_n\}$. Since $d_{p_Q}(A) = \operatorname{per}(Q \circ A) \geq 0$ for all $A \in H_n$, we see that $\mathcal{B} \subset \mathcal{C}$. In fact, more is true.

THEOREM 2. *The cone $\mathcal{C}$ is the dual cone of $\mathcal{B}$. Moreover, $\mathcal{B} \subsetneq \mathcal{C}^+$.*

PROOF. Order the elements of S_n in some convenient way. Then p_Q may be expressed as an $n!$-tuple with $\prod_\sigma(Q)$ in the σ-th place. In the same way, any $c \in \mathcal{C}$ may be written as an $n!$-tuple with $c(\sigma)$ in the σ-th place. Now, $c \in \mathcal{C}$ if and only if $\sum_{\sigma \in S_n} c(\sigma) \prod_\sigma(Q) \geq 0, Q \in H_n$. And $c \in \mathcal{B}^*$, the dual cone of $\mathcal{B}$, if and only if $\sum_{\sigma \in S_n} c(\sigma) \prod_\sigma(\overline{Q}) \geq 0$, i.e., if and only if the scalar product, $c \circ p_Q \geq 0, Q \in H_n$. Since H_n is invariant under complex conjugation, $\mathcal{C} = \mathcal{B}^*$.

To prove that $\mathcal{B} \subsetneqq \mathcal{C}^+$, we first take $Q \in H_n$. Then

$$\sum_{\sigma,\tau \in S_n} \overline{x(\sigma)} p_Q(\sigma\tau^{-1}) x(\tau)$$

$$= \sum_{\sigma,\tau \in S_n} \overline{x(\sigma)} \prod\nolimits_{\sigma\tau^{-1}}(Q) x(\tau)$$

$$= \sum_{\sigma,\tau \in S_n} \overline{x(\sigma)} (\prod_{t=1}^{n} q_{\tau(t),\sigma(t)}) x(\tau).$$

But, this is a value of the quadratic form afforded by a principal submatrix of the n-th Kronecker power of Q.

It remains to show that $\mathcal{B} \neq \mathcal{C}^+$. Note that $\varepsilon \in \mathcal{C}^+$: If $x \in \mathcal{A}$, then

$$\sum_{\sigma,\tau \in S_n} \overline{x(\sigma)} \varepsilon(\sigma\tau^{-1}) x(\tau) = \sum_{\sigma,\tau \in S_n} \overline{y(\sigma)} y(\tau),$$

where $y(\sigma) = x(\sigma)\varepsilon(\sigma), \sigma \in S_n$. But, this is just the sum of the elements of a rank 1 matrix in $H_{n!}$.

Suppose next that $\varepsilon \in \mathcal{B}$. Then there would be a finite set $K \subset H_n$ such that $\varepsilon = \sum_{Q \in K} p_Q$. Hence

$$\det(A) = \sum_{Q \in K} \operatorname{per}(Q \circ A),$$

for all $A \in H_n$. We now make two special choices for A. If $A = J$, the matrix each of whose entries is 1, then $\sum \operatorname{per}(Q) = 0$. Therefore, $\operatorname{per}(Q) = 0$ for all $Q \in K$. So, every Q in K must have a row of zeros. But then, letting $A = I$, the identity matrix, we conclude

$$1 = \det(I) = \sum_{Q \in K} h(Q) = 0.$$

To facilitate the further study of $\mathcal{C}$, it is convenient to define

$$\mathcal{C}_0 = \{c \in \mathcal{C} | c(e) = 0\}$$

and

$$\mathcal{C}_1 = \{c \in \mathcal{C} | c(e) = 1\}.$$

COROLLARY 1. *The convex set $\mathcal{C}_1$ is a translation of the cone $\mathcal{C}_0$. Specifically, $\mathcal{C}_1 = \mathcal{C}_0 + \varepsilon$, where ε is the signum function.*

PROOF. This is largely a restatement of Theorem 1 in different notation. Let $c \in \mathcal{C}_0$. Since $\varepsilon \in \mathcal{C}$ and $\mathcal{C}$ is a cone, $c+\varepsilon \in \mathcal{C}$. But, $c+\varepsilon$ takes the value 1 on the identity e. Thus, $c+\varepsilon \in C$. Conversely, suppose $c \in \mathcal{C}_1$. By Theorem 1, $d_c(A) \geq c(e)\det(A) = \det(A), A \in H_n$. But then $d_c(A) - \det(A) \geq 0$, $A \in H_n$, and $c - \varepsilon \in \mathcal{C}$. Since the value of $c-\varepsilon$ on e is $0, c - \varepsilon \in \mathcal{C}_0$.

Note that if $c \notin \mathcal{C}_0$, then $c/c(e) \in \mathcal{C}_1$. Thus, we may write

$$\mathcal{C} = \mathcal{C}_0 \cup (\cup_{r>0}\, r\mathcal{C}_1). \tag{7}$$

Further progress depends on the following technical lemmas whose long and tedious proofs will be omitted.

LEMMA 1. *Suppose $c \in \mathcal{A}$. If $d_c(A) = 0$ for all $A \in H_n$, then c is identically zero.*

LEMMA 2. *Suppose $c \in \mathcal{A}$. If $d_c(A)$ is real for all $A \in H_n$, then $c \in \mathcal{H}$, i.e., $c(\sigma^{-1}) = \overline{c(\sigma)}$, $\sigma \in S_n$.*

There is a natural partial order on $\mathcal{C}$. If $a, b \in \mathcal{C}$, then $a \geq b$ simply means that $a - b \in \mathcal{C}$. Note that $a \geq b$ if and only if $d_a(A) \geq d_b(A)$ for all $A \in H_n$.

THEOREM 3. *The unique extreme point of $\mathcal{C}_1$ is ε.*

PROOF. We begin by showing that ε is an extreme point. Suppose $\varepsilon = \theta a + (1-\theta)b$ for some $a, b \in \mathcal{C}$ and some θ satisfying $0 < \theta < 1$. By Theorem 1, $a \geq \varepsilon$ and $b \geq \varepsilon$. Thus, $\theta a + (1-\theta)b \geq \varepsilon$. Now, if there is a single $A \in H_n$ such that $d_a(A) > \det(A)$, we have a contradiction. It follows that $d_c(A) = 0$ for all $A \in H_n$ where $c = a - \varepsilon$. Appealing to Lemma 1, we conclude that $a = \varepsilon$. Similarly, $b = \varepsilon$.

To see that no other element of $\mathcal{C}_1$ is extreme it suffices to observe that $a = \frac{1}{2}(2a - \varepsilon) + \frac{1}{2}\varepsilon$ for all $a \in \mathcal{C}_1$. By Theorem 1, $a - \varepsilon \in \mathcal{C}_0$ for all $a \in \mathcal{C}_1$. Thus, both $2a - \varepsilon$ and ε are elements of $\mathcal{C}_1$.

COROLLARY 2. *The only extreme ray of the cone $\mathcal{C}$ not contained in $\mathcal{C}_0$ is $\langle\varepsilon\rangle = \{r\varepsilon | r > 0\}$.*

EXAMPLE 2. Recall that p denotes the constant function 1 on S_n. If we confuse the identity of S_n with the identity of $\mathcal{A}$, we may write e for the function which is 1 on the identity permutation and 0 on the rest of S_n. In particular, $d_p(A) = \text{per}\,(A)$ and $d_e(A) = h(A)$. Both p and e are elements of $\mathcal{C}_1$. It follows from Theorem 3 that neither p nor e is an extreme point of $\mathcal{C}_1$. We claim that neither is even on the boundary of $\mathcal{C}_1$. First consider e. Let $b \neq e$ be a fixed but arbitrary element of $\mathcal{C}_1$. We wish to show that there is an element $a \in \mathcal{C}_1$ such that $a \neq e$ but e is on the line segment joining a to b. Since the main diagonal product of $A \in H_n$ dominates (in absolute value) any other diagonal product, there is a number $r > 1$ such that $rh(A) \geq d_b(A), A \in H_n$. (If b happend to be p, then r could be taken to be $n!$) It follows that $re - b \in \mathcal{C}$. Should it happen that $re - b \in \mathcal{C}_0$, then replace r with $r+1$. Now, we have that

$$\frac{re-b}{r-1} = e + \frac{1}{r-1}(e-b) = a \in \mathcal{C}_1$$

and

$$e = \frac{r-1}{r}a + \frac{1}{r}b.$$

Because $\text{per}\,(A) \geq h(A), A \in H_n$ [**5**]; a similar argument holds for p.

It follows from Lemma 2 that $\mathcal{C} \subset \mathcal{H}$. In fact, these two cones are more closely related.

THEOREM 4. *Let e be the identity of $\mathcal{A}$ and write $\langle -e \rangle = \{-re | r > 0\}$. Then $\mathcal{H} = \mathcal{C} + \langle -e \rangle$.*

PROOF. Since $\mathcal{C} \subset \mathcal{H}$ and $\langle -e \rangle \subset \mathcal{H}$, we need only show that the typical element $a \in \mathcal{H}$ can be written in the form $b - re$ for some $b \in \mathcal{C}$ and some $r > 0$. Since the main diagonal product of $A \in H_n$ is dominant, it suffices to choose $r = \sum_{\sigma \in S_n} |a(\sigma)|$. Then $a + re = b \in \mathcal{C}$.

In fact, Theorem 4 can be placed in a more general setting. Suppose $c \in \mathcal{H} \subset \mathcal{A}$. By Wedderburn's Theorem, we may view $\mathcal{A}$ as a direct sum of full matrix rings $M_{n_1}, M_{n_2}, \ldots, M_{n_k}$. Since c is hermitian we may perform a unitary similarity on each marix ring so that the i-th component of c is $\text{diag}\,(\lambda_{i,1}, \lambda_{i,2}, \ldots, \lambda_{i,n_i}), 1 \leq i \leq k$. Being eigenvalues of hermitian c, the λ's are real. Indeed, diagonalizing c amounts to choosing a particular system of unityar, inequivalent, irreducible representations $B^{(1)}, B^{(2)}, \ldots, B^{(k)}$, of $S_n$of degrees $n_1, n_2, \ldots, n_k$, respectively. Moreover, the spectral decomposition of c with respect to these representations may be expressed as

$$c = \sum_{i=1}^{k} \sum_{j=1}^{n_i} \lambda_{i,j} b_{jj}^{(i)}, \tag{8}$$

where $b_{jj}^{(i)}$ is the j-th main diagonal entry function of $B^{(i)}$. Moreover, $c \in \mathcal{C}^+$ if and only if all of the λ's are nonnegative. In particular,

$$\mathcal{C} \subset \mathcal{H} = \mathcal{C}^+ - \mathcal{C}^+, \tag{9}$$

where $\mathcal{C}^+ - \mathcal{C}^+ = \{a - b | a, b \in \mathcal{C}^+\}$.

In fact, (9) is another version of Theorem 4. Similar arguments can be used to show that $\mathcal{C}^+$ is closed under pointwise multiplication, i.e.,

$$\mathcal{C}^+ \circ \mathcal{C}^+ = \mathcal{C}^+. \tag{10}$$

It is interesting to pursue Identity (8) a step further when c is an element of the generating set of $\mathcal{B}$. As in Example 1, let $Q \in H_n$ be fixed but arbitrary. Then $p_Q(\sigma) = \prod_\sigma(Q)$, the σ-diagonal product of Q, and $d_{p_Q}(A) = \operatorname{per}(Q \circ A)$. From (8),

$$p_Q = \sum_{i=1}^{k} \sum_{j=1}^{n_i} \lambda_{i,j} b_{jj}^{(i)}, \tag{11}$$

To compute $\lambda_{i,j}$, we appeal to the "Schur Relations". (See, e.g., [**7**, p. 16].)

$$\begin{aligned}\lambda_{i,j} &= n_i(p_Q, b_{jj}^{(i)}) \\ &= \frac{n_i}{n!} \sum_{\sigma \in S_n} \prod{}_\sigma(Q) b_{jj}^{(i)}(\sigma^{-1}) \\ &= \frac{n_i}{n!} \sum_{\sigma \in S_n} b_{jj}^{(i)}(\sigma) \prod_{t=1}^{n} q_{t\sigma(t)}.\end{aligned}$$

Thus, if $c \in \mathcal{C}^+$ arises from the j-th main diagonal entry function of $\overline{B}^{(i)}$, then the eigenvalue $\lambda_{i,j}$ of p_Q satisfies the identity.

$$(n!/n_i)\lambda_{i,j} = d_c(Q). \tag{12}$$

Identities (11) and (12) can be viewed in another way. We can exhibit the eigenvalues and an orthogonal set of eigenvectors for the $n!$ square matrix $\prod(Q)$ whose (σ,μ) entry is given by

$$\begin{aligned}[\prod(Q)]_{\sigma\mu} &= \prod_{t=1}^{n} q_{\mu(t)\sigma(t)} \\ &= \prod_{\sigma\mu^{-1}}(Q),\end{aligned}$$

where $\sigma,\mu \in S_n$. For each $i = 1,\ldots,k$ and $j,s = 1,\ldots,n_i$, the (j,s) entry functions of $\overline{B}^{(i)}$ are orthogonal eigenvectors corresponding to the eigenvalue $d_c(Q)$ in Equation (12). We note, in this context, that the Lieb-Marcus-Minc conjecture, (6), would follow if it were known that $\operatorname{per} Q$ is the dominant eigenvalue of $\prod(Q)$. (See [**11**].)

References

1. R.B. Bapat and V.S.Sunder, *An extremal property of the permanent and the determinant,* Linear Algebra Apl. **76** (1986), 153-163.

2. J. Chollet, *Is there a permanental analogue to Oppenheim's inequality?,* Amer. Math. Monthly **89** (1982), 57-58.

3. J. Hadamard, *Résolution d'une question relative aux déterminants,* Bull. Sci. Math. **17** (1893), 240-246.

4. E.H. Lieb, *Proofs of some conjectures on permanent,* J. Math. Mechanics **16** (1966), 127-134.

5. M. Marcus, *The Hadamard theorem for permanents,* Proc. Amer. Math. Soc. **15** (1964), 967-973.

6. R. Merris and J.A. Dias da Silva, *Generalized Schur functions,* J. Algebra **35** (1975), 442-448.

7. M. Newman, *Matrix Representations of Groups,* National Bureau of Standards Applied Math. Series **60** (1968).

8. A. Oppenheim, *Inequalities connected with definite hermitian forms,* J. London Math. Soc. **5** (1930), 114-119.

9. I. Schur, *Über endliche Gruppen und Hermitesche Formen,* Math. Z. **1** (1918), 184-207.

10. J.A. Dias da Silva, *On the closed left ideals of the group algebra* $\mathbf{C}S_n$, Port. Math. **39** (1980). 441-453.

11. G.W. Soules, *Constructing symmetric nonnegative matrices,* Linear and Multilinear Algebra **13** (1983), 241-251.

Department of Mathematics, al Sciences, San Diego State University, San Diego CA 92182

Department of Mathematics and Computer Science, California State University, Hayward, CA 94542

DEPARTMENT OF MATHEMATICS, CALIFORNIA STATE UNIVERSITY, NORTHRIDGE, CA 91330