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ON ω-FILTERED VECTOR SPACES AND THEIR APPLICATION TO ABELIAN p-GROUPS: II

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1. Introduction. An *ω-filtered vector space* is ordinary vector space equipped with a descending chain of subspaces $\{X_n : n \in \omega\}$. A morphism between ω-filtered vector spaces X and Y is a linear map $f : X \to Y$ such that for all $n, f(X^n) \subseteq Y^n$. The key example of an ω-filtered vector space - over $\mathbf{Z}(p)$, the field of p elements - is, of course, the socle, $G[p] \stackrel{\text{def}}{=} \{x \in G : px = 0\}$, of an abelian p-group.

In [**10**] we began a systematic investigation of ω-filtered vector spaces over an arbitrary countable field. In this paper we continue that study, with emphasis this time on questions which can be answered with the help of additional set-theoretic axioms, but which cannot be settled on the basis of the usual, Zermelo-Franekel axioms of sets theory (denoted ZFC). We give applications of our results to the theory of abelian p-groups, particularly to Crawley's Problem.

§2 is concerned with the classification of ω_1-separable ω-filtered vector spaces of dimension $\aleph_1$. Our results parallel those which have been obtained for ω_1-separable abelian groups (cf. [**7, 8, 11**): there is a satisfactory structure theory when we assume Martin's Axiom (MA) plus the negation of the Continuum Hypothesis ($\neg$CH), and extreme pathology when we assume CH or the Axiom of Constructibility (V=L). Among the consequences of the structure theory (which holds under MA+$\neg$CH) are (1) every weakly ω_1-separable space of dimension $\aleph_1$ is ω_1-separable (Corollary 2.2) and C-decomposable (Corollary 2.3) and (2) every ω_1-separable ω-filtered vector space over $\mathbf{Z}(p)$ of dimension $\aleph_1$ is the socle of a $p^{\omega+1}$-projective p-group. None of these consequences are theorems of ZFC.

§3 deals with dense subspaces of small codimension in an ω-filtered vector space, and has application to Crawley's Problem on the unique ω-elongation of p-groups. The main theorems (3.3 and 3.8) construct large numbers of dense subspaces of codimension 1 in any non-

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projective space of dimension $\aleph_1$, under hypotheses consistent with but weaker than V=L. As a consequence, we obtain strengthenings of theorems of Megibben and Mekler-Shelah (Corollaries 3.7, 3.11). We also note that assuming only $2^{\aleph_0} < 2^{\aleph_1}$, every Crawley group of cardinality $\aleph_1$ must be weakly ω_1-separable (Corollary 3.10).

Throughout the paper we shall consider ω-filtered vector spaces over a fixed countable field K. We shall assume familiarity with the basic definitions and results of [**10**]. For an introduction to ZFC, and to the additional axioms we use, see, for example, [**6, 14, 13**]; also useful are [**3**] and [**23**].

2. The structure of ω_1-separable spaces. We shall say that two ω-filtered vector spaces X and Y of dimension ω_1 are *filtration-equivalent* if they have ω_1-filtrations $X = \cup_{\nu<\omega_1} X_\nu$ and $Y = \cup_{\nu<\omega_1} Y_\nu$ such that, for all $\mu < \omega_1$, there is a *level preserving isomorphism* $f_\mu : X_\mu \to Y_\mu$, i.e., an isomorphism such that, for every $\nu < \mu$, $f_\mu(X_\nu) = Y_\nu$. (For the definition of ω_1-filtration and ω_1-separable see [**10**; pp. 152f].)

We shall state our first two results without proof since the proofs follow closely the proofs of corresponding results for groups. (see, respectively, [**11**; Theorem 2.1 and Theorem 2.2].)

THEOREM 2.1. (MA+¬CH) *Filtration-equivalent weakly ω_1-separable ω-filtered vector spaces of dimension $\aleph_1$ are isomorphic.*

COROLLARY 2.2. (MA+¬CH) *Every weakly ω_1-separable space X of dimension $\aleph_1$ is C-decomposable, i.e., $X \simeq Y \oplus P$, where P is projective and* fin dim $(P) =$ fin dim (X).

COROLLARY 2.3. (MA+¬CH) *Every weakly ω_1-separable space of dimension $\aleph_1$ is ω_1-separable.*

PROOF. It suffices to prove that if W is a countable closed subspace of a weakly ω_1-separable space X of dimension ω_1, then W is a summand of X. Let $Y = X/W \oplus W$. By choosing an ω_1-filtration of $X = \cup_{\nu<\omega_1} X_\nu$ such that $W = X_1$ and letting $Y_\nu = (X_\nu/W) \oplus W$ it is easy to show that X and Y are filtration-equivalent, and moreover the identity map $W \to W$ is level-preserving. I follows then from Theorem 2.1 (or actually from its proof) that there is an isomorphism of X onto Y which is the identity on W. Hence W is a direct summand of X.

COROLLARY 2.4. (MA+¬CH) *Every ω_1-separable ω-filtered $\mathbf{Z}(p)$-vector space of dimension $\aleph_1$ is the socle of a (unique) ω_1-separable $p^{\omega+1}$-projective p-group G.*

PROOF. This follows immediately from Corollary 2.2, Theorem 5.4 of [**10**], Theorem 1.1 of [**16**], and the fact that, assuming MA+¬CH, every weakly ω_1-separable p−group is ω_1-separable [**16**; Theorem 2.2].

For the definition of $P(A)$ see [**10**] (before 3.12).

COROLLARY 2.5. (MA+¬CH) *If A is a $p^{\omega+1}$-injective p-group such that $P(A)$ is ω_1-separable of final dimension $\aleph_1$, then A is the direct sum of two subgroups of uncountable final rank.*

PROOF. Corollary 2.2 implies that $P(A) = X_1 \oplus X_2$ for some X_1 and X_2 of uncountable final dimension. Hence, if A_1 and A_2 are $p^{\omega+1}$-injective groups such that $P(A_i) = X_i$ for $i = 1, 2$, then A_1 and A_2 have uncountable final rank, and $A \simeq A_1 \oplus A_2$ (cf. 4.1 of [**10**]).

Corollary 2.2 shows that, in a model of MA+¬CH, every ω_1-separable ω-filtered vector space of final dimension $\aleph_1$ is the direct sum of two subspace of uncountable final dimension. We do not know if, in every model of MA+¬CH, every such space which is not projective is that direct sum of two non-projective subspaces, but this can be proved as a consequence of a stronger hypothesis, PFA (cf.[**7**, Theorem 2.10] and [**17**]). On the other hand, this result fails in model of V=L, as we now show. (The analogous result has not previously been observed for groups.)

THEOREM 2.6. (V=L) *There is an ω_1 separable ω-filtered vector space $X$of dimension $\aleph_1$ which is C-decomposable and non-projective, but which is not the direct sum of two non-projective subspaces.*

PROOF. We shall construct $X = A \oplus F$, where F is projective and of final dimension $\aleph_1$ and X has the property that if $X = Y \oplus Z$, then either $A \cap Y$ or $A \cap Z$ has countable final dimension (and hence is projective). Suppose that $X = Y \oplus Z$ and $A \cap Y$ has countable final dimension. Note that there is a monomorphism

$$Y/(A \cap Y) \to (A + Y)/A,$$

and $(A + Y)/A$ is projective because it is isomorphic to a subspace of

F. Therefore, by Proposition 1.4 of [**10**], $Y/(A \cap Y)$ is projective, and therefore Y is projective. So it suffices to construct X with the stated property.

The construction will be similar to that in Theorem 2.8 of [**10**], and just as in that theorem, we can construction X to have a given basic subspace B, and also a given Γ-invariant other than 0 or 1.

Let $S_\nu, B(\nu), \hat{B}(\nu)$ etc be as defined in [**10**; Theorem 2.8]. Let $E \subset \lim(\omega_1)$ such that $\tilde{E} \neq 0, 1$. Let P be a projective space such that

$$f_n(P) = \begin{cases} 0 & \text{if } f_n(B) < \aleph_1 \\ \aleph_0 & \text{otherwise.} \end{cases}$$

For each $\nu < \omega_1$, let F_ν be an isomorphic copy of P, and let $F = \oplus_{\nu<\omega_1} F_\nu \simeq P^{(\omega_1)}$. (Note that $B \simeq B \oplus F$). We shall inductively construct A_ν such that, for all $\nu \in \lim(\omega_1)$

$$B(\nu) \subseteq A_\nu \subseteq \cup_{\mu<\nu} \hat{B}(\mu),$$

and for all $\nu < \tau, \nu \notin E$ implies A_ν is closed in A_τ. For all $\nu < \omega_1$, let $X_\nu = A_\nu \oplus \oplus_{\mu<\nu} F_\mu$.

Now, $\diamondsuit_{\omega_1}(E)$ gives us a family of pairs (Y_ν, Z_ν) of subsets of X_ν, such that it will be the case that, for any pair (Y, Z) of subsets of X, there is a stationary set of $\nu \in E$ such that $Y \cap X_\nu = Y_\nu$ and $Z \cap X_\nu = Z_\nu$. (Define a chain of bijections of X_μ with $\omega\mu$ as the X_μ are defined.)

Suppose we have a constructed A_τ for every $\tau < \mu$. The crucial case is when $\mu = \nu + 1$ and

(*) $\nu \in E$ and there is a strictly increasing sequence τ_n approaching ν such that, for all $n, \tau_n \notin E$,

$$X_{\tau_n} = (Y_\nu \cap X_{\tau_n}) \oplus (Z_\nu \cap X_{\tau_n}),$$

and, for all m,

$$Y_\nu \cap A^m_{\tau_n} \not\subseteq A_{\tau_{n-1}},$$
$$Z_\nu \cap A^m_{\tau_n} \not\subseteq A_{\tau_{n-1}}.$$

Then recursively define $y_n \in (Y_\nu \cap A_{\tau_n}) - A_{\tau_{n-1}}$ and $z_n \in (Z_\nu \cap A_{\tau_n}) - A_{\tau_{n-1}}$ such that $v(y_n)$ and $v(z_n)$ are strictly greater than

$$k_n \stackrel{\text{def}}{=} \max\{v(y_{n-1} + (Y_\nu \cap X_{\tau_{n-2}})), v(z_{n-1} + (Z_\nu \cap X_{\tau_{n-2}}))\}.$$

(Note that k_n is finite). Then let a_ν be the limit in $\hat{B}(\nu)$ of $\{\sum_{i\leq n}(y_i + z_i) : n \in \omega\}$ and let

$$A_{\nu+1} = (A_\nu + Ka_\nu) \oplus S_\nu.$$

This completes the crucial inductive step of the construction. In cases where $(*)$ is not satisfied, let $A_{\nu+1} = A_\nu \oplus S_\nu$. Of course, for limit $\sigma \leq \omega_1, A_\sigma = \bigcup_{\nu<\sigma} A_\nu$ and $X_\sigma = \bigcup_{\nu<\sigma} X_\nu$.

Let $A = A_{\omega_1}, X = X_{\omega_1} = A \oplus F$, and suppose $X = Y \oplus Z$ where $A \cap Y$ and $A \cap Z$ have uncountable final dimension. Then, for every $m \in \omega$ and every ν, there exists $\mu > \nu$ such that $Y \cap A_\mu^m \not\subseteq A_\nu$ and $Z \cap A_\mu^m \not\subseteq A_\nu$. From this it follows easily that the set C, of μ such that, for all $m \in \omega$ and all $\nu < \mu$,

$$Y \cap A_\mu^m \not\subseteq A_\nu \text{ and } Z \cap A_\mu^m \not\subseteq A_\nu,$$

is a cub. (For the definition of a *cub*, see [**10**, p. 152]). Moreover, w.l.o.g., for every $\mu \in C$,

$$X_\mu = (Y \cap X_\mu) \oplus (Z \cap X_\mu).$$

Since $\tilde{E} \neq 1, (\omega_1 - E) \cap C$ is unbounded, so its closure $\overline{(\omega_1 - E) \cap C}$ is a cub. Thus there exists a limit point ν of $\overline{(\omega_1 - E) \cap C}$ such that

$$Y \cap X_\nu = Y_\nu \text{ and } Z \cap X_\nu = Z_\nu.$$

Also, by choice of ν, ν is the limit of a sequence of ordinals $\tau_n \in (\omega_1 - E) \cap C$ (cf. [**6**. p. 92]). Thus we are in the case when $(*)$ holds.

Now, if a_ν is as in the construction, $a_\nu \in \overline{X}_\nu$ and $a_\nu = y' + z'$ where $y' \in Y, z' \in Z$. Since $\overline{X}_\nu / X_\nu$ has dimension ≤ 1 by construction,either $y' \in Y_\nu$ or $z' \in Z_\nu$. Say $y' \in Y_\nu$; fix n such that $y' \in Y_\nu \cap X_{\tau_{n-1}}$. Now by construction

$$v(a_\nu - \sum_{i\leq n}(y_i + z_i)) > k_{n+1}.$$

But

$$a_\nu - \sum_{i\leq n}(y_i + z_i) = (y' - \sum_{i\leq n} y_i) + (z' - \sum_{i\leq n} z_i),$$

so, since $Y \oplus Z$ is a direct sum (in $\mathcal{FV}$),

$$v(y' - \sum_{i\leq n} y_i) > k_{n+1}.$$

However, $(\sum_{i<n} y_i y' = y_n + ((\sum_{i<n} y_i) - y')$ belongs to $y_n + (Y_\nu \cap X_{\tau_{n-1}})$, which has value $\leq k_{n+1}$ by definition of k_{n+1}, a contradiction.

COROLLARY 2.7. (V=L) *There exists a weakly ω_1-separable $p^{\omega+1}$-projective p-group which is not $\sum$-cyclic and is not the direct sum of two non-$\sum$-cyclic subgroups.*

In a similar manner one can prove the following theorem (which follows from Corollary 1.10 of [**11**] for $K = \mathbf{Z}(p)$).

THEOREM 2.8 (V=L) *There is an ω_1-separable ω-filtered vector space X of dimension $\aleph_1$ which is not the direct sum of two subspaces of uncountable final dimension. In particular, X does not have a projective summand of uncountable final dimension.*

COROLLARY 2.9. (V=L) *There is an ω_1-separable ω-filtered vector space of dimension $\aleph_1$ which is not the socle of a $p^{\omega+1}$ projective p-group.*

COROLLARY 2.10. (V=L) *There is a $p^{\omega+1}$-injective p-group A such that $P(A)$ is ω_1-separable of final dimension $\aleph_1$, but A is not the direct sum of two subgroups of uncountable final rank.*

REMARK 2.11. Theorem 2.8 implies that 2.2 and hence 2.1, is not a theorem of ZFC. In fact, it can be shown that 2.1 fails in all models of ZFC+$2^{\aleph_0} < 2^{\aleph_1}$ (cf. the proof of an analogous result in [**7**; Theorem 3.2]). We do not know if 2.2 fails in all models of CH, but we shall show in the next section (3.5) that 2.3 fails in all models of ZFC+$2^{\aleph_0} < 2^{\aleph_1}$.

3.. Dense subspaces of codimension one. Recall that Crawley [**2**] and Hill and Megibben [**12**] proved that if $G$is a $\sum$-cyclic p-group, then it has the *unique ω-elongation property*, i.e., for any p-group B, any two ω-elongations of G by B are isomorphic. Later, Nunke [**21**] and Warfield [**24**] proved a converse to this result, i.e., a separable p-group with the unique ω-elongation property is $\sum$-cyclic. Crawley raised the question of a stronger converse. Call a p-group G a *Crawley group* if G is separable and any two ω-elongations of G by $\mathbf{Z}(p)$ are isomorphic; Crawley's Problem asks: is every Crawley group $\sum$-cyclic? Megibben [**15**] showed that this question is not decidable in ZFC for groups of cardinality $\aleph_1$: the answer is "yes" assuming V=L and "no" assuming MA+$\neg$CH. Mekler and Shelah obtained other results under V=L, including a result about "rigid systems" of elongations for groups of cardinality $\aleph_1$ which are not $\sum$-cyclic (cf. [**19**; Theorem 1.3 and Remark 3.12]).

By Richman's Criterion a separable p-group G is a Crawley group if and only if any two dense subspaces of $G[p]$ of codimension 1 are equivalent (cf. [**10**; Criterion 3.12]). Since equivalent subspaces of $G[p]$ are isomorphic as ω-filtered vector spaces, it is natural to raise the question of how many non-isomorphic dense subspaces of nondimension 1 an ω-filtered vector space may have; we shall show that the answer depends on additional set-theoretic hypotheses.

Megibben's results [**15**; Theorems 3.1 and 3.2]imply that, assuming MA+$\neg$CH, every ω_1-separable ω-filtered $\mathbf{Z}(p)$-vector space X of dimension $\aleph_1$ has the property that any two dense subspaces of codimension 1 are isomorphic. We shall now show that a more general result is a consequence of Corollary 2.3.

THEOREM 3.1 (MA+$\neg$CH) *If X is an ω_1-separable ω-filtered vector space of dimension $\aleph_1$, then any dense subspace of countable codimension is isomorphic to X.*

PROOF. Let Y be a dense subspace of X of countable codimension. Then Y is certainly weakly ω_1 separable since the closure in Y of any countable subset is contained in its closure in X. Hence by 2.3, Y is ω_1-separable.

By hypothesis there is a countable closed subspace X_0 of X such that $X = X_o + Y$. Denote $X_0 \cap Y$ by Y_0. We can assume Y_0 is dense in X_0 (cf. proof of Theorem 3.4 of [**10**]), and hence $X_0 \simeq Y_0$.

Now $Y = Y_0 \oplus W$ for some subspace W of Y since Y_0 is closed in Y and Y is ω_1-separable. But then $X = X_0 + Y = X_0 \oplus W$. (Note that the latter sum is direct by Lemma 2.9 of [**10**]). The result follows immediately.

REMARK 3.2. Note that, in fact, the argument proves (in ZFC) that if X is a space of dimension $\aleph_1$ which contains a dense subspace Y of countable codimension which is ω_1-separable, then X is isomorphic to Y. Moreover, it can be shown that if X contains any (weakly) ω_1-separable subspace Y of countable codimension, then X is (weakly) ω_1-separable (but not necessarily isomorphic to Y). (The proof of

Theorem 3.1 has been simplified by use of an idea of Alan Mekler's).

Now we shall show that Theorem 3.1 is very far from being a theorem of ZFC. ($\Phi_{\omega_1}(E)$ is the "weak-diamond" hypothesis: see [**4**] or [**6**, p. 34]).)

THEOREM 3.3 ($\Phi_{\omega_1}(E)$). *Let X be a weakly ω_1-separable ω-filtered vector space of dimension $\aleph_1$ such that $\Gamma(X) = \tilde{E}$. Then there are $2^{\aleph_1}$ dense subspaces $Y_i (i < 2^{\aleph_1})$ of codimension* 1 *in X such that for $i \neq j$* (a) *Y_i is not isomorphic to Y_j and* (b) *for any morphism $\theta : X \to X$, if $\theta(Y_i) \subseteq Y_j$, then $\theta(X) \subseteq Y_j$.*

PROOF. We may suppose that $E \subseteq \lim(\omega_1)$ and that there is an ω_1-filtration $X = \cup_{\nu<\omega_1} X_\nu$ such that X_1 is not closed in X and, for all $\nu > 1$, X_ν is closed in X if and only if $\nu \notin E$. Fix $z \in \overline{X}_1 - X_1$; note that $z \in X_2$.

Let ${}^\delta 2$ be the set of all functions from δ to $2 = \{0,1\}$. We shall construct, for each $\xi \in {}^\delta 2 (\delta \in \omega_1)$, a subspace Y_ξ of X_δ containing X_1 such that:

(i) $X_\delta = Y_\xi + Kz$, and $z \notin Y_\xi$;

(ii) for all $\mu < \delta, Y_{\xi\restriction\mu} \subseteq Y_\xi$, and if δ is a limit ordinal, then $Y_\xi = \cup_{\mu<\delta} Y_{\xi\restriction\mu}$; and

(iii) if $\delta \in E$, there exists $y_\delta \in \overline{X}_\delta - X_\delta$ such that $y_\delta \in Y_{\xi_0}, y_\delta - z \in Y_{\xi_1}$ (where $\xi_i = \xi \cup \{(\delta, i)\}$ for $i = 0, 1$).Suppose we can do this; note that, by(i), for any $\varphi : \omega_1 \to 2$, $Y_\varphi \stackrel{\text{def}}{=} \cup_{\delta<\omega_1} Y_{\varphi\restriction\delta}$ is a dense subspace of X of codimension 1. Our aim is to show that we can choose $2^{\aleph_1}$ different functions $\varphi : \omega_1 \to 2$ such that the corresponding subspaces Y_φ satisfy (a) and (b).

For each $\delta \in E$ and each triple (ξ, ρ, h) where $\xi, \rho \in {}^\delta 2$ and $h : Y_\xi \to Y_\xi$ is continuous, define

$$F_\delta(\xi, \rho, h) = \begin{cases} 1 & \text{if } \overline{h}(y_\delta) \in Y_{\rho_0} \\ 0 & \text{otherwise.} \end{cases}$$

(Here $\overline{h}$ is the extension of h to $\overline{X}_\delta = \overline{Y}_\xi$, taking values in $\hat{X}$, the completion of X.)

Now we can write E as a disjoint union, $E = \coprod_{\alpha<\omega_1} E_\alpha$ such that, for each $\alpha, \Phi_{\omega_1}(E_\alpha)$ holds (see, e.g., [**9**; Lemma 2.8]). Then $\Phi_{\omega_1}(E_\alpha)$ (plus a standard coding argument) imply that there is a function

$\psi_\alpha : E_\alpha \to 2$ such that, for all $\sigma, \tau : \omega_1 \to 2$ and $\theta : Y_\sigma \to Y_\tau$, $\{\delta \in E_\alpha : F_\delta(\sigma \restriction \delta, \tau \restriction \delta, \theta \restriction Y_{\sigma \restriction \delta}) = \psi_\alpha(\delta)\}$ is stationary in ω_1.

Let $\{S_i : i < 2^{\omega_1}\}$ be a family of subsets of ω_1 such that if $i \neq j$, then $S_i = S_j \neq \phi$ and $S_j - S_i \neq \phi$.

Define

$$\varphi_i(\delta) = \begin{cases} \psi_\alpha(\delta) & \text{if } \delta \in E_\alpha \text{ and } \alpha \in S_i \\ 0 & \text{otherwise.} \end{cases}$$

Let $Y_i = \cup_{\delta<\omega_1} Y_{\varphi_i \restriction \delta}$. We shall prove that (a) and (b) hold for this family of dense subspaces of codimension 1. Let $i \neq j$ and let $\alpha \in S_i - S_j$.

(a) Suppose there is an isomorphism $\theta : Y_i \to Y_j$. Then there exists $\delta \in E_\alpha$ such that if we let $\xi = \varphi_i \restriction \delta$, $\rho = \varphi_j \restriction \delta$, $h = \theta \restriction Y_\xi$, then $h(Y_\xi) \subseteq Y_\rho$ and $\psi_\alpha(\delta) = F_\delta(\xi, \rho, h)$. Notice that $\alpha \notin S_j$ implies $Y_j \cap X_{\delta+1} = Y_{\rho 0}$.

Case 1. $\psi_\alpha(\delta) = 0$. Then $\varphi_i(\delta) = 0$ and hence $y_\delta \in Y_{\xi_0} \subseteq Y_i$. But $\theta(y_\delta) = \overline{h}(\delta) \notin Y_{\rho 0}$ by definition of $F_\delta(\xi, \rho, h)$. This is a contradiction since $\theta(y_\delta) \in Y_j \cap \overline{Y}_\rho \subseteq Y_j \cap X_{\delta+1} = Y_{\rho 0}$.

Case 2. $\psi_\alpha(\delta) = 1$. Then $y_\delta - z \in Y_i$. Also $\overline{h}(y_\delta) \in Y_{\rho 0} \subseteq Y_j$, so $\theta^{-1}(\overline{h}(y_\delta)) \in Y_i$. But $\theta^{-1}(\overline{h}(y_\delta)) = y_\delta$ since $y_\delta \in \overline{Y}_\xi$ and $\overline{h} \restriction Y_\xi = \theta \restriction Y_\xi$. This is a contradiction since $z \notin Y_i$.

(b) Suppose there is a morphism $\theta : X \to X$ such that $\theta(Y_i) \subseteq Y_j$ but $\theta(z) \notin Y_j$. Let δ, ξ, ρ, h be as in (a), and consider the two cases. Case 1 is exactly as before.

Case 2. $\psi_\alpha(\delta) = 1$. Then $y_\delta - z \in Y_i$. But also $\theta(y_\delta - z) = \theta(y_\delta) - \theta(z) = \overline{h}(y_\delta) - \theta(z) \notin Y_{\rho 0}$, since $\overline{h}(y_\delta) \in Y_{\rho 0}$ but $\theta(z) \notin Y_j$. This is a contradiction since $\theta(Y_i) \subseteq Y_j$ implies $\theta(y_\delta - z) \in Y_j \cap \overline{Y}_\rho \subseteq Y_{\rho 0}$.

All that remains now is to do the construction of the Y_ξ. In fact, the construction, by induction on δ, is not difficult. If $\delta = \nu + 1$, where $\nu \notin E$, and $\xi \in {}^\delta 2$, let $Y_{\xi_0} = Y_{\xi_1}$ =a maximal extension of Y_ξ in $X_{\delta+1}$ with the property that it doesn't contain z. If $\delta = \nu + 1$, where $\nu \in E$, choose $y_\delta \in \overline{X}_\delta - X_\delta$. (Note that also $y_\delta - z \in \overline{X}_\delta - X_\delta$). For any $\xi \in {}^\delta 2$, we claim that $z \notin Y_\xi + K y_\delta$. If not, then $z = y + \lambda y_\delta$, where $y \in Y_\xi$ and $\lambda \in K$; since $z \notin Y_\xi$, $\lambda \neq 0$; but then $y_\delta = \lambda^{-1} z - \lambda^{-1} y \in \overline{X}_1 + X_\delta = X_\delta$, contradicting the choice of y_δ. Hence we can define Y_{ξ_0} to be a maximal extension of $Y_\xi + K y_\delta$ in $X_{\delta+1}$ which doesn't contain z. Similarly, we define Y_{ξ_1} to be a maximal extension of $Y_\xi + K(y_\delta - z)$ in $X_{\delta+1}$ which doesn't contain z.

REMARK 3.4. Part (b) of Theorem 3.3 - as well as part (b) of 3.8 - was proved in order to give an alternate derivation (based on Richman's Criterion) of the "rigid system" result of Mekler and Shelah, see 3.11 below.

The following corollary shows that Corollary 2.3 is not a theorem of ZFC.

COROLLARY 3.5. ($2^{\aleph_0} < 2^{\aleph_1}$) *There exist weakly ω_1-separable ω-filtered vector spaces of dimension $\aleph_1$ which are not ω_1-separable.*

PROOF. Let X be any weakly ω_1-separable space with $\Gamma(X) = 1$. Since $2^{\aleph_0} < 2^{\aleph_1}$ implies $\Phi_{\omega_1}(\omega_1)$, Theorem 3.3 and Remark 3.2 show that X has $2^{\aleph_1}$ dense subspaces of codimension 1 which are not ω_1-separable.

COROLLARY 3.6. ($2^{\aleph_0} < 2^{\aleph_1}$) *There is a $p^{\omega+1}$-projective p-group G of cardinality $\aleph_1$ which is weakly ω_1-separable but not ω_1-separable.*

PROOF. Let G be the $p^{\omega+1}$-projective group whose socle is $X \oplus B$, where X is weakly ω_1-separable and of dimension ω_1 but not ω_1-separable, and B is a basic subspace of X. Then G is weakly ω_1-separable because $X \oplus B$ is (cf. Theorem 2.12 of [**10**]); but G is not ω_1-separable because if it were, $X \oplus B$ would be ω_1-separable; but then X would be ω_1-separable because any closed countable subspace of X is closed in $X \oplus B$ and hence would be a direct summand.

The following corollary generalizes Theorem 3.2 of [**16**]. We say that a subset E of ω_1 is *non-small* if $\Phi_{\omega_1}(E)$ holds.

COROLLARY 3.7. *Assume that every stationary subset of ω_1 is non-small. Then every ω_1-separable p-group of cardinality $\aleph_1$ which is not $\sum$-cyclic contains $2^{\aleph_1}$ pairwise non-isomorphic pure subgroups $H_i (i < 2^{\aleph_1})$ such that $G/H_i \simeq Z(p^\infty)$ and H_i is not ω_1-separable.*

Obviously, Theorem 3.3 also has implications for ω-elongations of weakly ω_1 separable p-groups, but, in view of Crawley's problem, we wish to deal also with separable p-groups which are not weakly ω_1-separable.

THEOREM 3.8 (CH) *Let X be a separated ω-filtered vector space of* *dimension* $\aleph_1$ *which is not weakly* ω_1*-separable. Then there are* $2^{\aleph_1}$ *dense subspaces* $Y_i (i < 2^{\aleph_1})$ *of codimension* 1 *in* X *such that, for* $i \neq j$, *(a) and (b) of 3.3 hold.*

PROOF. Let $X = \cup_{\nu<\omega_1} X_\nu$ be an ω_1-filtration of X such that, for all $\nu \geq 1, \overline{X}_\nu$ is uncountable and such that there exists $z \in (\overline{X}_1 \cap X_2) - X_1$. Fix a function $\mathcal{E}$ on ω_1 such that, for every triple (f, γ_0, γ_1), where $f : Y \to Y'$ is a morphism of countable subspaces of X and $\gamma_0, \gamma_1 \in {}^\mu 2$ for some $\mu \in \omega_1$, there exists $\nu > \mu$ such that $\mathcal{E}(\nu) = (f, \gamma_0, \gamma_1)$. (This is possible since there are $2^{\aleph_0} = \aleph_1$ such triples.)

We shall define, by induction on δ, an ordinal $\tau_\delta \geq \delta$ and for each $\sigma \leq \tau_\delta$ and each $\xi \in {}^\sigma 2$, a subspace Y_ξ of X_σ containing X_1 and satisfying:

(0) if $\gamma < \delta$, then $\tau_\gamma < \tau_\delta$;

(i) $X_\sigma = Y_\xi + Kz$, and $z \notin Y_\xi$; and

(ii) for all $\mu < \sigma, Y_{\xi \restriction \mu} \subseteq Y_\xi$, and if $\sigma \in \lim(\omega_1)$, then $Y_\xi = \cup_{\mu<\sigma} Y_{\xi\restriction\mu}$.

Then, for all $\varphi : \omega_1 \to 2, Y_\varphi = \cup_{\delta<\omega_1} Y_{\varphi\restriction\delta}$ will be a dense subspace of X of codimension 1. We shall do the construction so that, for $\varphi_0 \neq \varphi_1, Y_{\varphi_0}$ and Y_{φ_1} satisfy (a) and (b).

Suppose that we have defined τ_ν and Y_ξ as above for all $\nu < \delta$ and all $\xi \in {}^\sigma 2$, where $\sigma \leq \tau_\nu$.

If δ is a limit ordinal we let $\tau_\delta = \sup\{\tau_\nu : \nu < \delta\}$; then τ_δ is a limit ordinal (by (0)) and, for each $\xi : \tau_\delta \to 2$, we let $Y_\xi = \cup_{\mu<\tau_\delta} Y_{\xi\restriction\mu}$.

If $\delta = \nu + 1$ for some ν, we will choose $\tau_\delta > \tau_\nu$, and then, for any $\rho : \tau_\delta \to 2$, define Y_ρ extending $Y_{\rho\restriction\tau_\nu}$. (Then for any $\sigma \leq \tau_\delta$, let $Y_{\rho\restriction\sigma} = Y_\rho \cap X_\sigma$).

Say $\mathcal{E}(\nu) = (f, \gamma_0, \gamma_1)$, where for some $\mu \leq \tau_\nu, \gamma_0, \gamma_1 \in {}^\mu 2, \gamma_0 \neq \gamma_1$ and $f : Y_{\gamma_0} \to Y_{\gamma_1}$. (If $\mathcal{E}(\nu)$ is not of this form, let $\tau_\delta = \tau_\nu + 1$, and let the Y_ρ be defined in any way which satisfies (i) and (ii).) Let $\hat{f} : \hat{Y}_{\gamma_0} \to \hat{Y}_{\gamma_1}$ be the (unique) extension of f to the closure, $\hat{Y}_{\gamma_0}$, of Y_{γ_0} in $\hat{X}$, the completion of X.

Case I. We can find τ_δ and $y \in (X_{\tau_\delta} \cap \overline{Y}_{\gamma_0}) - X_{\tau_\nu}$ such that $\hat{f}(y) \notin X$ or $\hat{f}(y) \in X_{\tau_\delta} - X_{\tau_\nu}$. Consider three subcases.

Subcase IA. $\rho \restriction \mu = \gamma_0$. Let Y_ρ be a maximal extension of $Y_{\rho\restriction\tau_\nu} + Ky$ in X_{τ_δ} with the property that $z \notin Y_\rho$. (Note that $z \notin Y_{\rho\restriction\tau_\nu} + Ky$.)

Subcase IB. $\rho \restriction \mu = \gamma_1$ and $\hat{f}(y) \in X_{\tau_\delta} - X_{\tau_\nu}$. Let Y_ρ be a maximal extension of $Y_{\rho\restriction\tau_\nu} + K(\hat{f}(y) - z)$ in X_{τ_δ} with the property that $z \notin Y_\rho$.

Subcase IC. (not Subcase A or B.) Let Y_ρ be a maximal extension of $Y_{\rho\restriction\tau_\nu}$ in X_{τ_δ} with the property that $z \notin Y_\rho$.

Case II. $\hat{f}(\overline{Y}_{\gamma_0}) \subseteq X_{\tau_\nu}$. Define an equivalence relation $\sim$ on $\overline{Y}_{\gamma_0} - X_{\tau_\nu}$ by $y_1 \sim y_2$ if and only if $y_1 - y_2 \in X_{\tau_\nu}$. Then, since $\overline{Y}_{\gamma_0} - X_{\tau_\nu}$ is uncountable and X_{τ_ν} is countable, there exist inequivalent y_1 and y_2 such that $\hat{f}(y_1) = \hat{f}(y_2)$. Hence $y_1 - y_2$ belongs to $\overline{Y}_{\gamma_0} - X_{\tau_\nu}$ and $\hat{f}(y_1 - y_2) = 0$. Let $y = (y_1 - y_2) + z$. We have $y \in \overline{Y}_{\gamma_0} - X_{\tau_\nu}$ and $\hat{f}(y) = \hat{f}(z)$. Let τ_δ be such that $y \in X_{\tau_\delta}$, and define Y_ρ such that if $\rho \restriction \mu = \gamma_0$, then $y \in Y_\rho$.

This completes the construction. Now we must verify (a) and (b). Let $\varphi_0 \neq \varphi_1 : \omega_1 \to 2$.

(a) Suppose, to obtain a contradiction, that there is an isomorphism $F : Y_{\varphi_0} \to Y_{\varphi_1}$. Then there is $\mu \in \omega_1$ such that $\varphi_0 \restriction \mu \neq \varphi_1 \restriction \mu$ and $F(Y_{\varphi_0 \restriction mu}) = Y_{\varphi_1 \restriction \mu}$. Let $\gamma_0 = \varphi_0 \restriction \mu, \gamma_1 = \varphi_1 \restriction \mu$ and $f = F \restriction Y_{\gamma_0} : Y_{\gamma_0} \to Y_{\gamma_1}$. Then there exists $\nu > \mu$ such that $\mathcal{E}(\nu) = (f, \gamma_0, \gamma)$. Now if $\delta = \nu + 1$, we constructed $Y_{\varphi_0 \restriction \tau_\delta}$ and $Y_{\varphi_1 \restriction \tau_\delta}$ so that there exists $y \in Y_{\varphi_0 \restriction \tau_\delta} \cap \overline{Y}_{\gamma_0}$ such that $\hat{f}(y) \notin Y_{\varphi_1}$. (Note that we are in Case I since $\hat{f}$ is one-one and $\overline{Y}_{\gamma_0}$ is uncountable.) But this is a contradiction since $\hat{f}(y) = F(y) \in Y_{\varphi_1}$.

(b) Suppose, to obtain a contradiction, that there exists $\theta : X \to X$ such that $\theta(Y_{\varphi_0}) \subseteq Y_{\varphi_1}$ but $\theta(z) \notin Y_{\varphi_1}$. Then there exists μ such that $\varphi_0 \restriction \mu \neq \varphi_1 \restriction \mu$, $\theta(Y_{\varphi_0 \restriction \mu}) \subseteq Y_{\varphi_1 \restriction \mu}$ and $\theta(z) \in X_\mu - Y_{\varphi_1 \restriction \mu}$. Let $\gamma_0 = \varphi_0 \restriction \mu$, $\gamma_1 = \varphi_1 \restriction \mu$, and $f = \theta \restriction Y_{\gamma_0} : Y_{\gamma_0} \to Y_{\gamma_1}$. There exists $\nu > \mu$ such that $\mathcal{E}(\nu) = (f, \gamma_0, \gamma_1)$. If we are in Case I, then the contradiction occurs exactly as in (a). Otherwise, in Case II, we have constructed Y_{φ_0} so that $y \in Y_{\varphi_0}$, but $\theta(y) = \hat{f}(y) = \hat{f}(z) = \theta(z) \notin Y_{\varphi_1}$. This contradicts the hypothesis.

For G with countable basic subgroup, the following result is contained in [**24**; Corollary 3.3].

COROLLARY 3.9. (CH) *If G is a separable p-group of cardinality $\aleph_1$ which is not weakly ω_1-separable, then there exist $2^{\aleph_1}$ ω-elongations of G by $\mathbf{Z}(p)$.*

We have not been able to prove Theorem 3.8 under the hypothesis $\Phi_{\omega_1}(\omega)$, i.e., $2^{\aleph_0} < 2^{\aleph_1}$. However, under this weaker hypothesis, using methods similar to those in the proof of Theorem 3.3, we can obtain the conclusion of Theorem 3.8 with $2^{\aleph_1}$ replaced by $\aleph_2$. As a consequence of this and Theorem 2.12 of [**10**], we have

COROLLARY 3.10. ($2^{\aleph_0} < 2^{\aleph_1}$) *Every Crawley group of cardinality $\aleph_1$ is weakly ω_1-separable.*

This corollary parallels Chase's result that, under $2^{\aleph_0} < 2^{\aleph_1}$, e very Whitehead group is strongly ω_1 - free [**1**]. Note also that - just as for Whitehead's problem - Crawley's problem (even for groups of cardinality $\aleph_1$) is not decidable in ZFC+GCH: see [**17**].

Theorems 3.3 and 3.8 yield (under weaker hypotheses than in [**19**]) the following "rigid system" theorem.

COROLLARY 3.11. *Assume that $2^{\aleph_0} = \aleph_1$ and that every stationary subset of ω_1 is non-small. Then, for every separable p-group G which is not $\sum$-cyclic, there exist $2^{\aleph_1}$ ω-elongations $H_i (i < 2^{\aleph_1})$ of G by $\mathbf{Z}(p)$ such that if $i \neq j$, then, for every homomorphism $f : H_i \to H_j, f(p^\omega H_i) = \{0\}$.*

PROOF. This follows from 3.3(b) and 3.8(b) since if $f : H_i \to H_j$, then f induces $\theta : X \to X$ with $\theta(Y_i) \subseteq Y_j$, where $X = G[p]$ and $Y_\ell = P(H_\ell)(\ell = i, j)$; see [**10**, p. 163f]. Then it is not hard to see that $\theta(X) \subseteq Y_j$ implies that $f(p^\omega H_i) = \{0\}$.

REMARK 3.12. Recently, Mekler and Shelah [**20**] have completely solved Crawley's problem in L. However, for groups G of cardinality $> \aleph_1$, the result is much weaker than that for groups of cardinality $\aleph_1$; specifically, they show that if G is not $\sum$-cyclic then there exist at least two ω-elongations of G by $\mathbf{Z}(p)$.

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