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CLOSED FILTERS AND GRAPH–CLOSED MULTIFUNCTIONS IN CONVERGENCE SPACES

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ABSTRACT. We study closed filters which generalize notions of regular filters and closed sets. Applying closed filters, we refine some results of G. Choquet and R.E. Smithson on graph–closed multifunctions.

0. Introduction. It is well known that every graph–closed multifunction $\Gamma : Y \to X$ is closed–valued, has the closed–valued inverse Γ^{-1} and is compact–to–closed, i.e., $\Gamma(K)$ is closed whenever K is a compact subset of Y [**1**]. However, in general compact–to–closed (multi) functions (with closed–valued inverses) need not be graph–closed (see, e.g., [**7**; Example 3.5]). The equivalence may be obtained under additional assumptions, e.g., that Y is a Hausdorff locally compact space (Smithson [**10**]). Besides, it follows from [**8**] that if Y is Hausdorff and for every multifunction $\Gamma : Y \to X$ (with the closed–valued inverse Γ^{-1}) this equivalence holds, then Y is locally compact.

In what follows, it is shown (in greater generality) that graph–closedness can be expressed in terms of some corresponding properties of the images of filters. Namely, graph–closed multifunctions turn out to be exactly those multifunctions with closed–valued inverses that map compact filters into closed filters.

This characterization theorem is in line with some recent results (e.g., [**3**]) which show that the investigation of certain properties of multifunctions (e.g., subcontinuity, upper semi–continuity) can be reduced to the study of filters

1. Terminology and notation. Let X be a nonempty set. Denote by φX the collection of all filters on X and let $\overline{\varphi}X = \varphi X \bigcup \{2^X\}$. We say that filters $\mathcal{F}, \mathcal{G} \in \varphi X$ *meet* [**3**] if $F \bigcap G \neq \emptyset$ for every $F \in \mathcal{F}$ and $G \in \mathcal{G}$. If filters $\mathcal{F}$ and $\mathcal{G}$ meet, then the family $\{F \bigcap G : F \in \mathcal{F}, G \in \mathcal{G}\}$ is a base of the *supremum* filter $\mathcal{F} \vee \mathcal{G}$. Note that $\mathcal{F}$ and $\mathcal{G}$ meet if and

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only if $\mathcal{F} \vee \mathcal{G} \neq 2^X$ [**2**]. We shall denote by $\mathcal{F} \vee A$ the supremum of $\mathcal{F}$ and the *discrete filter* $\mathcal{N}_L(A) = \{B \subset X : A \subset B\}$. Of course, $\mathcal{N}_L(A) \in \varphi X$ if and only if $A \neq \emptyset$.

Every mapping $\pi : \overline{\varphi}X \to 2^X$, such that $\pi(2^X) = \emptyset$, is called a *convergence* on X. The pair (X, π) is called a *convergence space.* Instead of $\pi(\mathcal{F})$ we will write $\lim^\pi \mathcal{F}$ or $\lim \mathcal{F}$. We say that a filter $\mathcal{F}$ is *convergent* ([*convergent to* x]) if $\lim \mathcal{F} \neq \emptyset (x \in \lim \mathcal{F})$.

A convergence π is said to be *constants–preserving* [**2**] if

$$x \in \lim^\pi \mathcal{N}_L(x) \quad \text{for each } x \in X,$$

where $\mathcal{N}_L(x) = \{A \subset X : x \in A\}$ is the *discrete filter* of x. A convergence π is *isotone* [**2**] if $\mathcal{F} \subset \mathcal{G} \in \varphi X$ implies $\lim^\pi \mathcal{F} \subset \lim^\pi \mathcal{G}$. A convergence π satisfying the condition

$$\lim^\pi \mathcal{F} \cap \lim^\pi \mathcal{G} \subset \lim^\pi (\mathcal{F} \cap \mathcal{G})$$

is said to be *finite–stable* [**2**].

A convergence π is called a *pseudotopology* [**6**] if it is constants–preserving, isotone and finite–stable.

If π is a constants–preserving convergence then the intersection of all filters convergent to x is said to be the *neighborhood filter* of $x : \mathcal{N}_\pi(x) = \bigcap (\lim^\pi)^{-1} x$.

If π is a constants–preserving and $x \in \lim^\pi \mathcal{N}_\pi(x)$ for every $x \in X$, then π is called a *pretopology* [**2.6**]. Every pretopology is a pseudotopology. A convergence space (X, π) is said to be *constants–preserving* [*isotone*, etc], if the convergence π is constants–preserving [isotone, etc]. A convergence space (X, π) is *Hausdorff* if

$$x, y \in \lim^\pi \mathcal{F} \text{ implies } x = y,$$

for every $\mathcal{F} \in \varphi X$.

Let (X, π) be a convergence space. The *adherence Adh*$^\pi$ is the mapping $\mathrm{Adh}^\pi : \overline{\varphi}X \to 2^X$ defined as follows

$$\mathrm{Adh}^\pi \mathcal{F} = \begin{cases} \bigcup_{\mathcal{G} \vee \mathcal{F} \neq 2^X} \lim^\pi \mathcal{G}, & \text{if } \mathcal{F} \in \varphi X, \\ \emptyset, & \text{if } \mathcal{F} = 2^X. \end{cases}$$

Let $\beta \mathcal{F}$ denote the family of all ultrafilters finer than $\mathcal{F}$. If (X, π) is isotone, then $\mathrm{Adh}^\pi \mathcal{F} = \bigcup_{\mathcal{U} \in \beta \mathcal{F}} \lim^\pi \mathcal{U} = \bigcup_{\mathcal{G} \supset \mathcal{F}} \lim^\pi \mathcal{G}$. The adherence

$\mathrm{Adh}^\pi \mathcal{N}_L(A)$ of a discrete filter of A is called the *closure* of A and is denoted by $\mathrm{cl}^\pi A$. A set $A \subset X$ is *closed* if $\mathrm{cl}^\pi A \subset A$.

Let $(X_t, \pi_t), t \in T$, be a collection of convergence spaces. The product convergence $\pi = \prod_{t\in T} \pi_t$ on $\prod_{t\in T} X_t$ is defined as follows [**6**]:

$$x \in \lim^\pi \mathcal{F} \text{ if and only if } p_t(x) \in \overset{\sigma}{\lim} p_t(\mathcal{F}) \text{ for each } t \in T.$$

(p_s denotes the projection $p_s : \prod_{t\in T} X_t \to X_s$).

A mapping f from a convergence space (X, π) to a convergence space (Y, σ) is called *continuous at* x if

$$x \in \lim^\pi \mathcal{F} \text{ implies } f(x) \in \overset{\sigma}{\lim} f(\mathcal{F}).$$

(For more information on convergence structures see [**1,2** and **6**]).

2. Closed filters. A filter $\mathcal{F}$ on a convergence space (X, π) is called *closed* if $\mathrm{Adh}^\pi \mathcal{F} \subset \bigcap_{F\in\mathcal{F}} F (= \mathrm{Adh}^L \mathcal{F}$, where L denotes the discrete convergence on X). Note that if π is constants–preserving then $\bigcap_{F\in\mathcal{F}} F \subset \mathrm{Adh}^\pi \mathcal{F}$ for every filter $\mathcal{F}$ on X. Observe that a subset $A \subset X$ is closed if and only if the filter $\mathcal{N}_L(A)$ is closed. Closed filters have the following properties:

(a) *If filters $\mathcal{F}$ and $\mathcal{G}$ are closed, then the filter $\mathcal{F} \vee \mathcal{G}$ is closed.*

(b) *A product of closed filters is closed.*

(c) *The preimage $f^{-1}(\mathcal{F})$ of a closed filter $\mathcal{F}$ by a continuous mapping f is closed.*

Recall that a filter $\mathcal{F}$ on X is called *compactoid* [**3**] if every ultrafilter $\mathcal{U}$ finer than $\mathcal{F}$ is convergent. A filter $\mathcal{F}$ is *compact* [**3**] if for every ultrafilter $\mathcal{U}$ finer than $\mathcal{F}$, $\lim \mathcal{U} \cap F \neq \emptyset$ for each $F \in \mathcal{F}$. A subset $A \subset X$ is *compactoid* [*compact*] if its discrete filter $\mathcal{N}_L(A)$ is compactoid [compact]. (More details about compact filters can be found in [**3** and **8**].)

We have the following:

PROPOSITION 2.1. *Every compactoid and closed filter is compact. In an isotone Hausdorff convergence space every compact filter is closed (and compactoid).*

Note that every closed filter finer than a compact filter is compact. It follows that a closed subset of a compact set is compact.

THEOREM 2.2. *Let X be a Hausdorff pseudotopological space. A filter $\mathcal{F}$ on X is closed if and only if for every compact filter $\mathcal{G}$ that meets $\mathcal{F}$ the filter $\mathcal{F} \vee \mathcal{G}$ is closed (equivalently: compact)*

PROOF. Assume that $\mathcal{F} \vee \mathcal{G}$ is closed for every compact filter $\mathcal{G}$ that meets $\mathcal{F}$. If $x \in \mathrm{Adh}\mathcal{F}$ then $x \in \lim \mathcal{H}$ for some filter $\mathcal{H}$ such that $\mathcal{H} \vee \mathcal{F} \neq 2^X$. Since the filter $\mathcal{G} = \mathcal{N}_L(x) \cap \mathcal{H}$ is compact and $\mathcal{G}$ meets $\mathcal{F}$, the filter $\mathcal{F} \vee \mathcal{G}$ is closed. Hence

$$x \in \mathrm{Adh}(\mathcal{F} \vee \mathcal{G}) \subset \bigcap_{F \in \mathcal{F}, G \in \mathcal{G}} F \cap G \subset \bigcap_{F \in \mathcal{F}} F.$$

It is known [**5**,p.201] that in a k–space X a subset $A \subset X$ is closed if and only if $B \cap X$ is closed (compact) for every compact subset $B \subset X$. Equivalently, A is closed if and only if, for every compact discrete filter $\mathcal{F}$ that meets A, the filter $\mathcal{F} \vee A$ is closed (compact).

More generally, we have:

COROLLARY 2.3. *Let X be a hausdorff pseudotopological space. A subset $A \subset X$ is closed if and only if for every compact filter that meets A the filter $\mathcal{F} \vee A$ is closed (equivalently: compact).*

Following J.–P. Penot [**9**] a filter $\mathcal{F}$ on a convergence space (X, π) is called *regular* if $\mathcal{F} = \overline{\mathcal{F}}$, where $\overline{\mathcal{F}}$ denotes the filter generated by the family $\{\mathrm{cl}^{\pi} F : F \in \mathcal{F}\}$. Every regular filter is closed but not conversely.

EXAMPLE 2.4. Let $X = R$ (with the usual topology) and $A = \{1/n : n \in N\}$. The filter $\mathcal{F}$ generated by

$$\{[0, t] \backslash A : 0 < t \leq 1\}$$

is closed but not regular.

PROPOSITION 2.5. *If $\mathcal{F}$ is a closed compactoid filter on a convergence space X, then Adh$\mathcal{F}$ is compactoid.*

PROOF. If $\mathcal{U}$ is an ultrafilter containing $\mathrm{Adh}\mathcal{F}$, then it contains $\bigcap_{F\in\mathcal{F}} F$. It follows that $\mathcal{F} \subset \mathcal{U}$ and, by assumption $\lim \mathcal{U} \neq \emptyset$.

The above proposition generalizes a result of J.–P. Penot proved in [**9**] for regular compactoid filters in topological spaces (cf. also [**4**]). Another generalization was obtained in [**3**] where it was proved that $\mathrm{Adh}\mathcal{F}$ is compactoid whenever $\mathcal{F}$ is a subregular filter. Recall that $\mathcal{F}$ is *subregular* if its closure filter $\overline{\mathcal{F}}$ is compactoid. Every subregular filter is compactoid but it need not be closed. On the other hand, there are closed compactoid filters which are not subregular.

EXAMPLE 2.6. Let $X = R$ and let $\mathcal{T}$ be the usual topology on R. Define the convergence π as follows:

$$\lim^{\pi} \mathcal{F} = \begin{cases} R & \text{if } \mathcal{F} = \mathcal{N}_L(x) \text{ for some } x \neq 0, \\ \{0\} & \text{if } \mathcal{F} \supset \mathcal{N}_{\mathcal{T}}(0), \\ \emptyset & \text{otherwise.} \end{cases}$$

Then the filter $\mathcal{N}_{\mathcal{T}}(0)$ is closed compactoid but it is not subregular. Indeed, we have

$$\mathrm{Adh}^{\pi} \mathcal{N}_{\mathcal{T}}(0) = \{0\} = \bigcap_{Q\in\mathcal{N}_{\mathcal{T}}(0)} Q$$

but the filter $\overline{\mathcal{N}_{\mathcal{T}}(0)} = \{R\}$ is not compactoid because an ultrafilter finer than the filter generated by the family $\{(t, +\infty) : t \in R\}$ is not convergent.

3. Graph–closed multifunctions. Let X and Y be convergence spaces and let $\Gamma : Y \to X$ be a multifunction. For $A \subset Y$ denote $\Gamma A = \bigcup_{a\in A} \Gamma a$. Let $\mathcal{F}$ be a filter on Y. Then the family $\{\Gamma F : F \in \mathcal{F}\}$ is a base of the *image filter* $\Gamma\mathcal{F}$. If $\Gamma F \neq \emptyset$ for every $F \in \mathcal{F}$ then $\Gamma\mathcal{F} \in \varphi X$; otherwise $\Gamma\mathcal{F} = 2^X$. If $\mathcal{H}$ is a filter on X then $\Gamma^{-1}\mathcal{H}$ will denote the image filter by the inverse multifunction $\Gamma^{-1}x = \{y \in Y : x \in \Gamma y\}$.

A multifunction $\Gamma : Y \to X$ is said to be *graph–closed at* $y \in Y$ if $\mathrm{Adh}\Gamma\mathcal{F} \subset \Gamma y$, whenever $y \in \lim \mathcal{F}$. Γ is *graph–closed* if it is graph–closed at every $y \in Y$. If Y is a pretopological space, then Γ is graph–closed at y if and only if $\mathrm{Adh}\Gamma\mathcal{N}(y) \subset \Gamma y$. Note that if Y is constants–preserving and Γ is graph–closed at y, then

$\mathrm{Adh}\,\mathcal{N}_L(\Gamma y) = \mathrm{Adh}\Gamma\mathcal{N}_L(y) \subset \Gamma y$, i.e., the set Γy is closed. Observe that Γ is graph–closed at y if and only if

$$(y,x) \in \mathrm{cl}\, G(\Gamma) \text{ implies } (y,x) \in G(\Gamma),$$

for every $x \in X$, where $G(\Gamma) = \{(y,x) : x \in \Gamma y\}$ denotes the graph of Γ. Hence a multifunction $\Gamma : Y \to X$ is graph–closed if and only if its graph $G(\Gamma)$ is closed in $Y \times X$.

THEOREM 3.1. *If a multifunction $\Gamma : Y \to X$ is graph–closed then $\Gamma\mathcal{F}$ is closed for every compact filter $\mathcal{F}$ on Y.*

PROOF. If $x \in \mathrm{Adh}\Gamma\mathcal{F}$, then there is a filter $\mathcal{H}$ on X such that $\mathcal{H} \vee \Gamma\mathcal{F} \neq 2^X$ and $x \in \lim\mathcal{H}$. Thus $\mathcal{F}$ and $\Gamma^{-1}\mathcal{H}$ meet and if $\mathcal{U}$ is an ultrafilter finer than $\mathcal{F} \vee \Gamma^{-1}\mathcal{H}$, then $\lim\mathcal{U} \cap F \neq \emptyset$ for each $F \in \mathcal{F}$. Let $F \in \mathcal{F}$ and take $y \in \lim\mathcal{U} \cap F$. Then $\mathrm{Adh}\Gamma\mathcal{U} \subset \Gamma y$. Moreover, $\Gamma\mathcal{U}$ meets $\mathcal{H}$ and the filter $\mathcal{H}$ is convergent to x. Therefore, $x \in \mathrm{Adh}\Gamma\mathcal{U} \subset \Gamma y \subset \Gamma F$, and consequently

$$\mathrm{Adh}\Gamma\mathcal{F} \subset \bigcap_{F\in\mathcal{F}} \Gamma F.$$

Applying the above theorem to a discrete filter $\mathcal{N}_L(A)$ we get the following result due to G. Choquet (for topological spaces):

COROLLARY 3.2. (G. Choquet [**1**]). *If a multifunction $\Gamma : Y \to X$ is graph–closed, then for each compact set $A \subset Y$ the set ΓA is closed.*

Let $\mathcal{F}$ be a compact filter on a convergence space Y. It follows from Theorem 3.1 that for every convergence space X and every graph–closed function $f : Y \to X$, the filter $f(\mathcal{F})$ is closed. We have also the converse.

THEOREM 3.3. *Let (Y,π) be an isotone [constants–preserving, pretopological] Hausdorff convergence space. A filter $\mathcal{F} \neq \{Y\}$ is compact if and only if, for every graph–closed bijective mapping f of Y onto an isotone [constants–preserving, pretopological] Hausdorff convergence space (X,σ), the filter $f(\mathcal{F})$ is closed.*

PROOF. Suppose that $f(\mathcal{F})$ is closed for every bijective graph–closed mapping $f : Y \to X$ but it is not compact. Then $\mathcal{F}$ is not compactoid, i.e., there is an ultrafilter $\mathcal{U} \supset \mathcal{F}$ such that $\lim^{\pi}\mathcal{U} = \emptyset$. Let $F_o \in \mathcal{F}$ be such that $F_o \neq X$ and choose an element $p \in X\backslash F_o$. Now define $(\lim^{\sigma})^{-1}x = (\lim^{\pi})^{-1}x$ for $x \neq p$ and $(\lim^{\sigma})^{-1}p = \{\mathcal{U}\} \cup \{\mathcal{H} \in \sigma X : \mathcal{G} \cap \mathcal{U} \subset \mathcal{H}$ for some $\sigma \in (\lim^{\pi})^{-1}p\}$. The convergence φ is isotone [constants–preserving, pretopological]. Moreover, the space (Y, σ) is Hausdorff. Indeed, if $x, y \in \lim^{\sigma}\mathcal{H}$ and $x \neq p \neq y$, then $x, y \in \lim^{\pi}\mathcal{H}$ and, consequently, $x = y$. Now let $p, y \in \lim^{\sigma}\mathcal{H}$ and suppose that $p \neq y$. Then $\lim^{\pi}\mathcal{H} = \{y\}$ and it implies that $\mathcal{H} \neq \mathcal{U}$. Thus $\mathcal{U} \cap \mathcal{G} \subset \mathcal{H}$ for some filter $\mathcal{G} \in (\lim^{\pi})^{-1}p$. Hence $\mathcal{G} \not\subset \mathcal{H}$ and $\mathcal{H}$ does not meet $\mathcal{U}$. Consequently, $\mathcal{U} \cap \mathcal{G} \not\subset \mathcal{H}$ – a contradiction. Now it is enough to note that the identity mapping $f : (Y, \pi) \to (Y, \sigma)$ is graph–closed but

$$p \in \mathrm{Adh}^{\sigma}\mathcal{F}\backslash \bigcap_{F \in \mathcal{F}} F,$$

contrary to the assumption.

In the case when Y is a topological space we can consider bijective mappings onto topological spaces:

THEOREM 3.4. *Let $(Y, \mathcal{T})$ be a topological Hausdorff space. A filter $\mathcal{F} \neq \{Y\}$ is compact if and only if for every graph–closed bijective mapping f of Y onto a Hausdorff topological space (X, ϱ), the filter $f(\mathcal{F})$ is closed.*

PROOF. (cf. [**7**; Theorem 2.1]). Suppose that $f(\mathcal{F})$ is closed for every graph–closed bijective mapping $f : Y \to X$, but $\mathcal{F}$ is not compact. Then $\mathcal{F}$ is closed and since $\mathcal{F} \neq \{Y\}$, there is a closed set $F_o \in \mathcal{F}$ such that $F_o \neq Y$. Moreover, $\mathcal{F}$ is not compactoid and applying [**3**; Theorem 3.8] we conclude that there is an open cover $\mathcal{H} = \{H_\alpha : \alpha \in A\}$ of Y such that, for every finite subfamily $\mathcal{H}' \subset \mathcal{H}$, we have

$$F \not\subset \bigcup \mathcal{H}' \text{ for every } F \in \mathcal{F}.$$

Let $p \in Y\backslash F_o$ and define the topology ϱ on Y by taking the family

$$\{W\backslash\{p\} : W \in \mathcal{T}\} \cup \{(F\backslash H_\alpha) \cup \{p\} : \alpha \in A, F \in \mathcal{F}\}$$

as its subbase. The topology ϱ is Hausdorff and the identity $f: (Y, \mathcal{T}) \to (Y, \varrho)$ is graph–closed. It is also easy to verify that

$$p \in \mathrm{Adh}^{\varrho}\mathcal{F} \setminus \bigcap_{F \in \mathcal{F}} F,$$

i.e., $f(\mathcal{F})$ is not closed–a contradiction.

From the above theorem we infer

COROLLARY 3.5. [**7**; Theorem 2.1]. *A Hausdorff topological space Y is compact if and only if every bijective mapping of Y onto a Hausdorff space X with a closed graph is closed.*

PROOF. Let A be a closed proper subset of Y. If every bijective graph–closed mapping $f: Y \to X$ is closed then applying Theorem 3.4 to the discrete filter $\mathcal{N}(A) \neq \{Y\}$, we obtain that $\mathcal{N}_L(A)$ is compact, i.e., the set A is compact. Consequently Y is compact.

Note that a multifunction Γ is graph–closed if and only if Γ^{-1} is graph–closed. Hence, if Γ is graph–closed and X is constants–preserving, then $\Gamma^{-1}x$ is closed for every $x \in X$, i.e. Γ^{-1} is closed–valued.

THEOREM 3.6. *Let Y be a pseudotopological space and X a constants–preservi convergence space. Then the following statements are equivalent:*
(a) *A multifunction $\Gamma: Y \to X$ is graph–closed.*
(b) *For all compact filters $\mathcal{F}$ on Y and $\mathcal{G}$ on X, the filters $\Gamma\mathcal{F}$ and $\Gamma^{-1}\mathcal{G}$ are closed.*
(c) *For every compact filter $\mathcal{F}$ on Y, the filter $\Gamma\mathcal{F}$ is closed and Γ^{-1} is closed–valued.*

PROOF. The implication (a)$\Rightarrow$(b) follows from Theorem 3.1 and (b)$\Rightarrow$(c) is obvious.

Now suppose that (c) holds but Γ is not graph–closed. Then there is a filter $\mathcal{F}$ convergent to y_o such that $\mathrm{Adh}\Gamma\mathcal{F} \not\subset \Gamma y_o$. Let $x_o \in \mathrm{Adh}\,\Gamma\mathcal{F} \setminus \Gamma y_o$. Then $y_o \notin \Gamma^{-1}x_o$, and since the set $\Gamma^{-1}x_o$ is closed, the filter $\mathcal{F}$ is disjoint from $\Gamma^{-1}x_o$. Consequently, $x_o \notin \Gamma F_o$ for some

$F_o \in \mathcal{F}$. Since the filter $\mathcal{F} \cap \mathcal{N}_L(y_o)$ is compact, the filter $\Gamma(\mathcal{F} \cap \mathcal{N}_L(y_o))$ is closed. Hence

$$\mathrm{Adh}\Gamma\mathcal{F} \subset \mathrm{Adh}\Gamma(\mathcal{F} \cap \mathcal{N}_L(y_o)) \subset \Gamma y_o \cup \bigcap_{F \in \mathcal{F}} \Gamma F.$$

But on the other hand,

$$x_o \in \mathrm{Adh}\Gamma\mathcal{F} \setminus (\Gamma y_o \cup \bigcap_{F \in \mathcal{F}} \Gamma F),$$

a contradiction.

THEOREM 3.7. *Let Y be a pretopological space and X a constants–preserving convergence space. A multifunction $\Gamma : Y \to X$ is graph–closed if and only if Γ^{-1} is closed–valued and the filter $\Gamma\mathcal{N}(y)$ is closed for each $y \in Y$.*

PROOF. If Γ is graph–closed, then $\Gamma\mathcal{N}(y)$ is closed because the filter $\mathcal{N}(y)$ is compact.

Conversely, assume that $\Gamma\mathcal{N}(y)$ and $\Gamma^{-1}x$ are closed for every $y \in Y$ and $x \in X$. If $x \notin \Gamma y$, then $x \notin \Gamma Q$ for some $Q \in \mathcal{N}(y)$. Hence $x \notin \bigcap_{Q \in \mathcal{N}(y)} \Gamma Q = \mathrm{Adh}\Gamma\mathcal{N}(y)$.

Let Y be a pretopological space. A multifunction $\Gamma : Y \to X$ is called *locally closed at y* if the filter $\Gamma\mathcal{N}(y)$ has a base consisting of closed sets [**10**]. If Γ is locally closed at y then the filter $\Gamma\mathcal{N}(y)$ is closed. It is also regular, whenever X is a constants–preserving convergence space. Consequently, the above theorem generalizes the following result due to R.E. Smithson (obtained for topological spaces).

COROLLARY 3.8. [**10**; Theorem 3.4]. *Let Y be a pretopological space and X be a constants–preserving convergence space. If a multifunction $\Gamma : Y \to X$ is locally closed at each $y \in Y$ and $\Gamma^{-1}x$ is closed for each $x \in X$, then Γ is graph–closed.*

From Theorem 3.7 and Corollary 3.2 we can infer also the following.

COROLLARY 3.9. [**10**; Theorem 3.7] *Let X be a topological space and Y a regular locally compact topological space. A multifunction $\Gamma : Y \to X$ is graph–closed if and only if ΓK and $\Gamma^{-1}x$ are closed for each compact set $K \subset Y$ and $x \in X$.*

A filter $\mathcal{E}$ on X is said *elementary,* if there is a sequence that generates it. The set of all elementary filters on X is denoted by εX. Let (X, π) be a convergence space and let $\pi \vee \varepsilon X$ denote the *upper restriction* [**2**] of the convergence π to the set εX, i.e.,

$$\overset{\pi\vee\varepsilon X}{\lim}\mathcal{F} = \begin{cases} \lim^{\pi}\mathcal{F}, & \text{if } \mathcal{F} \in \varepsilon X, \\ \emptyset, & \text{otherwise.} \end{cases}$$

Sequential adherence (cf. [**2**]) of a filter $\mathcal{F}$ is defined by the formula

$$\mathrm{Adh}^{\pi}_{\mathrm{seq}}\mathcal{F} = \bigcup_{\mathcal{G}\vee\mathcal{F}\neq 2^X} \overset{\pi\vee\varepsilon X}{\lim}\mathcal{G}.$$

A filter $\mathcal{F}$ is called *sequentially closed* if

$$\mathrm{Adh}^{\pi}_{\mathrm{seq}}\mathcal{F} \subset \bigcap_{F\in\mathcal{F}} F.$$

A subset $A \subset X$ is *sequentially closed* if its sequential closure $\mathrm{cl}^{\pi}_{\mathrm{seq}}A = \mathrm{Adh}^{\pi}_{\mathrm{seq}}\mathcal{N}_L(A)$ is included in A.

A multifunction $\Gamma : Y \to X$ is *sequentially graph–closed at $y_o \in Y$* if $\mathrm{Adh}_{\mathrm{seq}}\Gamma\mathcal{E} \subset \Gamma y_o$, whenever $y_o \in \lim \mathcal{E}$ and $\mathcal{E} \in \varepsilon Y$. One can prove that if Y and X are isotone spaces, then $\Gamma : Y \to X$ is sequentially graph–closed (i.e., sequentially graph–closed at every $y \in Y$) if and only if the set $G(\Gamma)$ is sequentially closed in $Y \times X$.

A filter $\mathcal{F}$ is said to be *sequentially compact* if, for every elementary filter $\mathcal{E}$ that meets $\mathcal{F}, \mathrm{Adh}_{\mathrm{seq}}\mathcal{E} \cap F \neq \emptyset$ for each $F \in \mathcal{F}$. A subset $A \subset X$ is *sequentially compact* if its discrete filter $\mathcal{N}_L(A)$ is such (cf. [**6**]).

Recall that a filter $\mathcal{G}$ is called *countably based* if it possesses a countable base. We say that a filter $\mathcal{F}$ is *countably compact* [**3**] if for every countably based filter $\mathcal{G}$ that meets $\mathcal{F}, \mathrm{Adh}\mathcal{G} \cap F \neq \emptyset$ for each $F \in \mathcal{F}$. Note that every countably based and sequentially compact filter is countably compact.

Proceeding as in the proof of Theorem 3.1 we get

THEOREM 3.10. *If a multifunction* Γ *is sequentially graph–closed, then* $\Gamma\mathcal{F}$ *is sequentially closed for every sequentially compact and countably based filter* $\mathcal{F}$.

THEOREM 3.11. *Let* Y *and* X *be constants–preserving and isotone spaces. If* $\Gamma^{-1}x$ *and* ΓK *are sequentially closed for every* $x \in X$ *and every compact set* K, *then* Γ *is sequentially graph–closed.*

PROOF. Suppose that $\mathrm{cl}_{\mathrm{seq}}G(\Gamma)\backslash G(\Gamma) \neq \emptyset$. If $(y_o, x_o) \in \mathrm{cl}_{\mathrm{seq}}G(\Gamma)\backslash G(\Gamma)$, then there is an elementary filter $\mathcal{E}$ convergent to (y_o, x_o) and such that $\mathcal{E} \vee G(\Gamma) \neq 2^{Y\times X}$. Consequently, we can find a sequence (y_n) convergent to y_o and a sequence (x_n) convergent to x_o with the property

$$x_n \in \Gamma y_n \text{ for } n \in N.$$

Since $y_o \notin \Gamma^{-1}x_o$ and $\Gamma^{-1}x_o$ is sequentially closed, the elementary filter of the sequence (y_n) is disjoint from $\Gamma^{-1}x_o$, i.e., $\{y_n : n \geq n_o\} \cap \Gamma^{-1}x_o = \emptyset$ for some $n_o \in N$. Now define $K = \{y_n : n \geq n_o\} \cup \{y_o\}$. Then K is compact but $x_o \in \mathrm{cl}_{\mathrm{seq}}\Gamma K\backslash\Gamma K$, a contradiction.

A convergence space (X, π) is *first countable* [**6**] if for every $x \in X$ and every filter $\mathcal{F}$ convergent to x there is a countably based filter $\mathcal{G} \subset \mathcal{F}$ such that $x \in \lim \mathcal{G}$.

REMARK. Applying terminology introduced in [**2**] we can say that a convergence π is first countable if and only if it is equal to the lower isotonization of the upper restriction of π to the set of all countably based filters.

If (X, π) is a first countable and isotone space, then, for every countably based filter $\mathcal{F}$ on X, we have

$$\mathrm{Adh}^{\pi}_{\mathrm{seq}}\mathcal{F} = \mathrm{Adh}^{\pi}\mathcal{F}.$$

Consequently, $\mathrm{cl}^{\pi}_{\mathrm{seq}}A = \mathrm{cl}^{\pi}A$ for every $A \subset X$. Thus, applying [**2**; Corollary 11.5], we infer that every first countable pretopology is a

Fréchet pretopology [**2**]. One can prove that, in first countable and isotone spaces, a countably based filter is countably compact if and only if it is sequentially compact.

From Theorems 3.10 and 3.11 we get the following

THEOREM 3.12. *Let Y and X be first countable, constants–preserving and isotone spaces. A multifunction $\Gamma : Y \to X$ is a graph–closed if and only if ΓK and $\Gamma^{-1}x$ are closed for each compact set $K \subset Y$ and $x \in X$.*

The above Theorem generalizes the following result:

COROLLARY 3.13. (L.L.Herrington [**7**]). *Let $f : Y \to X$ be a mapping from a first countable topological space Y into a first countable topological T_1 space X. Then the following statements are equivalent:*
(a) *f has a closed graph.*
(b) *$f^{-1}(K)$ is closed in Y for each compact $K \subset X$.*
(c) *f has closed point inverses and $f(K)$ is closed in X for each compact $K \subset Y$.*

L.L. Herrington has also shown that the assumption of the first countability of the range (and the domain) space cannot be relaxed in the above Corollary [**7**; Example 3.5].

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