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AUTOMORPHISM GROUPS OF 3–NODAL RATIONAL CURVES

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While much is known about automorphisms of smooth projective curves, relatively few results apparently exist concerning automorphisms of singular curves. In this note, we consider the simplest type of irreducible singular curves– rational curves with only nodes as singularities. We give a bound for the order of the automorphism group of such a curve and we determine which groups occur as automorphism groups of 2–nodal and 3–nodal rational curves. We will work over the complex numbers and we will let $\mathbf{P}$ denote the complex projective line, which we will also view as the Riemann sphere. By a "curve", we will mean a reduced and irreducible algebraic variety of (complex) dimension one.

A rational nodal curve of arithmetic genus g (i.e., with g nodes) is isomorphic to the quotient of $\mathbf{P}$ obtained by identifying g pairs of distinct points. If a node Q on such a curve is formed by identifying the points a and b of $\mathbf{P}$, then the local ring of Q is

$$O_Q = \{f \in O_a \bigcap O_b : f(a) = f(b)\}$$

(cf.[4]). Let

$$X = (a_1, b_1; \ldots; a_g, b_g)$$

denote the rational nodal curve of arithmetic genus g obtained by identifying the points a_i and b_i of $\mathbf{P}$ for $i = 1, \ldots g$ and let $\pi : \mathbf{P} \to X$ denote the quotient map. We will assume throughout that $g \geq 2$. Let $\mathrm{Aut}\,(X)$ denote the automorphism group of X and let S_n denote the symmetric group of degree n.

PROPOSITION 1. $\mathrm{Aut}\,(X)$ *is isomorphic to the group of all automorphisms* ϕ *of* $\mathbf{P}$ *which satisfy the following property:*

$$(*) \qquad \textit{There exists } \sigma_\phi \in S_g \textit{ such that } \phi(\{a_i, b_i\}) = \{a_{\sigma_\phi(i)}, b_{\sigma_\phi(i)}\} \textit{ for } i = 1, \ldots, g.$$

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PROOF. It is not hard to see that if ϕ is an automorphism of $\mathbf{P}$ (a Möbius transformation) satisfying the above condition, then ϕ descends to an automorphism $\overline{\phi}$ of X and that the map $\phi \to \overline{\phi}$ is a group homomorphism. Conversely, since an automorphism of an algebraic variety induces an automorphism of its normalization, every automorphism of X must be of the form $\overline{\phi}$, where ϕ is a Möbius transformation satisfying the above property, so this homomorphism is onto. Since a Möbius transformation is determined by its action on three points, this homomorphism is also injective.

We will now identify $\mathrm{Aut}\,(X)$ with the set of Möbius transformations satisfying $(*)$. It is easy to see how to get a faithful permutational representation of $\mathrm{Aut}\,(X)$ of degree $2g$.

COROLLARY 1. *There exists an injective group homomorphism* ρ : $\mathrm{Aut}\,(X) \to S_{2g}$.

PROOF. Suppose $\phi \in \mathrm{Aut}\,(X)$. Let $\rho(\phi)$ be the element of S_{2g} determined by the action of ϕ on the set $\{a_1, b_1, \ldots a_g, b_g\}$. It is easy to see that this defines a group homomorphism. The injectivity follows since a Möbius transformation with more than two fixed points must be the identity.

PROPOSITION 2. $|\mathrm{Aut}\,(X)1 \leq 4g(g-1)$.

PROOF. A Möbius transformation ϕ is completely determined by its action on the set $\{a_1, b_1, a_2\}$. If ϕ is going to be in $\mathrm{Aut}\,(X)$, then, by $(*)$, there are $2g$ possibilities for $\phi(a_1)$, $\phi(b_1)$ is determined once $\phi(a_1)$ is known, and there are then $2g-2$ possibilities for $\phi(a_2)$. Of course, not every such choice of $\phi(a_1)$ and $\phi(a_2)$ will necessarily produce an element of $\mathrm{Aut}\,(X)$. Thus the order of $\mathrm{Aut}\,(X)$ is at most $2g(2g-2)$.

Since $\mathrm{Aut}\,(X)$ is a finite group of Möbius transformations, the classical work of F. Klein **[2]** gives the following result.

PROPOSITION 3. Aut (X) *is a cyclic, dihedral, tetrahedal, octahedral or icosahedral group.*

It is a simple matter to classify the automorphism groups of 2–nodal rational curves. We may suppose that $X = (0, \infty; 1, a)$, where $a \notin \{0, \infty, 1\}$. There are seven non–trivial Möbius transformations which are candidates for automorphisms of X.

(1) Suppose $\phi(0) = 0, \phi(\infty) = \infty$, and $\phi(1) = a$. Then $\phi(z) = az$ and $\phi \in \text{Aut}(X) \Leftrightarrow \phi(a) = 1 \Leftrightarrow a = -1$. Note that $\rho(\phi) = (34)$.

(2) Suppose $\phi(0) = \infty, \phi(\infty) = 0$, and $\phi(1) = 1$. Then $\phi(z) = 1/z$ and $\phi \in \text{Aut}(X) \Leftrightarrow \phi(a) = a \Leftrightarrow a = -1$. Note that $\rho(\phi) = (12)$.

(3) Suppose $\phi(0) = \infty, \phi(\infty) = 0$, and $\phi(1) = a$. Then $\phi(z) = a/z$ and $\phi \in \text{Aut}(X)$ since $\phi(a) = 1$. Note that $\rho(\phi) = (12)(34)$.

(4) Suppose that $\phi(0) = 1, \phi(\infty) = a$, and $\phi(1) = 0$. Then $\phi(z) = (-az + a)/(-z + a)$ and $\phi \in \text{Aut}(X)$ since $\phi(a) = \infty$. Note that $\rho(\phi) = (13)(24)$.

(5) Suppose that $\phi(0) = 1, \phi(\infty) = a$, and $\phi(1) = \infty$. Then $\phi(z) = (az - 1)/(z - 1)$ and $\phi \in \text{Aut}(X) \Leftrightarrow \phi(a) = 0 \Leftrightarrow a = -1$. Note that $\rho(\phi) = (1324)$.

(6) Suppose that $\phi(0) = a, \phi(\infty) = 1$, and $\phi(1) = 0$. Then $\phi(z) = (-az + a)/(-az + 1)$ and $\phi \in \text{Aut}(X) \Leftrightarrow \phi(a) = \infty \Leftrightarrow a = -1$. Note that $\rho(\phi) =(1423)$.

(7) Suppose that $\phi(0) = a, \phi(\infty) = 1$, and $\phi(1) = \infty$. Then $\phi(z) = (z - a)/(z - 1)$ and $\phi \in \text{Aut}(X)$ since $\phi(a) = 0$. Note that $\rho(\phi) = (14)(23)$.

We have thus established

PROPOSITION 4. *Suppose* $X \simeq (0, \infty; 1, a)$. *Then*

(1) *If* $a \neq -1$, *then* Aut (X) *is the dihedral group* D_2 *(also known as the Klein four group).*

(2) *If* $a = -1$, *then* Aut (X) *is the dihedral group* D_4. *The group* $\rho(Aut(X))$ *is generated by* (1423) *and* (12).

We note that the generators of the automorphism group of $(0, \infty; 1, -1)$ may be easily visualized. They are: rotation of the sphere by $\pi/2$ with axis the diameter through i and $-i$ and rotation of the sphere by π with axis the diameter through 1 and –1. Notice that the Möbius transformation $\phi(z) = (-z+i)/(z+i)$ induces an isomorphism from $(0, \infty; 1, -1)$

onto $(1,-1;i,-i)$. This example generalizes to give the following result.

PROPOSITION 5. *Suppose*

$$X \simeq (1,-1;\xi_1,-\xi_1;\ldots;\xi_{g-1},-\xi_{g-1})$$

where $\xi_k = \exp(2\pi ik/g)$ *for* $k = 1,2,\ldots,g-1$. *Then* Aut (X) *is the dihedral group* D_{2g}.

PROOF. It is easy to see that the following two Möbius transformations are in Aut (X): rotation of the sphere by π/g with axis the diameter through 0 and ∞ and rotation of the sphere by π with axis the diameter through 1 and -1. These two transformations generate the dihedral group D_{2g} and, as was noted by Klein, this group is the full group of symmetries of this configuration of points (which Klein called a dihedron with $2g$ vertices).

We now proceed to the more difficult task of determining the automorphism groups of 3–nodal rational curves. We may assume that $X = (0,\infty;1,a;b,c)$. We consider the 23 non–trivial Möbius transformations which are candidates for automorphisms of X.

(1) Suppose $\phi(0) = 0$ (hence $\phi(\infty) = \infty$), and $\phi(1) = a$. Then $\phi(z) = az$ and, since $\phi(a)$ must be $1, a = -1$. It follows that $b = -c$ and $\rho(\phi) = (34)(56)$.

(2) Suppose $\phi(0) = 0$ and $\phi(1) = b$. Then $\phi(z) = bz$ and it follows that $ab = c$. There are two subcases to consider.

(i) Suppose $\phi(b) = b^2 = 1$. Then $b = -1$ and $c = -a$. In this case, $\rho(\phi) = (35)(46)$.

(ii)Suppose $\phi(b) = b^2 = a$. It follows that $b^3 = c$, hence $b^4 = 1$, since $\phi(c) = 1$. Since $a \neq 1$ and $a = b^2$, we must have that $b = \pm i$ and $c = \mp i$. In this case, $\rho(\phi) = (3546)$.

(3) The case when $\phi(0) = 0$, and $\phi(1) = c$ is completely similar to the previous case with the roles of b and c interchanged.

(4) Suppose $\phi(0) = \infty$ (hence $\phi(\infty) = 0$) and $\phi(1) = 1$. Then $\phi(z) = 1/z$ and, since $\phi(a) = a$, we must have $a = -1$ and $bc = 1$. Then $\rho(\phi) = (12)(56)$.

(5) Suppose $\phi(0) = \infty$, and $\phi(1) = a$. Then $\phi(z) = a/z$. There are two subcases to consider.

(i) Suppose $\phi(b) = b$. Then b and c are the two square roots of a and $\rho(\phi) = (12)(34)$.

(ii) Suppose $\phi(b) = c$. Then $a = bc$ and $\rho(\phi) = (12)(34)(56)$.

(6) Suppose $\phi(0) = \infty$ and $\phi(1) = b$. Then $\phi(z) = b/z, b = ac$, and $\rho(\phi) = (12)(35)(46)$.

(7) The case when $\phi(0) = \infty$ and $\phi(1) = c$ is completely similar to the previous case with the roles of b and c interchanged.

(8) Suppose $\phi(0) = 1$ (hence $\phi(\infty) = a$) and $\phi(1) = 0$. Then $\phi(z) = a(1-z)/(a-z)$. There are two subcases to consider.

(i) Suppose $\phi(b) = b$. Then b and c are the two roots of the equation $X^2 - 2aX + a = 0$ and $\rho(\phi) = (13)(24)$.

(ii)Suppose $\phi(b) = c$. Then b may be arbitrary so long as b is not a root of $X^2 - 2aX + a$, and $c = a(1-b)/(a-b)$. Then $\rho(\phi) = (13)(24)(56)$.

(9) Suppose $\phi(0) = 1$ and $\phi(1) = \infty$. Then $\phi(z) = (az-1)/(z-1)$. Since $\phi(a) = 0$, we have $a = -1$.

(i) If $\phi(b) = b$, then $b = \pm i$ and $c = \mp i$. In this case, $\rho(\phi) = (1324)$.

(ii)If $\phi(b) = c$ and $\phi(c) = b$, then one may see that $b^2 = -1$. But i and $-i$ are the fixed points of ϕ, so this subcase does not occur.

(10) Suppose $\phi(0) = 1$ and $\phi(1) = b$. Then $\phi(z) = [a(1-b)z + (b-a)]/[(1-b)z + (b-a)]$.

(i) Suppose $\phi(b) = 0$. Then $a = b/(1-b+b^2)$. Since $\phi(c) = \infty$, we have $c = (a-b)/(1-b)$. Note that it follows from these two equations that $c = ab$. In this case, $\rho(\phi) = (135)(246)$.

(ii) If $\phi(c) = 0$ and $\phi(b) = \infty$, then we have $c = (a-b)/a(1-b)$ and $a = b(2-b)$. From these two equations, it follows that $ac = b$. Note that $\rho(\phi) = (135246)$. Now, since the cube of this permutation in $(12)(34)(56)$, by case 5(ii), we must also have $a = bc$. Hence $c = -1$ and it follows that $a = -3$ and $b = 3$. (This can also be seen by solving the equations $\phi(a) = c, \phi(b) = \infty$ and $\phi(c) = 0$ for a, b, and c.)

(11) The case when $\phi(0) = 1$ and $\phi(1) = c$ is completely similar to the previous case with the roles of b and c interchanged.

(12) Suppose $\phi(0) = a$ (hence $\phi(\infty) = 1$) and $\phi(1) = 0$. Then $\phi(z) = (az-a)/(az-1)$. Since $\phi(a) = \infty$, we must have $a = -1$. It follows as in case 9, that $b = \pm i$. Note that $\rho(\phi) = (1423)$.

(13) Suppose $\phi(0) = a$ and $\phi(1) = \infty$. Then $\phi(z) = (z-a)/(z-1)$.

(i) If $\phi(b) = b$, then b and c are the roots of $X^2 - 2X + a = 0$ and $\rho(\phi) = (14)(23)$.

(ii)If $\phi(b) = c$, then b may be arbitrary as long as b is not a root of $X^2 - 2X + a$, and $c = (b-a)/(b-1)$. Note that $\rho(\phi) = (14)(23)(56)$.

(14) Suppose $\phi(0) = a$ and $\phi(1) = b$. Then $\phi(z) = [(b-a)z + a(1-$

$b)]/[(b-a)z+(1-b)]$.

(i) Suppose $\phi(b) = \infty$. Then $a = (b^2-b+1)/b$ and $c = a(b-1)/(b-a)$. Note that it follows that $c = ab$ and $\rho(\phi) = (146)(235)$.

(ii) Suppose $\phi(b) = 0$. Then, as in case 10(ii) it follows that $a = -1/3, b = 1/3$, and $c = -1$. Here $\rho(\phi) = (146235)$.

(15) The case when $\phi(0) = a$ and $\phi(1) = c$ is completely similar to the previous case with the roles of b and c interchanged.

(16) Suppose $\phi(0) = b$ (hence $\phi(\infty) = c$) and $\phi(1) = 0$. Then $\phi(z) = bc(z-1)/(bz-c)$ and, since $\phi(a) = \infty, c = ab$.

(i) If $\phi(b) = 1$, then $c = b^2/(b^2-b+1)$. In this case, $\rho(\phi) = (153)(264)$.

(ii)If $\phi(b) = a$, then it can be seen that $a = -3, b = -1$, and $c = 3$. Here, $\rho(\phi) = (154263)$.

(17) Suppose $\phi(0) = b$ and $\phi(1) = \infty$. Then $\phi(z) = cz - b/(z-1)$ and $ca = b$.

(i) If $\phi(b) = 1$, then it can be seen that $a = -1/3, b = 1/3$, and $c = -1$. In this case, $\rho(\phi) = (153264)$.

(ii)If $\phi(b) = a$, then $b = c^2 - c + 1$ and $\rho(\phi) = (154)(263)$.

(18) Suppose $\phi(0) = b$ and $\phi(1) = 1$. Then $\phi(z) = [c(b-1)z + b(1-c)]/[(b-1)z+1-c]$.

(i) If $\phi(b) = 0$, then $b = (2c-1)/c$ and $a = 2c-1$. In this case, $\rho(\phi) = (15)(26)$.

(ii)If $\phi(b) = \infty$, then $c = b^2 - b + 1$ and $\rho(\phi) = (1526)$. But since the square of this permutation is $(12)(56)$, it follows from case 4 that $a = -1$ and $bc = 1$. Hence $b = \pm i, c = \mp i$.

(19) Suppose $\phi(0) = b$ and $\phi(1) = a$. Then $\phi(z) = [c(b-a)z + b(a-c)]/[(b-a)z+a-c]$ and $\phi(a) = 1$.

(i) If $\phi(b) = 0$, then $a = c(b-1)/(c-1)$. In this case, $\rho(\phi) = (15)(26)(34)$.

(ii) If $\phi(b) = \infty$, then $b^2 - ab + a - c = 0$ and $\rho(\phi) = (1526)(34)$. But the square of this permutation is $(12)(56)$, so by case 4 we have $a = -1$ and $bc = 1$. It then follows that $b = 1$ or $b = -1$, a contradiction. Therefore, this subcase does not occur.

The final four cases are completely similar to the previous four with the roles of b and c interchanged.

We can now determine the groups which occur as automorphism groups of 3–nodal rational curves.

THEOREM. *Suppose* $X \simeq (0,\infty;1,a;b,c)$. *Then* $Aut(X)$ *is one of the* *following groups:*

(1) *the trivial group (this is the generic case)*
(2) *the cyclic group of order* 2
(3) *the dihedral group* D_2
(4) *the dihedral group* D_3
(5) *the dihedral group* D_6 *(if* $X \simeq (0,\infty;1,-3;-1,3)$*)*
(6) *the octahedral group (if* $X \simeq (0,\infty;1,-1;i,-i)$*).*

PROOF. It is clear that Aut (X) will be trivial unless a, b, and c satisfy at least one of the relations which we derive above.

It is not hard to see from our above computations that Aut (X) will be cyclic of order 2 for a "generic" choice of a, b, and c such that $a = bc$. An example of this would be $X_1 = (0,\infty;1,-1;-2,1/2)$; here, $\rho(\text{Aut}\,(X_1)) = \{(1),(12)(34)(56)\}$. This is an example of a "quasihyperelliptic" curve (a singular curve admitting a degree 2 morphism onto $\mathbf{P}$).

For special choices of a, b, and c such that $a = bc$, the automorphism group may be larger. For example, if $X_2 = (0,\infty;1,-1;2,-2)$, then an easy computation shows that Aut (X_2) is the dihedral group D_2 and $\rho(\text{Aut}\,(X_2)) = \{(1),(34)(56),(12)(35)(46),(12)(36)(45)\}$.

An inspection of our computations above also shows that whenever X admits an automorphism of degree 3, then either $c = ab$ or $b = ac$. But then, by cases 6 and 7, X must also admit an involution. Hence Aut (X) may not be cyclic of order 3. An example where Aut (X) is D_3 is $X_3 = (0,\infty;1,2/3;2,4/3)$; here, $\rho(\text{Aut}\,(X_3))$ is generated by $(153)(264)$ and $(12)(36)(45)$.

Next, we show that Aut (X) is never the tetrahedral group (also known as A_4). The tetrahedral group may be generated by an element σ of order 3 and an element τ of order 2 such that $\sigma\tau$ has order 3 [**1**]. Suppose that Aut (X) is the tetrahedral group and that $\rho(\text{Aut}\,(X))$ is generated by $\rho(\phi)$, of order 3, and $\rho(\psi)$, of order 2, where $\rho(\phi)\rho(\psi)$ has order 3. Then $\rho(\psi)$ must be an even permutation. But, as noted above, we must have either $c = ab$ or $b = ac$. Hence, by cases 6 and 7, Aut (X) contains an element θ such that $\rho(\theta)$ is an odd permutation. But this is a contradiction since the generators of $\rho(\text{Aut}\,(X))$ are even.

We also see from our computations that the only cases in which X admits an automorphism of order 4 correspond to $a = -1, b = \pm i$ and $c = \mp i$. As the reader may check, the automorphism group of the curve $X_4 = (0,\infty;1,-1;i,-i)$ has order 24 and $\rho(\text{Aut}\,(X_4))$ is generated by

(135)(246),(13)(24)(56), and (3546). Thus Aut (X_4) is the octahedral group (also known as S_4); in fact, as noted by Klein, $0, \infty, 1, -1, i, -i$ are the vertices of an octahedron inscribed in the sphere and we have identified diametrically opposite vertices. We remark that the curve X_4 also has the remarkable property that it has no nonsingular Weierstrass points [**3**]. Note that we have shown that Aut (X) may not be cyclic of order 4.

Now consider the cases when X admits an automorphism of order 6. Inspection of our computations shows that X must then be isomorphic to $X_5 = (0, \infty; 1, -3; -1, 3) \simeq (1, -1; \exp(2\pi i/3), -\exp(2\pi i/3); \exp(4\pi i/3), -\exp(4\pi i/3))$. We have seen that the automorphism group of this curve is D_6 (explicitly, (154263) and (12)(36)(45) generate $\rho(\mathrm{Aut}\,(X_5))$). Hence Aut (X) may not be cyclic of order 6.

Since the order of an element of Aut (X) may only be 1,2,3,4,or 6, the proof of the Theorem is complete.

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