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ABSOLUTELY CONTINUOUS SPECTRA OF PERTURBED PERIODIC HAMILTONIAN SYSTEMS

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ABSTRACT. This paper compares the spectrum of the Hamiltonian system $J\vec{y}\,' = (\lambda R(x) + Q(x))\vec{y}, -\infty < x < \infty$, with periodic coefficient matrices $R(x)$ and $Q(x)$, to that of a perturbed system $J\vec{y}\,' = (\lambda R(x) + Q(x) + P(x))\vec{y}$, where $P \in L^1_{\mathbf{R}}(-\infty, \infty)$. We show that the perturbation can introduce at most eigenvalues into the gaps between the endpoints of the stability intervals of the periodic system. We prove that the spectral function is continuously differentiable across the continuous spectrum. Further, it follows from the results here that the essential spectrum, the absolutely continuous spectrum and the singular continuous spectrum are invariant.

1. Introduction. We will study the 2×2 Hamiltonian system

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\vec{y}\,' = \left(\lambda \begin{pmatrix} r_1(x) & r_{12}(x) \\ r_{12}(x) & r_2(x) \end{pmatrix} + \begin{pmatrix} q_1(x) & q_{12}(x) \\ q_{12}(x) & q_2(x) \end{pmatrix}\right)\vec{y}\,', \quad -\infty < x < \infty, \tag{1.1}$$

which will be assumed to have real and piecewise continuous coefficient matrices which are periodic. We shall write $\vec{y}(x) = \binom{y(x)}{\hat{y}(x)}$ for a solution of (1.1), but otherwise our notation agrees with that of [**8**] and [**24**]. If the coefficients have period T, then $\vec{z}(t) = \vec{y}(tT)$ satisfies an equation of the form (1.1) with coefficients of period 1. Thus we will assume, without loss of generality, that $T = 1$.

Our objective is to examine spectral properties of operators associated with (1.1) as compared with those of operators arising from the

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perturbed system

$$\begin{aligned}(1.2)\qquad \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\vec{y}' = \Bigg(&\lambda \begin{pmatrix} r_1 & r_{12} \\ r_{12} & r_2 \end{pmatrix} \\ &+ \begin{pmatrix} q_1 & q_{12} \\ q_{12} & q_2 \end{pmatrix} + \begin{pmatrix} p_1(x) & p_{12}(x) \\ p_{12}(x) & p_2(x) \end{pmatrix}\Bigg)\vec{y},\end{aligned}$$

for a suitably small perturbation term

$$P(x) = \begin{pmatrix} p_1(x) & p_{12}(x) \\ p_{12}(x) & p_2(x) \end{pmatrix}.$$

Specifically we look at the questions of whether the continuous spectrum of (1.1) is stable under the perturbation, when eigenvalues can be introduced by it and what order of smoothness of the spectral functions may be expected from (1.1) and (1.2). Introducing obvious notation we write (1.1) and (1.2) in matrix form as

$$(1.3)\qquad J\vec{y}' = (\lambda R(x) + Q(x))\vec{y},$$

$$(1.4)\qquad J\vec{y}' = (\lambda R(x) + Q(x) + P(x)\vec{y}.$$

We are going to assume that our systems divide into the two categories, distinguished as follows:

$$(1.5)\qquad \begin{cases} \text{Case I:} & R(x) > 0; \text{ i.e., } R(x) \text{ is positive definite.} \\ \text{Case II:} & R(x) = \begin{pmatrix} r_1(x) & 0 \\ 0 & 0 \end{pmatrix}, r_1(x) > 0. \end{cases}$$

In Case I (1.4) is known as a *Dirac system* [**8,24,18,15,8**]. A detailed stability interval analysis for unperturbed periodic Dirac systems has been given by Harris in [**8**]. (The matrix $J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ is replaced in [**8**] by $q(x)\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, q(x) > 0$. However, the change of variable $t = \int_0^x (1/q)ds$ transforms the system in [**8**] to the form (1.3), having period $\int_0^1 (1/q)ds$.) Harris considers the selfadjoint operator $T_0(\vec{y}) = R^{-1}(J\vec{y}' - Q\vec{y})$ in the space $L_R^2(-\infty, \infty)$ of square–integrable vector functions, relative to the weight $R(x)$, and finds that the spectrum of

T_0 is purely continuous and consists of the stability intervals associated with the Floquet theory of (1.3). Therefore the classical analysis of Eastham in [**5**] for scalar equations carries over to Dirac systems.

Case II encompasses the scalar perturbed Hill's equation $-y'' + qy + py = \lambda y$ by means of the imbedding

(1.6)
$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} y \\ y' \end{pmatrix}' = \left(\lambda \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} -q & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} -p & 0 \\ 0 & 0 \end{pmatrix} \right) \begin{pmatrix} y \\ y' \end{pmatrix}.$$

In [**14**] we investigated how the stability intervals (i.e., spectrum) of Hill's equation $-y'' + qy = \lambda y, q(x)$ periodic, are altered when a perturbation term $p(x) \in L^1(-\infty, \infty)$ is included. We established absolute continuity of the continuous parts of the spectra of both the perturbed and unperturbed equations. The rationale for taking $L^1(-\infty, \infty)$ perturbations is discussed at length in [**14**].

The present paper is an effort to extend the results of [**14**] to the setting of Hamiltonian systems (1.3) and (1.4). We will require in particular the corresponding hypothesis

$$P(x) \in L^1(-\infty, \infty) \tag{1.7}$$

which means that each entry of $P(x)$ is absolutely integrable. Our results extend Theorems 2.7 and 2.8 of [**8**] in Case I, but we employ entirely different methods of spectral theory from those of [**8**]. Nevertheless, we will rely on many of the basic facts about Floquet theory from [**8**]. Our principal result, Theorem 4.1 in §4, asserts that the continuous spectra (definitions of all relevant terms from spectral theory will be given presently) of (1.3) and (1.4) have equal interiors and are, in fact, absolutely continuous spectra; i.e., there are no imbedded interoir eigenvalues. From an operator point of view we prove more. It follows from the results here that for the perturbation (1.4) of (1.3) the essential spectrum is invariant, the absolutely continuous spectrum is invariant, and the singular continuous spectrum is invariant (since it is the empty set in both cases). The latter conclusion is somewhat surprising since the singular continuous spectrum is not even stable under rank one perturbations [**22**, P. 140]. While abstract theorems using the resolvent are available to locate the absolutely continuous spectrum (cf. [**22**, p. 138]) we find it easier to use a scalar valued component of the resolvent–the Titchmarsh–Weyl m–coefficient. Even for location of the essential spectrum of (1.4), it should be noted that there are perturbations satisfying (1.7) which are not relatively compact perturbations of (1.3) so that results such as Weyl's theorem [**22**, p. 113] are not always applicable. As concerns the invariance of the absolutely continuous spectrum, the closest related result seems to be a paper of Thomas [**23**] on the Laplacian in $\mathbf{R}^3$. By using calculations similar to [**16**], one can handle large perturbations at one singular endpoint. Results of this type have been obtained by Carmona [**3**].

To place (1.3) and (1.4) in an operator–theoretic setting we introduce the space $L^2_R(-\infty,\infty)$ of equivalence classes of measurable functions $\vec{y}$ which satisfy

$$\int_{-\infty}^{\infty} \vec{y}^{\,*} R \vec{y}\, dx < \infty. \tag{1.8}$$

If $R > o$ then $L^2_R(-\infty,\infty)$ is a Hilbert space under the inner product $(\vec{f},\vec{g})_R = \int_{-\infty}^{\infty} \vec{f}^{\,*} R\vec{g}\, dx$. In this case we define an operator T with domain $D(T)$ by the conditions $\vec{y} \in D(T)$ if and only if $y \in L^2_R(-\infty,\infty)$, $\vec{y}$ is locally absolutely continuous, $R^{-1}(J\vec{y}\,' - (Q+P)\vec{y}) \in L^2_R(-\infty,\infty)$ and $T(\vec{y}) = R^{-1}(J\vec{y}\,' - (Q+P))\vec{y}$. Then $T : D(T) \to L^2_R(-\infty,\infty)$ is a symmetric Hilbert space operator and (1.4) is equivalent to the operator equation $T(\vec{y}) = \lambda\vec{y}$. Let T_0 be the "unperturbed operator" obtained from T by replacing $P(x)$ by 0; we will call T the "perturbed operator". Harris proved in [**8**] that T_0 is selfadjoint; i.e., (1.3) is of limit point type at $\pm\infty$, or in other words (1.3) has no nontrivial solutions in $L^2_R(-\infty,\infty)$ for any nonreal λ. Being selfadjoint, T_0 has a real spectrum $\sigma(T_0)$. The main conclusion of [**8**] is that $\sigma(T_0)$ is purely continuous and consists of the stability intervals of (1.3). One of the things we will prove is that the operator T under $L^1(-\infty,\infty)$ perturbations P is also selfadjoint; furthermore the continuous spectra of T and T_0 have equal interiors (so that no eigenvalues are introduced into stability intervals) and are absolutely continuous.

Under Case II of (1.5) the expression $||\vec{f}||_R = (\int_{-\infty}^{\infty} \vec{f}^{\,*} R\vec{f}\, dx)^{\frac{1}{2}}$ is only a seminorm, and we certainly cannot define the operator T as above. We will adopt the approach taken in [**11**] and [**12**] to define operators and their spectra. First we require F.V. Atkinson's "definiteness hypothesis" [**2**, p. 253]

$$\int_a^b \vec{y}^{\,*} R\vec{y}\, dx > 0, -\infty < a < b < \infty,\ \mathrm{Im}\,(\lambda) \neq 0, \tag{1.9}$$

for every solution $\vec{y}$ of (1.3) or (1.4) which does not vanish identically. (This condition is needed to guarantee existence of the Titchmarsh-Weyl coefficient defined below.) Let $\tilde{L}^2_R(-\infty,\infty) = \{\vec{y} \in L^2_R(-\infty,\infty) | \hat{y} = 0\}$ be the set of vectors in $L^2_R(-\infty,\infty)$ with vanishing second component, so that $||\vec{f}||_R$ is anorm when restricted to $\tilde{L}^2_R(-\infty,\infty)$. Let $D(T)$ be the set of locally absolutely continuous $\vec{y} \in L^2_R(-\infty,\infty)$ such that $\begin{pmatrix} r_1^{-1} & 0 \\ 0 & 0 \end{pmatrix}[J\vec{y}' - (Q+P)\vec{y}] \in \tilde{L}^2_R(-\infty,\infty)$ and set

$$(1.10)\quad T(\vec{y}) = \begin{pmatrix} r_1^{-1} & 0 \\ 0 & 0 \end{pmatrix}[J\vec{y}' - (Q+P)\vec{y}] : D(T) \to \tilde{L}^2_R(-\infty,\infty).$$

The vanishing of the second component of $J\vec{y} - (Q+P)\vec{y}$ is required to make (1.4) agree with the operator equation $T(\vec{y}) = \lambda\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\vec{y}$; see **[12]** and **[13]**. Note the special case (1.6) in which (1.10) is $T(\vec{y}) = \begin{pmatrix} -y'' + (q+p)y \\ 0 \end{pmatrix}$. Again let T_0 be the operator obtained from T by setting $P(x) = 0$.

Even though T, given by (1.10), is not in the strict sense a Hilbert space operator, we can still define its spectrum by following the method of **[12]** and **[13]**. The set $\rho(T)$ of all λ such that $R_\lambda(T) = \left(T - \lambda\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\right)^{-1} : \tilde{L}^2_R(-\infty,\infty) \to L^2_R(-\infty,\infty)$ exists and is a bounded operator in the norm $||\vec{f}||_R$ will be called the *resolvent set* of T. The spectrum $\sigma(T)$ of T is the complement of $p(T)$ in the complex numbers. The set of isolated points of $\sigma(T)$ is called the *point spectrum* of T, denoted by $P(T)$. The set $E(T) = \sigma(T) - P(T)$ is the *essential spectrum.* The set $PC(T) \subset E(T)$ consisting of eigenvalues in the essential spectrum, those λ for which there is a nontrivial solution $\vec{y} \in D(T)$ of $T(\vec{y}) = \lambda\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\vec{y}$, will be termed the *point-continuous* spectrum. Finally, $C(T) = E(T) - PC(T)$ is called the *continuous spectrum* of T.

The asymptotics we develop in §3 will show that (1.4) lies in the limit point case at $\pm\infty$. By **[10]** and **[21]** we can then construct the resolvent operator $R_\lambda(T)$, by means of a Green's function, for any nonreal λ. Hence the spectrum of T as given by (1.10) is real.

The definitions of the parts of the spectrum of (1.10) listed above are exactly the usual ones for Hilbert space operators provided we replace $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ by the identity matrix, $\begin{pmatrix} r_1^{-1} & 0 \\ 0 & 0 \end{pmatrix}$ by R^{-1} in (1.10) and $\tilde{L}^2_R(-\infty,\infty)$ by $L^2_R(-\infty,\infty)$. Henceforth in this paper T and T_0 will refer to the perturbed and unperturbed operators defined as above under either of the alternatives (1.5). It will not be necessary to specify which case actually prevails. In this way Theorem 4.1 (§4) will apply to both Dirac systems and also to non-positive definite systems, including scalar equations via (1.6).

In §2 we work with the Floquet representation relevant to (1.3) and define the stability intervals, §3 contains asymptotic forms of solutions and calculation of Titchmarsh–Weyl coefficients, and in §4 we prove the main theorem.

Finally, we mention some related works of interest. The paper [**17**] of Khrabustovskii takes up vector equations, of arbitrary order, with periodic coefficients and perturbations thereof. The coefficient of the highest derivative is required to be nonsingular and, when fitted to (1.3) and (1.9), the perturbation term $P(x)$ must satisfy $\int_x^{x+1} ||P(t)||^2 dt \to 0, x \to \infty$. It is noted that the essential spectrum is unchanged, and then further conditions which guarantee finiteness of the number of eigenvalues introduced into gaps between stability intervals are given. Further references to similar studies may be found in [**17**]. Maksudov and Veliev are concerned in [**19**] with spectral singularities of (1.3), where $R = I$ and Q is diagonal and complex valued, and their characterization in terms of the (regular) eigenvalues of (1.3) for the interval $0 \leq x \leq 1$. Their results are extended to systems of arbitrary order in [**20**]. Misjura has given in [**21**] necessary and sufficient conditions that a given sequence comprise the eigenvalues of a Dirac system with periodic or antiperiodic boundary values.

F.V. Atkinson has given in [**1**] a method for establishing continuously differentiable spectra for L^1–perturbations of Sturm–Liouville operators.

2. Floquet representation. The beginning part of this section is largely a restatement of the corresponding one in [**8**]. The presentation is given here for the sake of completeness.

Let $\vec{\theta}(x,\lambda)$ and $\vec{\varphi}(x,\lambda)$ be the unique solutions of (1.4) which satisfy the initial conditions

$$\vec{\theta}(0,\lambda) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \vec{\varphi}(0,\lambda) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad \text{for all complex } \lambda. \tag{2.1}$$

We let $\vec{\theta}_0(x,\lambda)$ and $\vec{\varphi}_0(x,\lambda)$ be the corresponding solutions of the unperturbed equation (1.3). We note along with [**8**] and [**14**] that these solutions are entire functions of λ for each fixed x, so that the "discriminant" function

$$D(\lambda) = \theta_0(1, \lambda) + \hat{\varphi}_0(1, \lambda) \tag{2.2}$$

is also entire. Just as in the scalar case, the "stability intervals" of (1.3) are defined as those open real λ–intervals for which $-2 < D(\lambda) < 2$. Harris [**8**] proved that the interior of $\sigma(T_0)$ consists of the union, S, of the stability intervals in Case I. Following [**14**], we make the simplifying assumption that

(2.3) the stability intervals are disjoint.

If two intervals in S should have a common endpoint λ_0, a special argument is needed to show $\lambda_0 \notin PC(T)$; we give this in §4.

Harris' proof in [**8**] that

$$D'(\lambda) \neq 0 \text{ in stability interval}$$

carries over to the nonnegative definite Case II of (1.5). Thus an easy proof [**14**] establishes that

$$\nu \operatorname{Im} D(\lambda + i\nu) \text{ has a constant sign, } \lambda \in S, \tag{2.4}$$

for all ν sufficiently small.

Now let I be an open interval whose closure is contained in the interior of one of the stability intervals. Using (2.4) we can find a region $\Omega_I = \{\lambda = \lambda_1 + i\lambda_2 | \lambda_1 \in I, -\delta < \lambda_2 < \delta, \delta > 0\}$ such that $\lambda_2 \operatorname{Im} D(\lambda_1 + i\lambda_2)$ is of constant sign for $\lambda \in \Omega_I$. Let $D(\Omega_I)$ be the image of Ω_I under $D(\lambda)$.

The Floquet multipliers $\rho = \rho(\lambda)$ are solutions of $\rho^2 - D\rho + 1 = 0$, which can write as $D(\lambda) = \rho(\lambda) + \rho(\lambda)^{-1}$. The roots ρ_1 and ρ_2 satisfy $\rho_1\rho_2 = 1$, but $\rho_1 = \pm 1$ and $\rho_2 = \pm 1$ only when $D = \pm 2$. Under the mapping $D = \rho + \rho^{-1}$, from the "ρ–plane" to the "D–plane", the inverse image of $D(\Omega_I)$ consists of two disjoint open sets (for δ sufficiently small), one in each of the upper and lower half planes, each of which encloses an arc of the unit circle. We can therefore determine an analytic branch of $\rho(\lambda)$ in which satisfies $|\rho(\lambda)| > 1$ for $\lambda_2 > 0, | \rho(\lambda) | < 1$, for $\lambda_2 < 0$ and $|\rho(\lambda_1)| = 1$ for $\lambda = \lambda_1 + i\lambda_2 \in \Omega_I$. If we let $\alpha(\lambda) = \mu(\lambda) + i$ and $v(\lambda)$ be an analytic logarithm of $\rho(\lambda)$

in Ω_I, then there follows $u(\lambda) > 0$ for $\lambda_2 > 0$, and $u(\lambda) < 0$ for $\lambda_2 < 0, e^{\alpha(\lambda)} = \rho(\lambda), \lambda = \lambda_1 + i\lambda_2 \in \Omega_I$. This discussion parallels the corresponding one for Hill's equation in [**14**].

The usual Floquet representation [**8**] gives independent solutions of (1.3) in the forms

$$\vec{\psi}_1(x,\lambda) = e^{\alpha(\lambda)x}\vec{p}_1(x,\lambda), \vec{\psi}_2(x,\lambda) = e^{-\alpha(\lambda)x}\vec{p}_2(x,\lambda),$$

for $\lambda \in \Omega_I$ and $-\infty < x < \infty$, where $\vec{p}_K(x,\lambda) = \binom{p_K(x,\lambda)}{\hat{p}_K(x,\lambda)}$ are periodic in x and analytic in λ. The quantity $\Delta(\lambda) = p_1(0,\lambda)\hat{p}_2(0,\lambda)p_- \hat{p}_1(0,\lambda)p_2(0,\lambda)$ will be used for further representation of the solutions of (1.3).

LEMMA 2.1. *For $\lambda \in \Omega_I, \Delta(\lambda) \neq 0$.*

PROOF. The function $\Delta(\lambda) = 0$ if and only if

$$\begin{aligned} p_1(0,\lambda)c_1 + p_2(0,\lambda)c_2 &= 0 \\ \hat{p}_1(0,\lambda)c_1 + \hat{p}_2(0,\lambda)c_2 &= 0 \end{aligned} \tag{2.5}$$

has a nontrivial solution (c_1, c_2), for $\Delta(\lambda)$ is the determinant of this system. But (2.5) is equivalent to $c\vec{\psi}_1(0,\lambda) + c_2\vec{\psi}_2(0,\lambda) = \vec{0}$ and, by uniqueness of solutions to initial value problems, this is equivalent to $c_1\vec{\psi}_1(x,\lambda) + c_2\vec{\psi}_2(x,\lambda) = \vec{0}$, contradicting the independence of the $\vec{\psi}_k$. This contradiction completes the proof.

LEMMA 2.2. *For $\lambda \in \Omega_I$,*

$$\begin{aligned} \vec{\theta}_0(x,\lambda) &= (\hat{p}_2(0)e^{\alpha x}\vec{p}_1(x) - \hat{p}_1(0)e^{-\alpha x}\vec{p}_2(x))/\Delta(\lambda), \\ \vec{\varphi}_0(x,\lambda) &= (-p_2(0)e^{\alpha x}\vec{p}_1(x) + p_1(0)e^{-\alpha x}\vec{p}_2(x))/\Delta(\lambda), \end{aligned} \tag{2.6}$$

where we have suppressed the λ–dependent in the numerators.

PROOF. This follows from writing $\vec{\theta}_0(x) = c_1e^{\alpha x}\vec{p}_1(x)+c_2e^{-\alpha x}\vec{p}_2(x)$, $\vec{\varphi}_0(x) = d_1e^{\alpha x}\vec{p}_1(x) + d_2e^{-\alpha x}\vec{p}_2(x)$, setting $x = 0$ and solving for the constants using the definition of $\Delta(\lambda)$.

For $\lambda \in \Omega_I$, $\lambda_2 > 0$, we obviously have $\vec{\psi}_2 \in L^2_R[0,\infty), \vec{\psi}_1 \notin L^2_R[0,\infty), \vec{\psi}_1 \in L^2_R(-\infty,0], \vec{\psi}_2 \notin L^2_R(-\infty,0]$. Therefore (1.3) is of limit point type at each of $\pm\infty$. According to [**10**], [**11**] or [**12**] we may form the Titchmarsh–Weyl functions

$$m_0^{(+)}(\lambda) = -\lim_{x\to\infty} \frac{\theta_0(x,\lambda)}{\varphi_0(x,\lambda)},\ m_0^{(-)}(\lambda) = -\lim_{x\to-\infty} \frac{\theta_0(x,\lambda)}{\varphi_0(x,\lambda)},$$

where the limits exists and define unique analytic functions for $\mathrm{Im}\,(\lambda) \neq 0$. The relations

$$\lambda_2 \mathrm{Im}\, m_0^{(+)}(\lambda) > 0,\ \lambda_2 \mathrm{Im}\, m_0^{(-)}(\lambda) < 0,\ m_0^{(\pm)}(\overline{\lambda}) = \overline{m_0^{(\pm)}(\lambda)}, \tag{2.7}$$

are known to hold [**10, 11, 12**] for $\lambda = \lambda_1 + i\lambda_2$, $\lambda_2 \neq 0$. H. Weyl (see [**6**]) derived the m–coefficient in order to exhibit solutions of integrable square of a scalar equation. In our case, the solutions $\vec{X}_0^{(+)}(x,\lambda) = \vec{\theta}_0(x,\lambda) + m_0^{(\pm)}(\lambda)\vec{\varphi}_0(x,\lambda)$ satisfy

$$\vec{X}_0^{(\pm)} \in L^2_R[0,\infty),\ \vec{X}_0^{(-)} \in L^2_R(-\infty,0],\ \mathrm{Im}\,(\lambda) \neq 0, \tag{2.8}$$

and uniquely up to constant multiples. In view of (2.6), and suppressing λ dependence on the right,

$$\begin{aligned}\Delta(\lambda)\vec{X}^{(+)}(x,\lambda) &= (\hat{p}_2(0) - m^{(+)}p_2(0))e^{\alpha x}\vec{p}_1(x) \\ &\quad + (-\hat{p}_1(0) + m_0^{(+)}p_1(0))e^{-\alpha x}\vec{p}_2(x),\end{aligned} \tag{2.9}$$

for $\mathrm{Im}\,\lambda \neq 0$. Taking $\lambda = \lambda_1 + i\lambda_2 \in \Omega_I, \lambda_2 > 0$, we see that the term $e^{\alpha x}\vec{p}_1(x)$ becomes unbounded as $x \to \infty$ by the property $u(\lambda) > 0$ for $\alpha = u + iv$. In order that (2.8) not be violated, it must be that $\hat{p}_2(0) - m_0^{(+)}p_2(0) = 0$ in (2.9) with the (+). From this equation we see that $p_2(0) \neq 0$, as otherwise we would have $\hat{p}_2(0) = 0$ and therefore $\Delta(\lambda) = 0$, contradicting Lemma 2.1. Therefore $m_0^{(+)}(\lambda) = \hat{p}_2(0,\lambda)/p_2(0,\lambda)$ for $\lambda = \lambda_1 + i\lambda_2 \in \Omega_I, \lambda_2 > 0$. Working with the $(-)$ version of (2.9), making $x \to -\infty$ and using the second part of (2.8) yields similarly $m_0^{(-)}(x) = \hat{p}_1(0,\lambda)/p_1(0,\lambda), \lambda = \lambda + i\lambda_2 \in \Omega_I, \lambda_2 > 0$.

The same idea can be used for the case $\lambda_2 < 0$ if we switch the order in which the $(\pm)$ signs are used in (2.9). We record all these properties as follows; compare [**14**].

LEMMA 2.3. *For $\lambda = \lambda_1 + i\lambda_2 \in \Omega_I, \lambda_2 \neq 0$, we have $p_1(0,\lambda) \neq 0 \neq p_2(0,\lambda)$ and*

$$
(2.10) \qquad \begin{aligned}
m_0^{(+)}(\lambda) &= \hat{p}_2(0,\lambda)/p_2(0,\lambda), \qquad \lambda_2 > 0,\\
m_0^{(+)}(\lambda) &= \hat{p}_1(0,\lambda)/p_1(0,\lambda), \qquad \lambda_2 < 0,\\
m_0^{(-)}(\lambda) &= \hat{p}_1(0,\lambda)/p_1(0,\lambda), \qquad \lambda_2 > 0,\\
m_0^{(-)}(\lambda) &= \hat{p}_2(0,\lambda)/p_2(0,\lambda), \qquad \lambda_2 < 0.
\end{aligned}
$$

The local representations (2.10) give rise to an analytic continuation phenomenon, whose scalar equation counterpart was discussed in [**14**], for the m–coefficients. We will prove in the next lemma that $p_1(0,\lambda_1) \neq 0 \neq p_2(0,\lambda)$ for $\lambda_1 \in I$, but accepting this for the moment the functions $\hat{p}_2(0,\lambda)/p_2(0,\lambda)$ and $\hat{p}_1(0,\lambda)/p_1(0,\lambda)$ are analytic throughout Ω_I. Then the first equation of (2.10) represents $m_0^{(+)}$ in the upper part of Ω_I and provides its analytic continuation into the lower part. But its analytic continuartion into the lower part does not equal the Titchmarsh–Weyl coefficient for $\lambda_2 < 0$; its value there is the second equation of (2.10) and these quantities cannot agree for a real value $\lambda_1 \in I$ as otherwise $\Delta(\lambda_1) = 0$. Similar remarks apply to the coefficient $m_0^{(-)}(\lambda)$.

LEMMA 2.4. *For $\lambda_1 \in I$, we have $p_1(0,\lambda_1) \neq 0 \neq p_2(0,\lambda_2)$.*

PROOF. We begin with the formulas

$$
(2.11) \qquad \begin{aligned}
\int_0^\infty \vec{X}_0^{(+)*} R \vec{X}_0^{(+)}\, dx &= \frac{\operatorname{Im} m_0^{(+)}(\lambda)}{\lambda_2},\\
\int_{-\infty}^0 \vec{X}_0^{(-)*} R \vec{X}_0^{(-)}\, dx &= -\frac{\operatorname{Im} m_0^{(-)}(\lambda)}{\lambda_2},
\end{aligned}
$$

for $\lambda = \lambda_1 + i\lambda_2 \in \Omega_I, \lambda_2 > 0$, which are proved in [**11**], for the functions $\vec{X}_0^{(\pm)} = \vec{\theta}_0 + m_0^{(\pm)}\vec{\varphi}_0$ of (2.7). If it happens that $p_1(0,\lambda_1) = 0$, then

$\hat{p}_1(0,\lambda_1) \neq 0$ by Lemma 2.1 and the definition of $\Delta(\lambda)$. Therefore (2.10) implies that $m_0^{(-)}(\lambda)$ has a pole at λ_1 which, by (2.7), must be simple. Letting r be its residue there, multiplying the second equation of (2.11) by λ_2^2 and letting $\lambda_2 \to 0$ results in

$$\int_{-\infty}^{0} \vec{\varphi}_0^{\,*} R \vec{\varphi}_0 dx = (\frac{1}{r}). \tag{2.12}$$

But from the second equation of (2.6) $\vec{\varphi}_0^{\,*} R \vec{\varphi}_0 = |p_2(0)|^2 \vec{p}_1^{\,*} R \vec{p}_1$ is periodic which means that (2.12) cannot possibly be finite. Thus $p_1(0,\lambda_1) = 0$ is entenable, and so is $p_2(0,\lambda_1) = 0$ by similar reasoning.

3. Asymptotics of solutions. We now want to think of the solutions $\vec{\theta}$ and $\vec{\varphi}$, from (2.1), as perturbations of the Floquet solutions (2.5). Let $Y_0(x,\lambda)$ and $Y(x,\lambda)$ be the fundamental matrix solutions of (1.3) and (1.4), respectively, for which $Y_0(0,\lambda) = Y(0,\lambda) = I$ for all λ; i.e.,

$$Y_0 = \begin{pmatrix} \theta_0 & \varphi_0 \\ \hat{\theta}_0 & \hat{\varphi}_0 \end{pmatrix} = [\vec{\theta}_0, \vec{\varphi}_0],\ Y = \begin{pmatrix} \theta & \varphi \\ \hat{\theta} & \hat{\varphi} \end{pmatrix} = [\vec{\theta}, \vec{\varphi}].$$

The variation of parameters formula for (1.4) reads, suppressing the λ,

$$\begin{aligned} Y(x) &= Y_0(x) + Y_0(x) \int_0^x Y_0^{-1}(t) J^{-1} P(t) Y(t) dt \\ &= Y_0(x) + \int_0^x K(x,t) P(t) Y(t) dt, \end{aligned} \tag{3.1}$$

where

$$\begin{aligned} K(x,t,\lambda) &= Y_0(x) Y_0^{-1}(t) J^{-1} \\ &= \begin{bmatrix} \theta_0(x)\varphi_0(t) - \varphi_0(x)\theta_0(t) & \theta_0(x)\hat{\varphi}_0(t) - \varphi_0(x)\hat{\theta}_0(t) \\ \hat{\theta}_0(x)\varphi_0(t) - \hat{\varphi}_0(x)\theta_0(t) & \hat{\theta}_0(x)\hat{\varphi}_0(t) - \hat{\varphi}_0(x)\hat{\theta}_0(t) \end{bmatrix}. \end{aligned}$$

The Kernel $K(x,t,\lambda)$ may be formulated using (2.5). Beginning with the upper left entry, with $\lambda \in \Omega_I$,

$$\begin{aligned} &K_{11}(x,t,\lambda) = \theta_0(x)\varphi_0(t) - \varphi_0(x)\theta_0(t) \\ &= (\hat{p}_2(0) e^{\alpha x} p_1(x) - \hat{p}_1(0) e^{-\alpha x} p_2(x))(-p_2(0) e^{\alpha t} p_1(t) + p_1(0) e^{-\alpha t} p_2(t)) \Delta^{-2} \\ &-(-p_2(0) e^{\alpha x} p_1(x) + p_1(0) e^{-\alpha x} p_2(x))(\hat{p}_2(0) e^{\alpha t} p_1(t) - \hat{p}_1(0) e^{-\alpha t} p_2(t)) \Delta^{-2} \\ &= (e^{\alpha(x-t)} p_1(x) p_2(t) - e^{-\alpha(x-t)} p_2(x) p_1(t)) \Delta^{-1}, \end{aligned}$$

due to cancellation and the definition of $\Delta(\lambda)$. After computing the other entries of K, on sees that

$$
\begin{aligned}
K_{11}(x,t,\lambda) &= \Delta(\lambda)^{-1}(e^{\alpha(x-t)}p_1(x)p_2(t) - e^{-\alpha(x-t)}p_2(x)p_1(t)),\\
K_{12}(x,t,\lambda) &= \Delta(\lambda)^{-1}(e^{\alpha(x-t)}p_1(x)\hat{p}_2(t) - e^{-\alpha(x-t)}p_2(x)\hat{p}_1(t)),\\
K_{21}(x,t,\lambda) &= \Delta(\lambda)^{-1}(e^{\alpha(x-t)}\hat{p}_1(x)p_2(t) - e^{-\alpha(x-t)}\hat{p}_2(x)p_1(t)),\\
K_{22}(x,t,\lambda) &= \Delta(\lambda)^{-1}(e^{\alpha(x-t)}\hat{p}_1(x)\hat{p}_2(t) - e^{-\alpha(x-t)}\hat{p}_2(x)\hat{p}_1(t)).
\end{aligned} \tag{3.2}
$$

Having (3.2) in hand we can proceed with the asymptotic form of $Y(x,\lambda)$ for $\lambda = \lambda_1 + i\lambda_2 \in \Omega_I, \lambda_2 \geq 0$. Using $||\cdot||$ to denote the norm of a matrix, or vector, obtained by summing the magnitudes of its components, it is obvious from (2.6) that

$$||Y_0(x,\lambda)|| \leq M_1(\lambda)e^{u(\lambda)x} \tag{3.3}$$

for $\lambda = \lambda_1 + i\lambda_2 \in \Omega_I, \lambda_2 \geq 0, 0 \leq x < \infty$ and a certain constant $M_1(\lambda) > 0$. If we let $Y_1(x,\lambda) = e^{-u(\lambda)x}Y(x,\lambda)$, then (3.1) is the same as

$$Y_1(x) = e^{-ux}Y_0(x) + \int_0^x e^{-u(x-t)}K(x,t)P(t)Y_1(t)dt. \tag{3.4}$$

We will apply the norm $||\cdot||$ to (3.4), but observe first that

$$||e^{-u(\lambda)(x-t)}K(x,t,\lambda)|| \leq M_2(\lambda), 0 \leq t \leq x < \infty, \lambda_2 \geq 0, \tag{3.5}$$

for some constant $M_2(\lambda)$, and that the norm satisfies $||\int_a^b N(x)dx|| \leq \int_a^b ||N(x)||dx$ and $||N_1N_2|| \leq ||N_1||||N_2||$ for matrix functions N, N_1, and N_2. Working with (3.3)-(3.5) it follows that

$$||Y_1(x)|| \leq M_1(\lambda) + \int_0^x M_2(\lambda)||P(t)||||Y_1(t)||dt,$$

and then, from Gronwall's inequality [**9**, p. 241], that

$$||Y_1(x)|| \leq M_1(\lambda)\exp(M_2(\lambda)\int_0^\infty ||P(t)||dt),$$

in view of (1.7). Therefore, we have

$$||Y(x,\lambda)|| \leq K(\lambda)e^{u(\lambda)x},\ 0 \leq x < \infty, \lambda = \lambda_1 + i\lambda_2 \in \Omega_I, \lambda_2 \geq 0, \tag{3.6}$$

where $K(\lambda)$ is a constant. Included in (3.6) are bounds of the form

$$\|\vec{\theta}(x,\lambda)\| \leq K_\theta(\lambda)e^{u(\lambda)x}, \|\vec{\varphi}(x,\lambda)\| \leq K_\varphi(\lambda)e^{u(\lambda)x}, \tag{3.7}$$

for appropriate constants K_θ and K_φ.

Note that equations (2.6), combined together and rewritten, may be expressed as

(3.8)

$$\Delta(\lambda)Y_0(x,\lambda) = e^{\alpha x}\vec{p}_1(x)(\hat{p}_2(0), -p_2(0)) - e^{-\alpha x}\vec{p}_2(x)(\hat{p}_1(0), -p_1(0)),$$

and in a similar way (3.2) becomes

$$\Delta(\lambda)K(x,t,\lambda) = e^{\alpha(x-t)}\vec{p}_1(x)p_2(t)^T - e^{2(x-t)}\vec{p}_2(x)\vec{p}_1(t)^T, \tag{3.9}$$

where T denotes the ordinary transpose.

We are now ready to derive the asymptotic forms of $Y(x,\lambda)$ in the separate cases of real and nonreal λ. If one substitutes (3.8) and (3.9) into (3.1) and takes λ to be real, the result is

(3.10)

$$\begin{aligned}
&\Delta(\lambda)Y(x,\lambda)\\
&\quad= e^{\alpha x}\vec{p}_1(x)(\hat{p}_2(0), -p_2(0)) - e^{-\alpha x}\vec{p}_2(x)(\hat{p}_1(0), -p_1(0))\\
&\qquad + \int_0^x (e^{\alpha(x-t)}\vec{p}_1(x)\vec{p}_2(t)^T - e^{-\alpha(x-t)}\vec{p}_2(x)\vec{p}_1(t)^T)P(t)Y(t)dt\\
&\quad= e^{\alpha x}\vec{p}_1(x)\Big(A_1^{(+)}(\lambda) - \int_x^\infty e^{-\alpha t}\vec{p}_2(t)^T P(t)Y(t)dt\Big)\\
&\qquad + e^{-\alpha x}\vec{p}_2(x)(A_2^{(+)}(\lambda) + \int_x^\infty e^{\alpha t}\vec{p}_1(t)^T P(t)Y(t)dt),
\end{aligned}$$

where

$$\begin{aligned}
A_1^{(+)}(\lambda) &= (\hat{p}_2(0), -p_2(0)) + \int_0^\infty e^{-\alpha t}\vec{p}_2(t)^T P(t)Y(t)dt,\\
A_2^{(+)}(\lambda) &= (-\hat{p}_1(0), p_1(0)) - \int_0^\infty e^{\alpha t}\vec{p}_1(t)^T P(t)Y(t)dt.
\end{aligned}$$

This may be written in the form

(3.11)

$$\Delta(\lambda)Y(x,\lambda) = e^{\alpha x}\vec{p}_1(x)A_1^{(+)}(\lambda) + e^{-\alpha x}\vec{p}_2(x)A_2^{(+)}(\lambda) + o(1) \quad (\lambda \text{ real})$$

where $0(1)$ is term which tends to 0 as $x \to \infty$.

Next let $\lambda = \lambda_1 + i\lambda_2 \in \Omega_I, \lambda_2 \geq 0$. Note that the term $A_1^{(+)}(\lambda)$ is defined for these nonreal λ on account of (3.6) and (1.7). Recall that $\alpha(\lambda) = u(\lambda) + iv(\lambda)$ and $u(\lambda) > 0$ for $\lambda_2 > 0$. The term $e^{-\alpha x}\vec{p}_2(x)(\hat{p}_1(0), -p_1(0))$ in (3.10) has norm order of magnitude $e^{-u(\lambda)x}, x \to \infty$. In the next term of the same equation $\int_0^x e^{-\alpha t}\vec{p}_2(t)^T P(t)Y(t)dt = \int_0^\infty e^{-\alpha t}\vec{p}_2(t)^T p(t)Y(t)dt - \int_x^\infty e^{-\alpha t}\vec{p}_2(t)^T P(t)Y(t)dt$, again using (3.6), which is $\int_0^\infty e^{-\alpha t}\vec{p}_2(t)^T P(t)Y(t)dt + o(1), x \to \infty$. Passing to the next term in (3.10) and using (3.6),

$$\begin{aligned}
&\left\|\int_0^x e^{-\alpha(x-t)}\vec{p}_1(t)^T P(t)Y(t)dt\right\| \leq K(\lambda)\int_0^x e^{u(2t-x)}\|\vec{p}_1(t)^T P(t)\|dt \\
&\leq K(\lambda)\int_0^{x/2} \|\vec{p}_1(t)^T P(t)\|dt + K(\lambda)e^{ux}\int_{x/2}^x \|\vec{p}_1(t)^T P(t)\|dt \\
&= e^{\alpha x}K_1(x,\lambda) = o(e^{\alpha x}), x \to \infty,
\end{aligned}$$

where $K_1(x,\lambda) \to 0$ as $x \to \infty$. Now, factoring out $e^{\alpha x}\vec{p}_1(x)$ from the right side of the first equation in (3.10),

$$\Delta(\lambda)Y(x,\lambda) = e^{\alpha x}(\vec{p}_1(x)A_1^{(+)}(\lambda) + o(1)), x \to \infty, \lambda_2 \geq 0. \tag{3.12}$$

We will require companion results to (3.11) and (3.12) for negative $x, x \to -\infty$. Beginning with a real $\lambda \in I$ and working with (3.10),

$$\begin{aligned}
\Delta(\lambda)Y(x,\lambda) =& e^{\alpha x}\vec{p}_1(x)\{(\hat{p}_2(0), -p_2(0)) \\
&+ \int_0^{-\infty} e^{-\alpha t}\vec{p}_2(t)^T p(t)Y(t)dt\} \\
&+ e^{-\alpha x}\vec{p}_2(x)\{(\hat{p}_1(0), p_1(0)) \\
&- \int_0^{-\infty} e^{\alpha t}\vec{p}_1(t)^T p(t)Y(t)\} + o(1) \\
=& e^{\alpha x}\vec{p}_1(x)A_1^{(-)}(\lambda) + e^{-\alpha x}\vec{p}_2(x)A_2^{(-)}(\lambda) \\
&+ o(1),\ x \to -\infty, \lambda \text{ real},
\end{aligned} \tag{3.13}$$

with $A_1^{(-)}(\lambda)$ and $A_2^{(-)}(\lambda)$ defined in the obvious way. The asymptotic estimate for $x \to -\infty$ which is complementary to (3.12) is

$$\Delta(\lambda)Y(x,\lambda) = e^{-\alpha x}(\vec{p}_2(x)A_2^{(-)}(\lambda) + o(1)),\ x \to -\infty, \lambda_2 \geq 0, \tag{3.14}$$

valid for $\lambda = \lambda_1 + i\lambda_2 \in \omega_I$.

We now close this section with a lemma giving an alternative way of expressing the functions $A_1^{(+)}(\lambda)$ and $A_2^{(-)}(\lambda)$ for real $\lambda \in I$.

LEMMA 3.1. *For real $\lambda \in I$ define*

$$B(\lambda) = I + \int_0^\infty Y_0^{-1}(t)J^{-1}P(t)Y(t)dt,$$

$$\tilde{B}(\lambda) = I + \int_0^{-\infty} \gamma_0^{-1}(t)J^{-1}P(t)Y(t)dt.$$

Then

$$B_{11}B_{22} - B_{12}B_{21} = \tilde{B}_{11}\tilde{B}_{22} - \tilde{B}_{12}\tilde{B}_{21} = 1, \tag{3.15}$$

$$A_1^{(+)}(\lambda) = [\hat{p}_2(0), -p_2(0)]B(\lambda), \quad A_2^{(-)}(\lambda) = [\hat{p}_1(0), p_1(0)]\tilde{B}(\lambda), \tag{3.16}$$

(3.17)
$$Y(x) = Y_0(x)B(\lambda) + o(1), x \to \infty; Y(x) = Y_0(x)\tilde{B}(\lambda) + o(1), x \to -\infty.$$

PROOF. Starting with the variation of parameters formula (3.1), we obtain

$$\begin{aligned} Y(x) &= Y_0(x)(I + \int_0^\infty Y_0^{-1}J^{-1}PY\,dt - \int_x^\infty Y_0^{-1}J^{-1}PY\,dt) \\ &= Y_0(x)(B(\lambda) + o(1)) \\ &= Y_0(x)B(\lambda) + o(1), \ x \to \infty, \end{aligned}$$

on noting that $Y_0(x, \lambda)$ and $Y(x, \lambda)$ are bounded for real $\lambda \in I$ and taking (1.7) into account. In (3.3) and (3.6) recall that $u(\lambda) = 0$ for real λ. The other equation in (3.17) can be derived similarly.

The identity $J = Y^*JY$ holds for real λ ([**2**,p. 268]) and this implies (3.17) that $J = (Y_0B + 0(1)^*J(Y_0B + o(1)) = B^*Y_0^*JY_0B + o(1) = B^*JB + o(1)$. Since B does not depend on x, it must be that $B^*JB = J$, which includes the first of (3.15). The proof for $\tilde{B}$ is the same.

As concerns (3.16) let $Z_0(t, \lambda) = (e^{\alpha t}\vec{p}_1(t), e^{-\alpha t}\vec{p}_2(t))$ so that (2.6) becomes

$$\Delta(\lambda)Y_0(x, \lambda) = Z_0(x, \lambda)\begin{bmatrix} \hat{p}_2(0) & -p_2(0) \\ -\hat{p}_1(0) & p_1(0) \end{bmatrix},$$

$$Z_0(x) = \Delta(\lambda)Y_0(x)\Delta(\lambda)^{-1}\begin{bmatrix} p_1(0) & p_2(0) \\ \hat{p}_1(0) & \hat{p}_2(0)\end{bmatrix} = Y_0(x)(\vec{p}_1(0), \vec{p}_2(0)).$$

This can be substituted into the definition of $A_1^{(+)}(\lambda)$ to lead to

$$\begin{aligned} A_1^{(+)} &= (\hat{p}_2(0), -p_2(0)) + \int_0^\infty (e^{-\alpha t}\vec{p}_2(t))^T p(t)Y(t)dt \\ &= (\hat{p}_2(0), -p_2(0)) + \int_0^\infty (Y_0(t)\vec{p}_2(0))^T p(t)Y(t)dt \\ &= (\hat{p}_2(0), -p_2(0)) + \vec{p}_2(0)^T\int_0^\infty JJ^{-1}Y_0^*(t)p(t)Y(t)dt \\ &= (\hat{p}_2(0), -p_2(0)) + (\hat{p}_2(0), -p_2(0))\int_0^\infty Y_0^{-1}(t)J^{-1}p(t)Y(t)dt \\ &= (\hat{p}_2(0), -p_2(0))B(\lambda), \end{aligned}$$

and the other identity in (3.16) follows similarly.

4. Perturbed and unperturbed spectra. The Titchmarsh–Weyl function for the whole line unperturbed operator T_0 is given by ([**11**], compare [**4**, p. 251])

(4.1)

$$\begin{aligned} M_0(\lambda) =& (m_0^{(-)}(\lambda) - m_0^{(+)}(\lambda))^{-1} \\ & \begin{pmatrix} 1 & (\frac{1}{2})(m_0^{(-)}(\lambda) + m_0^{(+)}(\lambda)) \\ (\frac{1}{2})(m_0^{(-)}(\lambda) + m_0^{(+)}(\lambda)) & m_0^{(-)}(\lambda)m_0^{(+)}(\lambda) \end{pmatrix}, \end{aligned}$$

for $\operatorname{Im}(\lambda) \neq 0$. It is related to the spectral function ρ_0 for T_0 by the Titchmarsh–Kodaira formula [**12, 13**]

$$\rho_0(\lambda_2) - \rho_0(\lambda_1) = \lim_{\delta\to 0}\pi^{-1}\int_{\lambda_1}^{\lambda_2}\operatorname{Im} M_0(\mu + i\delta)d\mu, \tag{4.2}$$

valid at points λ_1 and λ_2 of continuity of $\rho_0(\lambda)$, where $\operatorname{Im} M_0 = (M_0 - M_0^*)/2i$. The spectral function $\rho_0(\lambda)$ is a right–continuous and nondecreasing (in the nonnegative definite sense) matrix function whose points of increase comprise the spectrum $\sigma(T_0)$; see [**12, 13**]. Jump discontinuities of $\rho(\lambda)$ are the eigenvalues of T_0, both isolated and non-isolated, while continuous points of increase comprise $C(T_0)$. We are going to prove shortly that $\rho_0(\lambda)$ is continuously differentiable on the interior of $C(T_0)$.

Using (4.2) the points of $\sigma(T_0)$ may be classified in terms of M_0, as the following theorem from [**12**] shows (see also [**13**]).

THEOREM A. *For a given real number λ_0, we have*

(i) $\lambda_0 \in \rho(T_0)$ *if and only if* $M_0(\lambda)$ *is analytic at* λ_0*;*

(ii) $\lambda_0 \in P(T_0)$ *if and only if* $M_0(\lambda)$ *has a simple pole at* λ_0*;*

(iii) $\lambda_0 \in C(T_0)$ *if and only if* $M_0(\lambda)$ *is not analytic at* λ_0*, and* $lim_{\nu\to 0}\nu M_0(\lambda_0 + i\nu) = 0$

(iv) $\lambda_0 \in PC(T_0)$ *if and only if* $\lim_{\nu\to 0} \nu M_0(\lambda_0 + i\nu) = S \neq 0$*, and* $M_0(\lambda) - iS(\lambda - \lambda_0)^{-1}$ *is not analytic at* λ_0.

The first part of the following result, proved for Case I of (1.5) in [**8**], is the Hamiltonian system version of the classical result which links the spectrum of Hill's equation to its stability intervals.

PROPOSITION 4.1. *The spectrum* $\sigma(T_0)$ *is purely continuous and consists of the stability intervals of* (1.3)*. Moreover, the spectral function* $\rho_0(\lambda)$ *is absolutely continuous, in fact continuously differentiable, on the interior of* $C(T_0)$.

PROOF. Let I be an interval defined as in §2. By the remarks below (2.10) the functions $m_0^{(\pm)}(\lambda)$ are not analytic at points of I (although their local representations are). This implies that $M_0(\lambda)$ in (4.1) is not analytic at any point of I. If $\lambda = \lambda_1 + i\lambda_2$ and $\lambda_2 > 0$ in (4.1), then $(m_0^{(-)}(\lambda) - m_0^{(+)}(\lambda))^{-1} = -p_1(0\lambda)p_2(0, \lambda\Delta(\lambda)$, a quantity which has a finite and nonzero limit as $\lambda_2 \to 0$. Thus the entries of $M_0(\lambda)$ have finite and nonzero limits as $\lambda_2 \to 0$, so Theorem A (iii) implies that $I \subset C(T_0)$.

Suppose now that $\tilde{I}$ is an open interval whose closure lies in an *instability interval*; i.e., $|D(\lambda)| > 2$ for all $\lambda \in I$. The analytic functions $\rho(\lambda)$ and $\alpha(\lambda) = u(\lambda) + iv(\lambda)$ can be defined as in §2, with the only differences being that $u(\lambda) > 0, \lambda \in \Omega_{\tilde{I}}$, while $\nu(\lambda)$ changes sign with $\mathrm{Im}\,(\lambda)$. Lemma 2.1 and the first part of Lemma 2.3, $p_1(0, \lambda) \neq 0 \neq p_2(0, \lambda)$ for $\lambda \in \tilde{I}$, continue to hold, but (2.10) must be replaced by $m_0^{(+)}(\lambda) = \hat{p}_2(0, \lambda)/p_2(0, \lambda)$ and $m_0^{(-)}(\lambda) = \hat{p}_1(0, \lambda)/p_1(0, \lambda)$ for $\lambda \in \Omega_I$. Hence the functions $m_0^{(\pm)}(\lambda)$ are analytic throughout $\Omega_{\tilde{I}}$. It is still true that the denominator $(m_0^{(-)}(\lambda) - m_0^{(+)}(\lambda))^{-1} = -p_1(0,\lambda)p_2(0,\lambda)/\Delta(\lambda)$ in (4.1) is finite and nonzero in Ω_I, and so therefore is $M_0(\lambda)$. By Theorem A(i), $\tilde{I} \subset \rho(T_0)$.

If λ_0 is an endpoint of a stability interval, or even a common endpoint of two such intervals, then $\lambda_0 \in PC(T_0) \cup C(T_0)$ since the above proof shows that λ_0 cannot be either a regular point or isolated singularity of $M_0(\lambda)$. But $|D(\lambda_0)| = 2$ implies by [8] that the solutions of (1.3) are either periodic, linear functions times periodic functions or combinations of these. Thus there are no $L_R^2(-\infty,\infty)$ solutions for $\lambda = \lambda_0$, so $\lambda_0 \notin PC(T_0)$.

Since $M_0(\lambda_1+i\lambda_2)$ has a finite and continuous limit as $\lambda_2 \to 0, \lambda_1 \in I$, we may take the limit inside the integral in (4.2), yielding the result that $\rho_0(\lambda)$ is continuously differentiable.

In order to state and prove the principal result we begin by claiming that (1.4) is of limit point type at both $\pm\infty$. Letting $A_1^{(+)}(\lambda) = (A_{11}^{(+)}(\lambda), A_{12}^{(+)}(\lambda))$, (3.16) implies $A_{11}^{(+)} = \hat{p}_2(0)B_{11} - p_2(0)B_{21}$ and $A_{12}^{(+)} = \hat{p}_2(0)B_{12} - p_2(0)B_{22}$ for real $\lambda \in I$, where I is the interval of §2. Then $A_{11}^{+}/p_2(0) = B_{11}(\hat{p}_2(0)/p_2(0)) - B_{21}$ and $A_{12}^{(+)}/p_2(0) = B_{12}(\hat{p}(0)/p_2(0)) - B_{22}$, these equations also holding for real $\lambda \in I$. However $\hat{p}_2(0,\lambda)/p_2(0,\lambda) = m_0^{(+)}(\lambda + i\cdot 0) = \lim_{\varepsilon\to 0} m_0^{(+)}(\lambda + i\varepsilon) \neq 0$ by Lemma 2.3. Then, according to (2.7), $\operatorname{Im} m_0^{(+)}(\lambda + i0) > 0$, which implies that $A_{11}^{(+)}(\lambda)$ and $A_{12}^{(+)}(\lambda)$ are nonreal for real $\lambda \in I$. In particular the analytic functions $A_{11}^{(+)}$ and $A_{12}^{(+)}$ of not vanish identically, and a similar proof establishes this for $A_2^{(-)}(\lambda) = (A_{21}^{(-)}(\lambda), A_{22}^{(-)}(\lambda))$. By (3.12) and (3.14)

$$||Y(x,\lambda)|| \geq C_1 e^{u(\lambda)x}, \quad x \text{ near } \infty,$$
$$||Y(x,\lambda)|| \geq C_2^{-u(\lambda)x}, \quad x \text{ near } -\infty,$$

where λ is a nonreal complex number such that $A_1^{(+)}(\lambda) \neq 0 \neq A_2^{(-)}(\lambda)$. These inequalities show that (1.4) is of limit point type at $\pm\infty$.

Since the limit point case holds we may form the (perturbed) m-functions for (1.4),

$$m^{(+)}(\lambda) = -\lim_{x\to\infty} \frac{\theta(x,\lambda)}{\varphi(x,\lambda)}, \quad m^{(-)}(\lambda) = -\lim_{x\to-\infty} \frac{\theta(x,\lambda)}{\varphi(x,\lambda)}, \ \operatorname{Im}(\lambda) \neq 0,$$

by [**10**], [**11**] or [**12**]. As in (1.4) the m–function for the whole line operator T is

(4.3)

$$M(\lambda) =(m^{(-)}(\lambda) - m^{(+)}(\lambda))^{-1} \cdot P \begin{pmatrix} 1 & (\frac{1}{2})(m^{(-)}(\lambda) + m^{(+)}(\lambda)) \\ (\frac{1}{2})(m^{(-)}(\lambda) + m^{(+)}(\lambda)) & m^{(-)}(\lambda)m^{(+)}(\lambda) \end{pmatrix}$$

for $\operatorname{Im}(\lambda) \neq 0$. The perturbed spectral function $\rho_P(\lambda)$ bears the same relationship to $M(\lambda)$ as does $\rho_0(\lambda)$ to $M_0(\lambda)$ in (4.2). Theorem A also carries over to the perturbed whole line operator T.

According to (3.12) and (3.14),

$$m^{(+)}(\lambda) = -\frac{A_{11}^{(+)}(\lambda)}{A_{12}^{(+)}(\lambda)},$$

$$m^{(-)}(\lambda) = -\frac{A_{21}^{(-)}(\lambda}{A_{22}^{(-)}(\lambda)}, \quad \operatorname{Im}(\lambda) \neq 0.$$

We have noted above that these functions are not identically vanishing and are nonreal, thus nonzero, for real $\lambda \in I$. By (3.16) and (2.10), we therefore have

$$m^{(+)}(\lambda) = -\frac{B_{21} - B_{11}m_0^{(+)}}{B_{22} - B_{12}m_0^{(+)}}(\lambda), \quad \lambda \in I, \tag{4.4}$$

where all terms are evaluated at λ and where $m_0^{(+)}(\lambda)$ is written for $m_0^{(+)}(\lambda + i \cdot 0)$, and similarly

$$m^{(-)}(\lambda) = -\frac{\tilde{B}_{21} - \tilde{B}_{11}m_0^{(-)}}{\tilde{B}_{22} - \tilde{B}_{12}m_0^{(-)}}(\lambda), \quad \lambda \in I, \tag{4.5}$$

with the same notational convention. Equations (4.4) and (4.5) show that the functions $m^{(\pm)}(\lambda)$ have finite and nonzero limits for $\lambda \in I$. Property (2.7) shows that $\operatorname{Im} M^{(+)}(\lambda) > 0$ and $\operatorname{Im} m^{(-)}(\lambda) < 0$ for $\lambda \in I$, which we had established earlier for $m_0^{(\pm)}(\lambda)$. These results may be summarized by writing $m_0^{(+)} = G_0 + iF_0, m_0^{(-)} = \Gamma_0 + i\sum_0, m^{(+)} = G + iF, m^{(-)} = \Gamma + i\Sigma$ and then

$$F_0(\lambda) > 0, \;\; F(\lambda) > 0, \;\; \Sigma_{0(\lambda)} < 0, \;\; \Sigma(\lambda) < 0 \;\; \text{for } \lambda \in I. \tag{4.6}$$

CLAIM. *The limit* $\lim_{\lambda_2 \to 0} M(\lambda_1 + i\lambda_2)$ *exists and is nonreal for* $\lambda_1 \in I$.

PROOF. The numerators in (4.3) have finite limits and so it will be sufficient to prove that $(m^{(-)} - m^{(+)})^{-1}$ has a finite and nonreal limit. Using the notation of (4.6)

$$\operatorname{Im}(m^{(-)} - m^{(+)})^{-1} = (F - \Sigma)/((\Gamma - G)^2 + (F - \Sigma)^2),$$

and (4.6) shows that this expression has a finite and nonzero limit across I. This completes the proof of the claim.

We now state and prove our principal result.

THEOREM 4.1. *The continuous spectra* $C(T_0)$ *and* $C(T)$ *have equal interiors and* $\rho_P(\lambda)$ *is continuously differentiable there.*

PROOF. Let $I \subset C(T_0)$ be as above and form the region Ω_I. By the claim $M(\lambda_1 + i\lambda_2)$ has a nonreal and continuous limit as $\lambda_2 \to 0$ and $\lambda_1 \in I$. Then $M(\lambda)$ cannot be analytic at any point of I by (2.7), but $\lambda_2 M(\lambda_1 + i\lambda_2) \to 0$ as $\lambda_2 \to 0$. Thus $I \subset C(T)$ is a consequence of Theorem A(iii). This proves that the interior of $C(T_0)$ lies within that of $C(T)$. For the reverse inclusion take an interval $\tilde{I}$ lying in an instability interval of (1.3) and proceed as in the proof of Proposition 4.1. The functions $A_1^{(+)}(\lambda)$ and $A_2^{(-)}(\lambda)$ are analytic in $\Omega_{\tilde{I}}$, are real $\lambda \in \tilde{I}$ and still are not identically zero. Then $m^{(+)} = -A_{11}^{(+)}/A_{12}^{(+)}$ and $m^{(-)} = -A_{21}^{(-)}/A_{22}^{(-)}$ are analytic in $\Omega_{\tilde{I}}$ except for poles. Since the same holds for (4.3), then, by Theorem, A $\tilde{I} \subset \rho(T) \cup P(T)$. It therefore remains only to show that there are no eigenvalues imbedded in the interior of $C(T)$. By the claim we can take the limit in (4.2), for the perturbed data $\rho_P(\lambda)$ and $M(\lambda)$, under the integral sign yielding $\rho'_P(\lambda)_P = \pi^{-1}\operatorname{Im} M(\lambda), \lambda \in I$, so $\rho_P(\lambda)$ has no jump discontinuities in the interior of $C(T_0)$. This completes the proof of the theorem.

We have proceeded under the assumption (2.3) that the stability intervals in S have no common endpoints. We now remove this assumption. If λ_0 is the common endpoint of two stability intervals, say (λ_1, λ_0) and (λ_0, λ_2), then (λ_1, λ_0) and (λ_0, λ_2) consist of continuous spectrum and $|D(\lambda_0)| = 2$. Moreover, $D'(\lambda_0) = 0$ and the fundamental matrix $Y_0(x, \lambda_0)$ has entries which are periodic; see [**8**; Lemma 2.2] in which positive definiteness of $R(x)$ may be replaced by the definiteness assumption (1.9). In particular (1.3) has no $L^2_R(-\infty, \infty)$ solutions for $\lambda = \lambda_0$. By (3.17) (1.4) has no $L^2_R(-\infty, \infty)$ solutions either for $\lambda = \lambda_0$, and thus $\lambda_0 \notin PC(T)$. Hence $C(T_0)$ and $C(T)$ have equal interiors without assumption (2.3)

We remark finally that the results of this paper extend to halfline periodic problems, i.e., operators T_0 and perturbations T defined on the space $L^2_R(0, \infty) = \{\vec{y}\,|\,\vec{y}$ measurable and $\int_0^\infty \vec{y} R \vec{y}\, dx < \infty\}$. The results are easier to prove, in fact, for we can deal with the functions $m^{(\pm)}(\lambda)$ separately, instead of (4.3).

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