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THE ROGERS–RAMANUJAN IDENTITIES WITHOUT JACOBI'S TRIPLE PRODUCT

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ABSTRACT. We provide polynomial identities which converge to the Rogers–Ramanujan identities. These identities naturally involve the partial products for the related infinite products. Hence Jacobi's triple product identity is never required.

1. Introduction. For many years it was an open question whether a bijective proof could be given for the Rogers–Ramanujan identities. In 1980, A. Garsia and S. Milne [**6**], [**7**] gave the first bijective proof using what has since become called the Garsia–Milne Involution Principle. Subsequently D. Bressoud and D. Zeilberger [**5**] gave an alternative bijective proof; however it also relied on the Garsia– Milne Involution Principle. Indeed, given the known analytic proofs of the Rogers–Ramanujan identities it seems that the Involution Principle is inherently involved; this is because all the known proofs actually establish

$$\sum_{n=0}^{\infty} \frac{q^{n^2}}{(q;q)_n} = \frac{1}{(q;q)_\infty} \sum_{\lambda=-\infty}^{\infty} (-1)^\lambda q^{\lambda(5\lambda+1)/2} \tag{1.1}$$

and

$$\sum_{n=0}^{\infty} \frac{q^{n^2+n}}{(q;q)_n} = \frac{1}{(q;q)_\infty} \sum_{\lambda=-\infty}^{\infty} (-1)^\lambda q^{\lambda(5\lambda+3)/2}, \tag{1.2}$$

where

$$(A;q)_n = \prod_{m=0}^{\infty} (1-Aq^m)/(1-Aq^{m+n}) \tag{1.3}$$

$(= (1-A)(1-Aq)\cdots(1-Aq^{n-1})$, when n is a nonnegative integer),

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and

$$(A;q)_\infty = \prod_{m=0}^{\infty} (1 - Aq^m). \tag{1.4}$$

The standard infinite product form of the righthand sides of (1.1) and (1.2) is then deduced using Jacobi's Triple Product Indentity [**3**; p. 22, Cor. 2.9] in the following form:

$$\sum_{\lambda=-\infty}^{\infty} (-1)^\lambda q^{\lambda(5\lambda+2\alpha+1)/2} = (q^5;q^5)_\infty (q^{3+\alpha};q^5)_\infty (q^{2-\alpha};q^5)_\infty. \tag{1.5}$$

The natural bijective method to pass from the numerator products introduced by (1.5) to the standard forms:

$$\sum_{n=0}^{\infty} \frac{q^{n^2}}{(q;q)_n} = \frac{1}{(q;q^5)_\infty (q^4;q^5)_\infty}, \tag{1.6}$$

and

$$\sum_{n=0}^{\infty} \frac{q^{n^2+n}}{(q;q)_n} = \frac{1}{(q^2;q^5)_\infty (q^3;q^5)_\infty}, \tag{1.7}$$

appears to be the Involution Principle.

At the Colloque de Combinatoire Énumérative–U.Q.A.M 1985, D. Zeilberger asked whether it was possible to provide a proof of the Rogers–Ramanujan identities which makes no use of Jacobi's Triple Product. This might then provide the starting point for a bijective proof of the Rogers–Ramanujan identities that would avoid the Involution Principle.

The object of this paper is to provide such a proof. In the next section we outline how our proof goes. In §3 we provide the necessary lemmas from basic hypergeometric series. §4 provides the actual proof.

2. The background of the proof. We shall consider well–known families of polynomials [**8**; Sect. 7, Ch. 3], [**1**] that converge to the Rogers–Ramanujan identities. Namely

$$D_n = \sum_{o \le 2j \le n} q^{j^2} \begin{bmatrix} n-j \\ j \end{bmatrix}, \tag{2.1}$$

and

$$d_n = \sum_{0\le 2j\le n-1} q^{j^2+j}\begin{bmatrix} n-j-1 \\ j \end{bmatrix}, \tag{2.2}$$

where the Gaussian polynomial or q–binomial coefficient is defined by

$$\begin{bmatrix} n \\ m \end{bmatrix} = \begin{bmatrix} n \\ m \end{bmatrix}_q = \frac{(q;q)_n}{(q;q)_m(q;q)_{n-m}} = \prod_{j=0}^{m-1} \frac{(1-q^{n-j})}{(1-q^{j+1})}. \tag{2.3}$$

Now clearly

$$\lim_{n\to\infty} D_n = \sum_{j=0}^{\infty} \frac{q^{j^2}}{(q;q)_j}, \tag{2.4}$$

and

$$\lim_{n\to\infty} d_n = \sum_{j=0}^{\infty} \frac{q^{j^2+j}}{(q;q)_j}. \tag{2.5}$$

Let us now define

$$G_n(q) = \prod_{\substack{j\equiv 1,4 (\mathrm{mod}\ 5) \\ 0<j<n}} (1-q^j)^{-1} \tag{2.6}$$

and

$$H_n(q) = \prod_{\substack{j\equiv 2,3 (\mathrm{mod}\ 5) \\ 0<j<n}} (1-q^j)^{-1}. \tag{2.7}$$

Obviously there exist polynomials $P_n(q)$ and $R_n(q)$ such that

$$D_n = G_n(q)P_n(q), \tag{2.8}$$

and

$$d_n = H_n(q)R_n(q). \tag{2.9}$$

In §4 we shall derive closed forms for $P_n(q)$ and $R_n(q)$ which will imply

$$\lim_{n\to\infty} P_n(q) = 1, \tag{2.10}$$

and

$$\lim_{n\to\infty} R_n(q) = 1. \tag{2.11}$$

Equations (2.10) and (2.11) are adequate to establish the series-product forms of the Rogers–Ramanujan identities (i.e. (1.6) and (1.7)), namely we let $n \to \infty$ in (2.8) and (2.9) respectively.

3. The q–Hypergeometric Series Lemmas. First we want to represent both D_n and d_n as appropriate q–hypergeometric series. This is easily accomplished using the polynomial identities of [**1**]. We shall utilize both ordinary and bilateral q–hypergeometric series.

$$ {}_r\phi_s\left[\begin{matrix} a_1,\ldots,a_r; q, t \\ b_1,\ldots,b_s \end{matrix}\right] = \sum_{j=0}^{\infty} \frac{(a_1;q)_j(a_2;q)_j\cdots(a_r;q)_j t^j}{(q;q)_j(b_1;q)_j\cdots(b_s;q)_j}, \tag{3.1}$$

and

$$ {}_r\psi_s\left[\begin{matrix} a_1,\ldots,a_r; q, t \\ b_1,\ldots,b_s \end{matrix}\right] = \sum_{j=-\infty}^{\infty} \frac{(a_1;q)_j(a_2;q)_j\cdots(a_r;q)_j t^j}{(b_1;q)_j(b_2;q)_j\cdots(b_s;q)_j}. \tag{3.2}$$

We note that if one of the a_i is q^{-N} where N is a nonnegative integer then both ${}_r\phi_s$ and ${}_r\psi_s$ terminate above. If $b_i = q^{N+1}$ then ${}_r\psi_s$ terminates below. We refer the reader to the books by Bailey [**4**] or Slater [**14**] or the survey article [**2**] for the theoretical development of these series.

LEMMA 1.

$$D_{2n-1} = \frac{(q;q)_{2n}(1-q^3)}{(q;q)_{n-1}(q;q)_{n+2}}$$

$$\lim_{\tau\to 0} {}_8\psi_8\left[\begin{matrix} q^{5+\frac{3}{2}}, & -q^{5+\frac{3}{2}}, & q^{5-n}, & q^{4-n}, & q^{3-n}, & q^{2-n}, & q^{1-n}, & \tau; & q^5, & q^{5n+4}/\tau \\ q^{\frac{3}{2}}, & -q^{\frac{3}{2}}, & q^{n+3}, & q^{n+4}, & q^{n+5}, & q^{n+6}, & q^{n+7}, & \tau^{-1} & q^8 & \end{matrix}\right]. \tag{3.3}$$

for $n > 0$;

$$D_{2n} = \frac{(q;q)_{2n+1}(1-q^2)}{(q:q)_n(q;q)_{n+2}}$$

(3.4)
$$\cdot \lim_{\tau\to 0} {}_8\psi_8 \left[\begin{matrix} q^6, & -q^6, & q^{-n+4}, & q^{-n+3}, & q^{-n+2}, \\ q, & -q, & q^{n+3}, & q^{n+4}, & q^{n+5}, \end{matrix} \right. \left. \begin{matrix} q^{-n+1}, & q^{-n}, & \tau; & q^5, & q^{5n+6}/\tau \\ q^{n+6}, & q^{n+7}, & \tau^{-1} & q^7 & \end{matrix} \right]$$

for $n \geq 0$;

$$d_{2n-1} = \frac{(q;q)_{2n}(1-q)}{(q;q)_n(q;q)_{n+1}}$$

(3.5)
$$\cdot \lim_{\tau\to 0} {}_8\psi_8 \left[\begin{matrix} q^{5+\frac{1}{2}}, & -q^{5+\frac{1}{2}}, & q^{-n+4}, & q^{-n+3}, & q^{-n+2}, \\ q^{\frac{1}{2}}, & -q^{\frac{1}{2}}, & q^{n+2}, & q^{n+3}, & q^{n+4}, \end{matrix} \right. \left. \begin{matrix} q^{-n+1}, & q^{-n}, & \tau; & q^5, & q^{5n+3}/\tau \\ q^{n+5}, & q^{n+6}, & \tau^{-1} & q^6 & \end{matrix} \right]$$

for $n > 0$; and

$$d_{2n} = \frac{(q;q)_{2n+1}(1-q^4)}{(q;q)_{n-1}(q;q)_{n+3}}$$

(3.6)
$$\cdot \lim_{\tau\to 0} {}_8\psi_8 \left[\begin{matrix} q^7, & -q^7, & q^{-n+5}, & q^{-n+4}, & q^{-n+3}, \\ q^2, & -q^2, & q^{n+4}, & q^{n+5}, & q^{n+6}, \end{matrix} \right. \left. \begin{matrix} q^{-n+2}, & q^{-n+1}, & \tau; & q^5, & q^{5n+7}/\tau \\ q^{n+7}, & q^{n+8}, & \tau^{-1} & q^9 & \end{matrix} \right]$$

for $n > 0$.

PROOF. We start with the formulae

(3.7)
$$D_n = \sum_{\lambda=-\infty}^{\infty} (-1)^\lambda q^{\lambda(5\lambda+1)/2} \begin{bmatrix} n \\ \left[\frac{n-5\lambda}{2}\right] \end{bmatrix},$$

and

(3.8)
$$d_n = \sum_{\lambda=-\infty}^{\infty} (-1)^\lambda q^{\lambda(5\lambda-3)/2} \begin{bmatrix} n \\ \left[\frac{n-5\lambda}{2}\right]+1 \end{bmatrix},$$

where $[x]$ is the largest integer $\le x$. These formulae are established by means of simple recurrences in [**1**], [**9**].

Each of (3.3)-(3.6) is proved similarly. We shall go through the details for (3.3) and then briefly indicated the remainder. In (3.7) we split the sum into two parts: one with λ even, the other with λ odd.

$$\begin{aligned}
&D_{2n-1}\\
&= \sum_{\lambda=-\infty}^{\infty} q^{10\lambda^2+\lambda}\begin{bmatrix} 2n-1 \\ n-5\lambda-1\end{bmatrix}\\
&\quad - \sum_{\lambda=-\infty}^{\infty} q^{10\lambda^2+11\lambda+3}\begin{bmatrix} 2n-1 \\ n-5\lambda-3\end{bmatrix}\\
&= (q;q)_{2n-1}\sum_{\lambda=-\infty}^{\infty}\frac{q^{10\lambda^2+\lambda}}{(q;q)_{n-5\lambda-1}}\\
&\quad \cdot \frac{\{(1-q^{n+5\lambda+2})(1-q^{n+5\lambda+1}) - q^{10\lambda+3}(1-q^{n-5\lambda-1})(1-q^{5\lambda-2})\}}{(q;q)_{n+5\lambda+2}}\\
&= (q;q)_{2n-1}\sum_{\lambda=-\infty}^{\infty}\frac{q^{10\lambda^2+\lambda}(1-q^{10\lambda+3})(1-q^{2n})}{(q;q)_{n-5\lambda-1}(q;q)_{n+5\lambda+2}}\\
&= \frac{(q;q)_{2n}}{(q;q)_{n-1}(q;q)_{n+2}}\\
&\quad \cdot \sum_{\lambda=-\infty}^{\infty}\frac{(-1)^\lambda q^{10\lambda^2+\lambda+5\lambda n-\binom{5\lambda+1}{2}}(q^{-n+1};q)_{5\lambda}(1-q^{10\lambda+3})}{(q^{n+3};q)_{5\lambda}}\\
&= \frac{(q;q)_{2n}(1-q^3)}{(q;q)_{n-1}(q;q)_{n+2}}\\
&\quad \lim_{\tau\to 0} {}_8\psi_8\left[\begin{matrix} q^{\frac{13}{2}}, & -q^{\frac{13}{2}}, & q^{5-n}, & q^{4-n}, & q^{3-n}, \\ q^{\frac{3}{2}}, & -q^{\frac{3}{2}}, & q^{n+3}, & q^{n+4}, \end{matrix}\right.
\end{aligned}\tag{3.9}$$

$$\left.\begin{matrix} q^{2-n}, & q^{1-n}, & \tau; & q^5, & q^{5n+4}/\tau \\ q^{n+5}, & q^{n+6}, & q^{n+7}, & \tau^{-1} & q^8 \end{matrix}\right]$$

which is (3.3).

For the remaining three identities we provide only the key step:

(3.10)

$$\begin{aligned}
D_{2n} &= \sum_{\lambda=-\infty}^{\infty} q^{10\lambda^2+\lambda}\begin{bmatrix} 2n \\ n-5\lambda\end{bmatrix} - \sum_{\lambda=-\infty}^{\infty} q^{10\lambda^2-9\lambda+2}\begin{bmatrix} 2n \\ n-5\lambda+2\end{bmatrix}\\
&= (q;q)_{2n}\sum_{\lambda=-\infty}^{\infty}\frac{q^{10\lambda^2-\lambda}(1-q^{10\lambda+2})(1-q^{2n+1})}{(q;q)_{n+5\lambda+2}(q;q)_{n-5\lambda}}.
\end{aligned}$$

$$
\begin{aligned}
d_{2n-1} &= \sum_{\lambda=-\infty}^{\infty} q^{\lambda(10\lambda-3)} \begin{bmatrix} 2n-1 \\ n-5\lambda \end{bmatrix} \\
&\quad - \sum_{\lambda=-\infty}^{\infty} q^{(2\lambda+1)(5\lambda+1)} \begin{bmatrix} 2n-1 \\ n-5\lambda-2 \end{bmatrix} \\
&= (q;q)_{2n-1} \sum_{\lambda=-\infty}^{\infty} \frac{q^{\lambda(10\lambda-3)}(1-q^{10\lambda+1})(1-q^{2n})}{(q;q)_{n-5\lambda}(q;q)_{n+5\lambda+1}}.
\end{aligned} \tag{3.11}
$$

$$
\begin{aligned}
d_{2n} &= \sum_{\lambda=-\infty}^{\infty} q^{\lambda(10\lambda-3)} \begin{bmatrix} 2n \\ n-5\lambda+1 \end{bmatrix} \\
&\quad - \sum_{\lambda=-\infty}^{\infty} q^{(2\lambda-1)(5\lambda-4)} \begin{bmatrix} 2n \\ n-5\lambda+3 \end{bmatrix} \\
&= (q;q)_{2n} \sum_{\lambda=-\infty}^{\infty} \frac{q^{\lambda(10\lambda+3)}(1-q^{10\lambda+4})(1-q^{2n+1})}{(q;q)_{n+5\lambda+3}(q;q)_{n-5\lambda-1}}.
\end{aligned} \tag{3.12}
$$

Now each of the ${}_8\psi_8$'s appearing in Lemma 1 is terminating above and below. Furthermore, each is of the classical very well–poised type. We need now a transformation of such series that will yield the factorizations (2.8) and (2.9) and the limits (2.10) and (2.11).

Since our ${}_8\psi_8$'s are terminating, we can easily shift the index of summation to yield ${}_8\phi_7$'s. Then the q–analog of Whipple's theorem [**12**], [**11**; p. 100, eq (3.4.1.5)] provides us with the appropriate transformation. All this is encoded in the following result.

LEMMA 2. *Let $R \geq 0$ and $-R < \varepsilon$ be integers. Then*

$$
\begin{aligned}
&\lim_{\tau\to 0} {}_8\psi_8 \left[\begin{matrix} aq^{-R}, & q\sqrt{a}, & -q\sqrt{a}, & cq^{-R}, & dq^{-R}, & eq^{-R}, \\ q^{R+1}, & \sqrt{a}, & -\sqrt{a}, & \frac{aq^{R+1}}{c}, & \frac{aq^{R+1}}{d}, & \frac{aq^{R+1}}{e}, \end{matrix} \right. \\
&\qquad\qquad \left. \begin{matrix} q^{-R-\varepsilon}, & \tau q^{-R}; & q, & \frac{a^2q^{6R+2+\varepsilon}}{cde\tau} \\ aq^{R+1+\varepsilon}, & a\tau^{-1} & q^{R+1} & \end{matrix} \right] \\
&= F_R(q|\varepsilon;a;c,d,e)L_R(q|\varepsilon;a;c,d;e),
\end{aligned} \tag{3.13}
$$

where
(3.14)
$F_R(q|\varepsilon; a; c, d, e) =$

$$\frac{(q;q)_R(aq;q)_{R+\varepsilon}(a^{-1}q;q)_{R-\varepsilon}(a^{-1}q^{R+1-\varepsilon)},q)_{R+\varepsilon}}{(a^{-1}q^{R+1};q)_R(c^{-1}q^{R+1};q)_R(d^{-1}q^{R+1};q)_R(e^{-1}q^{R+1};q)_R(q^{\varepsilon+R+1};q)_R}$$
$$\times \frac{1}{(aq^{R+1}/d;q)_{R+\varepsilon}(aq^{R+1}/e;q)_{R+\varepsilon}(aq^{R+1}/c;q)_{R+\varepsilon}},$$

(3.15)
$L_R(q|\varepsilon; a; c, d; e)$

$$= \left(\frac{aq^{1+2R}}{de};q\right)_{2R+\varepsilon}\left(\frac{aq^{1+2R}}{ce};q\right)_{2R+\varepsilon}$$
$$\times\, 3\phi 2\left(\begin{matrix} q^{-2R-\varepsilon}, & eq^{-2R}, & \frac{eq^{-2R-\varepsilon}}{a}; & q, & q \\ & \frac{ceq^{-4-R\varepsilon}}{a}, & \frac{deq^{-4R-\varepsilon}}{a} & & \end{matrix}\right)$$
$$= \sum_{j=0}^{2R+\varepsilon}(-1)^j q^{[\frac{j}{2}]+2Rj}\left(\frac{a}{cd}\right)^j\binom{2R+\varepsilon}{j}(e^{-1}q^{2R-j+1};q)_j(\frac{a}{e}q^{2R+\varepsilon-j+1};q)_j$$
$$\times\left(\frac{aq^{1+2r}}{de};q\right)_{2R+\varepsilon-j}\left(\frac{aq^{1+2R}}{ce};q\right)_{2R+\varepsilon-j}.$$

REMARK. The expression F_R will contribute primarily the $G_n(q)$ or $H_n(q)$ while L_R will provide most of the $P_n(q)$ or $R_n(q)$.

PROOF. If we examine the series on the lefthand side of (3.13) we see that, in fact, it is a finite sum whose index j runs from $-R$ to $R+\varepsilon$. The first thing to do is shift j to $j-R$ so that the sum runs from 0 to $2R+\varepsilon$. To do this we make use of the fact that

$$(A;q)_{j-R} = \frac{(Aq^{-R};q)_j}{(Aq^{-R};q)_R}.$$

Hence

$$\lim_{\tau\to 0} {}_8\psi_8\left[\begin{matrix} aq^{-R}, & q\sqrt{a}, & -q\sqrt{a}, & cq^{-R}, & dq^{-R}, & \\ & q^{R+1}, & \sqrt{a}, & -\sqrt{a}, & \frac{aq^{R+1}}{c}, & \frac{aq^{R+1}}{d}, \\ eq^{-R-\varepsilon}, & \tau q^{-R}; & q, & \frac{a^2q^{6R+2+\varepsilon}}{cde\tau} & & \\ aq^{R+1+\varepsilon}, & a\tau^{-1} & q^{R+1} & & & \end{matrix}\right]$$

$$= \lim_{\tau\to 0}\left\{\frac{(q;q)_R(\frac{aq}{c};q)_R(\frac{aq}{d};q)_R(\frac{aq}{e};q)_R}{(aq^{-2R};q)_R(cq^{-2R};q)_R(dq^{-2R};q)_R}\right.$$

$$\left.\cdot\frac{(aq^{1+\varepsilon};q)_R(a\tau^{-1}q;q)_R}{(eq^{-2R};q)_R(q^{-2r-\varepsilon};q)_R(\tau q^{-2R};q)_R}\right\}$$

$$\times\frac{(1-aq^{-2R})}{(1-a)}\cdot\left(\frac{a^2q^{6R+2+\varepsilon}}{cde\tau}\right)^{-R}$$

$$\times\, {}_8\phi_7\left[\begin{matrix}aq^{-2R}, & \sqrt{a}q^{1-R}, & -\sqrt{a}q^{1-R}, & cq^{-2R}, & dq^{-2R}, \\ & \sqrt{a}q^{-R}, & -\sqrt{a}q^{-R}, & \frac{aq}{c}, & \end{matrix}\right.$$

$$\left.\begin{matrix}eq^{-2R}, & q^{-2R-\varepsilon}, & \tau q^{-2R}; & \tau q^{-2R}; & q, & \frac{a^2q^{6R+2+\varepsilon}}{cde\tau} \\ \frac{aq}{d}, & \frac{aq}{e}, & aq^{1+\varepsilon}, & a\tau^{-1}q & & \end{matrix}\right]$$

$$= \frac{(q;q)_R(\frac{aq}{c};q)_R(\frac{aq}{d};q)_R(\frac{aq}{e};q)_R(aq^{1+\varepsilon};q)_R(-1)^R a^r q^{R(R+1)/2}}{(aq^{-2R};q)_R(cq^{-2R};q)_R(dq^{-2R};q)_R(eq^{-2R};q)_R(q^{-2R-\varepsilon};q)_R}$$

$$\times\frac{(1-aq^{-2R})}{(1-a)}\left(\frac{cde}{a^2q^{6R+2+\varepsilon}}\right)^R\frac{(aq^{1-2R};q)_{2R+\varepsilon}(\frac{aq^{1+2R}}{de};q)_{2R+\varepsilon}}{(\frac{aq}{d};q)_{2R+\varepsilon}(\frac{aq}{e};q)_{2R+\varepsilon}}$$

$$\times\, {}_3\phi_2\left[\begin{matrix}eq^{-2R}, & dq^{-2R}, & q^{-2R-\varepsilon}; & q, & \frac{q^{2R}}{c} \\ \frac{aq}{c}, & \frac{edq^{-4R-\varepsilon}}{a} & & & \end{matrix}\right]$$

(by the q–analog of Whipple's Theorem [**12**])

$$= F_R(q|\varepsilon;a;c,d,e)$$

$$\times(\frac{aq}{c};q)_{2R+\varepsilon}\left(\frac{aq^{1+2R}}{de};q\right)_{2R+\varepsilon}{}_3\phi_2\left(\begin{matrix}eq^{-2R}, & dq^{-2R}, & q^{-2R-\varepsilon}; & q, & q \\ \frac{aq}{c}, & \frac{edq^{-4R-\varepsilon}}{a} & & & \end{matrix}\right)$$

(by algebraic simplication of the initial factors)

$$= F_R(q|\varepsilon;a;c,d,e)\times\left(\frac{aq^{1+2R}}{de};q\right)_{2R+\varepsilon}$$

$$\cdot\left(\frac{aq^{1+2R}}{ce};q\right)_{2R+\varepsilon}{}_3\phi_2\left[\begin{matrix}q^{-2R-\varepsilon}, & eq^{-2R}, & eq^{-2R-\varepsilon}; & q, & q \\ \frac{ceq^{1-4R-\varepsilon}}{a}, & \frac{deq^{-4R-\varepsilon}}{a} & & & \end{matrix}\right]$$

(by [**10**; p. 175, eq. (10.2)] with $p\to q, a=q^{-2R-\varepsilon}, b=eq^{-2R}, c\to eq^{-2R-\varepsilon}/a, e\to aq/c, f=edq^{-4R-\varepsilon}/a$)

$$= F_R(q|\varepsilon;a;c,d,e)L_R(q|\varepsilon;a;c,d;e),$$

which is our desired result using the first expression in (3.15) for $L_R(q|\varepsilon;a;c,d;e)$. The second expression for $L_R(q|\varepsilon;a;c,d;e)$ is easily derived from the first once we observe that

$$\begin{bmatrix} A \\ B \end{bmatrix} = (-1)^B q^{AB-B(B-1)/2} \frac{(q^{-A};q)_B}{(q;q)_B} \tag{3.16}$$

and

$$(cq^{-N};q)_j = (-1)^j q^{-Nj+j(j-1)/2} c^j (c^{-1}q^{N-j+1};q)_j. \tag{3.17}$$

Hence Lemma 2 is established.

LEMMA 3. *For* $|q| < 1$,

$$\lim_{R\to\infty} L_R(q|\varepsilon;a;c,d;e) = 1. \tag{3.18}$$

PROOF. For $L_R(q|\varepsilon;a;c,d;e)$ we use the second representation in (3.15), which we write as

$$\sum_{j=0}^{2R+\varepsilon} q^{2Rj} \cdot T_j(R). \tag{3.19}$$

As $R \to \infty$, we see that $T_j(R)$ is bounded by

$$\begin{aligned} &\left|\frac{a}{cd}\right|^j \frac{1}{(|q|;|q|)_j} (-|e^{-1}||q|^{1+\varepsilon};|q|)_\infty (-|\frac{a}{e}||q|;|q|)_\infty \\ &\times (-|\frac{aq}{de}|;|q|)_\infty (-|\frac{aq}{ce}|;|q|)_\infty. \end{aligned} \tag{3.20}$$

Hence as $R \to \infty$ every term of the sum in (3.19) goes to 0 except the first, and the first converges to 1.

4. The Rogers–Ramanujan Identities.

We are now prepared to give the main results outlined in §2.

THEOREM 1. *Equations* (2.8) *and* (2.9) *hold with* $P_n(q)$ *and* $R_n(q)$ *given by*

(4.1) $P_{10n-5}(q) = L_n(q^5|-1; q^3; q^7, q^6; q^4)$,

(4.2) $P_{10n-4}(q) = L_n(q^5|-1; q^2; q^6, q^4; q^3)$,

(4.3) $P_{10n-3}(q) = (1-q^{10n-3})L_n(q^5|-1; q^3; q^6, q^4; q^2)$,

(4.4) $P_{10n-2}(q) = (1-q^{10n-2})L_n(q^5|-1; q^2; q^4, q^3; q)$.

(4.5) $P_{10n-1}(q) = (1-q^{10n})L_n(q^5|-1; q^3; q, q^2; q^4)$,

(4.6) $P_{10n}(q) = L_n(q^5|0; q^2; q, q^3; q^4)$,

(4.7) $P_{10n+1}(q) = L_n(q^5|0; q^3; q, q^2; q^4)$,

(4.8) $P_{10n+2}(q) = (1-q^{10n+2})L_n(q^5|0; q^2; q^{-1}, q; q^3)$,

(4.9) $P_{10n+3}(q) = (1-q^{10n+3})L_n(q^5|0; q^3; q^{-1}, q; q^2)$,

(4.10) $P_{10n+4}(q) = (1-q^{10n+5})L_n(q^5|0; q^2; q^{-2}, q^{-1}; q)$;

and

(4.11) $R_{10n-6}(q) = L_n(q^5|-1; q^4; q^6, q^7; q^8)$,

(4.12) $R_{10n-4}(q) = L_n(q^5|-1; q^4; q^3, q^6; q^7)$,

(4.13) $R_{10n-3}(q) = L_n(q^5)|-1; q; q^2, q^3; q^4)$,

(4.14) $R_{10n-2}(q) = (1-q^{10n-1})L_n(q^5|-1; q^4; q, q^3; q^6)$,

(4.15) $R_{10n-1}(q) = L_n(q^5|0; q; q^2, q^3; q^4)$,

(4.16) $R_{10n}(q) = (1-q^{10n})(1-q^{10n+1})L_n(q^5|-1; q^4; q, q^2; q^3)$,

(4.17) $R_{10n+1}(q) = L_n(q^5|0; q; q^{-1}, q^2; q^3)$,

(4.18) $R_{10n+2}(q) = L_n(q^5|0; q^4; q, q^2; q^3)$,

(4.19) $R_{10n+3}(q) = (1-q^{10n+4})L_n(q^5|0; q; q^{-2}, q^{-1}; q^2)$,

(4.20) $R_{10n+5}(q) = (1-q^{10n+5})(1-q^{10n+6})L_n(q^5|0; q; q^{-3}, q^{-2}; q^{-1})$.

PROOF. These twenty results are merely straightforward applications of Lemma 2 to Lemma 1. We give the details for (4.1); the remainder are done in exactly the same way.

By (3.3)

$$D_{10n-5} = \frac{(q;q)_{10n-4}(1-q^3)}{(q;q)_{5n-3}(q;q)_{5n}} \cdot \lim_{\tau\to 0} {}_8\psi_8$$

(4.21)
$$\left[\begin{matrix} q^{\frac{13}{2}}, & -q^{\frac{13}{2}}, & q^{7-5n}, & q^{6-5n}, & q^{5-5n}, & q^{4-5n}, & q^{3-5n}, \tau; & q^5, & q^{25n-6}/\tau \\ q^{\frac{3}{2}}, & -q^{\frac{3}{2}}, & q^{5n+1}, & q^{5n+2}, & q^{5n+3}, & q^{5n+4}, & q^{5n+5}, & \tau^{-1}q^8 & \end{matrix}\right]$$

$$= \frac{(q;q)_{10n-4}(1-q^3)}{(q;q)_{5n-3}(q;q)_{5n}} F_n(-q^5|-1;q^3;q^7.q^6,q^4) L_n(q^5|-1;q^3;q^7,q^6;q^4)$$

(by Lemma 2).

Comparing (4.21) with (2.8), we see that to establish (4.1) we need only show that

$$\begin{aligned}
&\frac{(q;q)_{10n-4}(1-q^3)}{(q;q)_{5n-3}(q;q)_{5n}} F_n(q^5|-1;q^3;q^7,q^6,q^4) \\
&\qquad = \frac{(q;q)_{10n-4}(q^5;q^5)_n(q^3;q^5)_n}{(q;q)_{5n-3}(q;q)_{5n}(q^{5n+2};q^5)_n(q^{5n-2};q^5)_n} \\
&\qquad\quad \cdot \frac{(q^2;q^5)_{n+1}(q^{5n+7};q^5)_{n-1}}{(q^{5n-1};q^5)_n(q^{5n+1};q^5)_n(q^{5n};q^5)_n} \\
&\qquad\quad \times \frac{1}{(q^{5n+2};q^5)_{n-1}(q^{5n+4};q^5)_{n-1}(q^{5n+1};q^5)_{n-1}} \\
&\qquad = \frac{1}{(q;q^5)_n(q^4;q^5)_n(q^{5n+4};q^5)_{n-1}(q^{5n+1};q^5)_{n-1}} \\
&\qquad = \frac{1}{(q;q^5)_{2n-1}(q^4;q^5)_{2n-1}} = G_{10n-5}(q),
\end{aligned}$$

as desired.

The rest follow in the same way.

THEOREM 2. *Equation* (1.6) *and* (1.7) *are valid, i.e., the Rogers-Ramanujan identities hold.*

PROOF. By Theorem 1 and Lemma 3 we see immediately that

$$\lim_{n\to\infty} P_n(q) = 1, \tag{4.22}$$

$$\lim_{n\to\infty} R_n(q) = 1. \tag{4.23}$$

Hence

$$\begin{aligned}
\sum_{n=0}^{\infty} \frac{q^{n^2}}{(q;q)_n} &= \lim_{n\to\infty} D_n \\
&= \lim_{n\to\infty} G_n(q)\cdot P_n(q) \qquad \text{(by (2.4) and (2.8))} \\
&= \lim_{n\to\infty} G_n(q) \\
&= \frac{1}{(q;q^5)_\infty(q^4;q^5)_\infty} \qquad \text{(by (4.22) and (2.6))}
\end{aligned} \tag{4.24}$$

and

$$\begin{aligned}
\sum_{n=0}^{\infty} \frac{q^{n^2+n}}{(q;q)_n} &= \lim_{n\to\infty} d_n \\
&= \lim_{n\to\infty} H_n(q)\cdot R_n(q) \qquad \text{(by (2.5) and (2.9))} \\
&= \lim_{n\to\infty} H_n(q) \\
&= \frac{1}{(q^2;q^5)_\infty (q^3;q^5)_\infty} \qquad \text{(by (4.23) and (2.7)).}
\end{aligned} \tag{4.25}$$

5. Conclusion. It should be pointed out that the title of this paper is somewhat misleading. We have technically avoided the use of Jacobi's triple product; however the real engine of our proof is Lemma 2 a result much stronger than Jacobi's triple product. Indeed if we let $R \to \infty$ in (3.13) we obtain

$$\frac{\sum_{j=-\infty}^{\infty} \left(a^{2j} q^{\binom{2j}{2}} - a^{2j+1} q^{\binom{2j+1}{2}}\right)}{(1-a)} = (q;q)_\infty (aq;q)_\infty (a^{-1}q;q)_\infty$$

or

$$\sum_{j=-\infty}^{\infty} (-1)^j a^j q^{\binom{j}{2}} = (q;q)_\infty (a;q)_\infty (a^{-1}q;q)_\infty,$$

which is precisely Jacobi's triple product identity [**3**; p.21, Th. 2.8]. Thus in Lemma 2 we have a finite, rational function identity that converges to Jacobi's triple product in the limit.

On the other hand, it is well–known that the standard finite form of Jacobi's triple product identity [**3**; p. 49, Ex. 1] is equivalent to the q–binomial theorem. It would be unreasonable to expect that the Roger's–Ramanujan identities could be deduced without ever invoking a result as strong as the q–binomial theorem.

The real point of this paper lies in the fact that we have useful closed forms for $P_n(q)$ and $R_n(q)$ given by Theorem 1. If real combinatorial progress is to be made on the understanding of the Rogers–Ramunajan identities, then $P_n(q)$ and $R_n(q)$ deserve further study.

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