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LIFTING OF ROTUNDITY PROPERTIES FROM E TO $L^p(\mu, E)$

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ABSTRACT. We consider some rotundity properties which are extensions of the uniform rotundity and show that these properties lift from the Banach space E (or from the conjugate Banach space E^*) to the Lebesgue–Bochner function space $L^p(\mu, E)$ (or to $(L^p(\mu, E))^*$), $1 < p < \infty$. We make no assumption on E^*; in particular, we do not assume that E^* has the Radon–Nikodym property.

0. Introduction. In their paper **[16]** Smith and Turett give several interesting results about the geometry of the Lebesgue–Bochner function spaces $L^p(\mu, E)$. In particular they show that the following statement holds.

THEOREM 0. *Let $(S, \sum, \mu)$, a finite measure space, and E, a Banach space, be given. Assume that E^*, the conjugate space of E, satisfies the Radon–Nikodym property. Then $L^p(\mu, E)$, $1 < p < \infty$, is weakly uniformly rotund if and only if E is.*

One of the purposes of this paper is to prove the above result without any assumption on E^*. (By the way, it is unknown up to now whether the weak uniform rotundity of a Banach space E implies that E^* has the Radon–Nikodym property). Moreover, we consider three other geometric properties, namely *weak local uniform rotundity weak* uniform rotundity* (in a conjugate space) and *weak* local uniform rotundity,* and we show that they lift from E (or E^*) to $L^p(\mu, E)$ (or $(L^p(\mu, E))^*$). Also for these properties we will make no assumption on E^*. It is worth noting that assuming the Radon–Nikodym property for E^* would be an effective restriction in this case (see Remark 4 at the end of the paper).

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Th question of whether certain properties of a Banach space E are inherited by $L^p(\mu, E)$ has been studied extensively (see Day [**3**], Diestel [**4**], Leonard and Sundaresan [**10**] and [**11**], Mc Shane [**13**], Smith and Turett [**16**], Sundaresan [**17**], Turett and Uhl [**18**]).

1. Definitions. To begin with, let us recall the cited geometric properties. Two of them ((WUR) and (W*UR)) are directionalizations of the *uniform rotundity* (a property introduced by Clarkson in the famous paper [**1**]), as underlined by Smith [**14**]. The remaining properties are localizations of the preceding ones, as it is easily realized. Throughout E will be Banach space with norm $||\cdot||$ and E^* will be the conjugate space of E with norm $||\cdot||_*$.

DEFINITION 1. (see [**2**]). *E (resp. E^*) is said to be weakly uniformly rotund* (WUR) *(resp. weakly* uniformly rotund* (W*UR)) *provided that $||x_n|| = ||y_n|| = 1$ for every $n \in N$ and $||x_n + y_n|| \xrightarrow{w} 2$ imply that $x_n - y_n \xrightarrow{w} 0$ (resp. $||f_n||_* = ||g_n||_* = 1$ for every $n \in N$ and $||f_n + g_n||_* \to 2$ imply that $f_n - g_n \xrightarrow{w^*} 0$).*

DEFINITION 2. (see [**12**] and [**9**]). E*(resp. E^*) is said to be weakly locally rotund* (WLUR) *(resp. weakly* locally uniformly rotund* (W*LUR)*) provided that $||x|| = ||y_n|| = 1$ for every $n \in N$ and $||x + y_n|| \to 2$ imply that $y_n \xrightarrow{w} x$ (resp. $||f||_* = ||g_n||_* = 1$ for every $n \in N$ and $||f + g_n||_* \to 2$ imply that $g_n \xrightarrow{w^*} f$).*

The connections among the above rotundity properties (for a conjugate space) are contained in the chart below where an arrow denotes implication:

$$\begin{array}{ccc} \text{(WUR)} & \longrightarrow & \text{(W*UR)} \\ \downarrow & & \downarrow \\ \text{(WLUR)} & \longrightarrow & \text{(W*LUR).} \end{array}$$

Obviously, for a general Banach space, this chart reduces to: (WUR) $\to$ (WLUR). No implication can be reversed (see §4).

Let $(S, \sum, \mu)$ be a measure space. We will denote by $L^p(\mu, E), 1 < p < \infty$, the Lebesgue–Bochner function space of μ–equivalence classes of strongly measurable functions $f : S \to E$ with $\int_S ||f(s)||^p d\mu < \infty$, endowed with the norm

$$|||f||| = \left(\int_S ||f(s)||^p d\mu\right)^{1/p}.$$

We will denote by $|||\cdot|||_*$ the norm of the conjugate space $(L^p(\mu, E))^*$. For our purposes an integral representation theorem for the elements of $(L^p(\mu, E))^*$ will be useful. Its most general form is due to A. and C. Ionescu Tulcea [**8**] (see also Dinculeanu [**7**, p. 119]); as far as it is known (see [**5**, p. 116]), it is used here, in the study of the structure of $L^p(\mu, E)$, for the first time. We state it as a lemma.

LEMMA 0. *Let $(S, \sum, \mu)$ be a σ-finite measure space. Then for each linear continuous functional L on $L^p(\mu, E)$ there exists a function g, $g : S \to E^*$ such that:*

(a) *the function $s \to \langle g(s), f(s)\rangle$ is an element of $L^1(\mu, \mathbf{R})$ for every $f \in L^p(\mu, E)$;*

(b) $L(f) = \int_s \langle g(s), f(s)\rangle d\mu$ *for every $f \in L^p(\mu, E)$;*

(c) *the function $s \to ||g(s)||_*$ is an element of $L^q(\mu, \mathbf{R})$, $1/p + 1/q = 1$; and*

(d) $|||L|||_* = \left(\int_S ||g(s)||_*^q d\mu\right)^{1/q}$.

2. Reformulation of the rotundity properties. In order to obtain useful reformulations of the above rotudity conditions, we consider a class G of real functions $G(u, v, t)$ defined for $u \geq 0, v \geq 0, t \geq 0$. We say that $G \in \mathcal{G}$ provided that:

i) $G(u, v, t) > G(u, v, t')$ if $t < t'$;

ii) G is continuous;

iii) $G(u, v, u + v) \geq 0$; and

iv) $G(u, v, u + v) = 0$ if and only if $u = v$.

The class $\mathcal{G}$ contains, for example, the functions G_r, $1 < r < \infty$, given by

$$(*) \qquad G_r(u, v, t) = 2^{r-1}(u^r + v^r) - t^r.$$

LEMMA 1. *For every Banach space E the following are equivalent:*

j) *E is* (WUR);

jj) *any $G \in \mathcal{G}$ satisfies the condition:*

(1) *for bounded sequences $\{x_n\}, \{y_n\} \subset E$, $G(||x_n||, ||y_n||, ||x_n + y_n||) \to 0$ implies $x_n - y_n \xrightarrow{w} 0$; and*

jjj) *the condition* (1) *satisfied by some $G \in \mathcal{G}$.*

PROOF. j) $\Rightarrow$ jj). Arguing by contradiction, we assume that the exist $G \in \mathcal{G}, f \in E^*, \varepsilon > 0$ and two bounded sequences $\{x_n\}, \{y_n\} \subset E$ for which $G(||x_n||, ||y_n||, ||x_n + y_n||) \to 0$ and $|f(x_n - y_n)| \geq \varepsilon$. Since $\{x_n\}, \{y_n\}$ are bounded, we can suppose that $d_1, d_2, l \in [0, \infty[$ exist such that $||x_n|| \to d_1, ||y_n|| \to d_2, ||x_n + y_n|| \to l$; obviously, $l \leq d_1 + d_2$. On the other hand, i) and iii) imply

$$0 \leq G(||x_n||, ||y_n||, ||x_n|| + ||y_n||) \leq G(||x_n||, ||y_n||, ||x_n + y_n||)$$

and so, letting $n \to \infty$ and using ii), we have

$$0 \leq G(d_1, d_2, d_1 + d_2) \leq G(d_1, d_2, l) = 0.$$

Hence, by iv), $d_1 = d_2 = d$ follows, whereas i) gives $l = 2d$. Clearly $d > 0$.

Now, we consider the norm one sequences $\{x_n/||x_n||\}, \{y_n/||y_n||\}$ with n sufficiently large. We have

$$\begin{aligned} &\left|f\left(\frac{x_n}{||x_n||} - \frac{y_n}{||y_n||}\right)\right| \\ &\geq \left|f\left(\frac{x_n}{d} - \frac{y_n}{d}\right)\right| - \left|f\left(\frac{x_n}{||x_n||} - \frac{x_n}{d}\right) + f\left(\frac{y_n}{d} - \frac{y_n}{||y_n||}\right)\right| \\ &\geq \frac{\varepsilon}{d} - \left|f\left(\frac{x_n}{||x_n||} - \frac{x_n}{d}\right) + f\left(\frac{y_n}{d} - \frac{y_n}{||y_n||}\right)\right|; \end{aligned}$$

consequently, for n sufficiently large,

$$(\alpha) \qquad \left|f\left(\frac{x_n}{||x_n||} - \frac{y_n}{||y_n||}\right)\right| \geq \frac{\varepsilon}{2d}.$$

On the other hand,

(β)

$$2 \geq \left|\left|\frac{x_n}{||x_n||} + \frac{y_n}{||y_n||}\right|\right| \geq \left|\left|\frac{x_n}{d} + \frac{y_n}{d}\right|\right| - \left|\left|\left(\frac{x_n}{||x_n||} - \frac{x_n}{d}\right) - \left(\frac{y_n}{d} - \frac{y_n}{||y_n||}\right)\right|\right| \to 2.$$

(α) and (β) contradict j).

jj) $\Rightarrow$ jjj). This is trivial.

jjj) $\Rightarrow$ j). Consider two arbitrary norm one sequences $\{x_n\}, \{y_n\} \subset E$ such that $||x_n + y_n|| \to 2$. Then, for any $G \in \mathcal{G}$, we have

$$G(||x_n||, ||y_n||, ||x_n + y_n||) \to G(1, 1, 2) = 0.$$

Since we are assuming jjj), we get $x_n - y_n \xrightarrow{w} 0$.

The proofs of the following Lemmas 2-4 are analogous to that of Lemma 1. They will be therefore omitted.

LEMMA 2. *For every conjugate Banach space E^* the following are equivalent:*

j) *E^* is* (W*UR);

jj) *any $G \in \mathcal{G}$ satisfies the condition:*

(2) *for bounded sequences $\{f_n\}, \{g_n\} \subset E^*$, $G(||f_n||_*, ||g_n||_*, ||f_n + g_n||_*) \to 0$ implies $f_n - g_n \xrightarrow{w^*} 0$;*

jjj) *the condition* (2) *is satisfied by some $G \in \mathcal{G}$.*

LEMMA 3. *For every Banach space E the following are equivalent:*

j) *E is* (WLUR):

jj) *any $G \in \mathcal{G}$ satisfies the condition:*

(3) *for $x \in E$ and bounded sequence $\{x_n\} \subset E$, $G(||x||, ||x_n||, ||x + x_n||) \to 0$ implies $x_n \xrightarrow{w} x$;*

jjj) *the condition* (3) *is satisfied by some $G \in \mathcal{G}$.*

LEMMA 4. *For every conjugate Banach space E^* the following are equivalent:*

j) *E^* is* (W*LUR);

jj) *any $G \in \mathcal{G}$ satisfies the condition:*

(4) *for $f \in E^*$ and a bounded sequence $\{f_n\} \subset E^*, G(||f||_*, ||f_n||_*, ||f + f_n||_*) \to 0$ implies $f_n \xrightarrow{w^*} f$;*

jjj) *the condition* (4) *is satisfied by some $G \in \mathcal{G}$.*

3. Lifting of the rotundity conditions. In this section we prove our main results concerning the lifting of the considered rotundity conditions from E to $L^p(\mu, E)$ (or from E^* to $(L^p(\mu, E))^*$).

For the sake of brevity and clarity we prove them supposing that $(S, \sum, \mu)$ is a σ–finite measure space. We shall show later on (see Remarks 1 and 2) how this assumption can be dropped. Also, to avoid triviality, we always suppose the existence of a set $X \in \sum$ with $0 < \mu(X) < \infty$.

THEOREM 1. *$L^p(\mu, E)$ is* (WUR) *if and only if E is.*

PROOF. The "only if" part is clear since E is isometrically embedded in $L^p(\mu, E)$.

To prove the reverse implication, assume that E is (WUR). We show that jjj) of Lemma 1 is true for $L^p(\mu, E)$, by taking $G = G_p, G_p$ given by (*). All we need to prove is that, for bounded sequences $\{f_n\}, \{g_n\} \subset L^p(\mu, E\}, G_p(|||f_n|||, |||g_n|||, |||f_n + g_n|||) \to 0$ implies $L(f_n - g_n) \to 0$ for each $L \in (L^p(\mu, E))^*$.

Proceeding by contradiction, we assume the existence of bounded sequences $\{f_n\}, \{g_n\} \subset L^p(\mu, E)$, $\sigma > 0$ and $L \in (L^p(\mu, E))^*$ for which $G_p(|||f_n|||, |||g_n|||, |||f_n + g_n|||) \to 0$ and $L(f_n - g_n) \geq 2\sigma$.

Now, according to Lemma 0, L has an integral representation $L(f) = \int_S < h(s), f(s) > d\mu$, $f \in L^p(\mu, E)$, where h, $h : S \to E^*$, is weak* measurable and such that $\left(\int_S ||h(s)||_*^q d\mu \right)^{1/q} = |||L|||_*$, $1/p + 1/q = 1$.

Hence

$$\int_S \langle h(s), f_n(s) - g_n(s) \rangle d\mu \geq 2\sigma, \quad \text{for each } n \in N.$$

Since h satisfies condition c) of Lemma 0, for each $\eta > 0$ there exists a set $S_\eta \in \sum$, $0 < \mu(S_\eta) < \infty$, for which $\left(\int_{S \setminus S_\eta} ||h(s)||_*^q d\mu \right)^{1/q} \leq \eta$; it follows that for some $\tilde{S} \in \sum$, $0 < \mu(\tilde{S}) < \infty$,

$$\int_{\tilde{S}} \langle h(s), f_n(s) - g_n(s) \rangle d\mu > \sigma, \quad \text{for each } n \in N.$$

The inequalities

$$G_p(|||f_n|||, |||g_n|||, |||f_n + g_n|||) \geq$$
$$\int_{\tilde{S}} G_p(||f_n(s)||, ||g_n(s)||, ||f_n(s) + g_n(s)||) d\mu, \text{ for each } n \in N,$$

and the fact that the integrands are nonnegative (by properties i) and iii) of the class $\mathcal{G}$ allow us to suppose (by taking a subsequence if necessary) that $G_p(||f_n(s)||, ||g_n(s)||, ||f_n(s) + g_n(s)||) \to 0$ a.e. on $\tilde{S}$.

Let

$$P_n = \left\{ s \in \tilde{S} : \langle h(s), f_n(s) - g_n(s) \rangle \geq \frac{\sigma}{2\mu(\tilde{S})} \right\}, \quad n \in N.$$

We have for every $n \in N$,

$$\begin{aligned}\sigma &\leq \int_{\tilde{S}} \langle h(s), f_n(s) - g_n(s)\rangle d\mu \\ &= \int_{P_n} \langle h(s), f_n(s) - g_n(s)\rangle d\mu + \int_{\tilde{S}\setminus P_n} \langle h(s), f_n(s) - g_n(s)\rangle d\mu \\ &\leq \Big(\int_{P_n} \|h(s)\|_*^q d\mu\Big)^{1/q}\Big(\int_{P_n} \|f_n(s) - g_n(s)\|^p d\mu\Big)^{1/p} + \frac{\sigma}{2\mu(\tilde{S})}\mu(\tilde{S}\setminus P_n),\end{aligned}$$

hence

$$\int_{P_n} \|h(s)\|_*^q \geq \Big(\frac{\sigma}{2\gamma}\Big)^q,$$

where γ is any upper bound for the real sequence $\{| \, \|f_n|\, \| + | \, \|g_n|\, \|\}$.

This implies the existence of a positive lower bound η for the real sequence $\{\mu(P_n)\}$.

Let

$$Q_n = \Big\{s \in P_n : (\|f_n(s)\| + \|g_n(s)\|)^p \leq \frac{2\gamma^p}{\eta}\Big\}.$$

Then, we have, for every $n \in N$,

$$\begin{aligned}\frac{\mu(P_n)}{\eta}\gamma^p &\geq \int_{P_n} (\|f_n(s)\| + \|g_n(s)\|)^p d\mu \\ &\geq \int_{P_n\setminus Q_n} (\|f_n(s)\| + \|g_n(s)\|)^p d\mu \\ &\geq \int_{P_n\setminus Q_n} (\|f_n(s)\| + \|g_n(s)\|)^p d\mu \geq \mu(P_n\setminus Q_n)\frac{2\gamma^p}{\eta};\end{aligned}$$

hence $\mu(Q_n) \geq \frac{\eta}{2}$ for every $n \in N$.

Consequently, denoting $Q = \limsup_n Q_n$, we have $\mu(Q) \geq \eta/2$.

Then it is possible to take $t \in Q$ for which $G_p(\|f_n(t)\|, \|g_n(t)\|, \|f_n(t)+g_n(t)\|) \to 0$.

Since $t \in \bigcap_{n=1}^{\infty} Q_{h(n)}$, for a suitable subsequence $\{Q_{h(n)}\}$ of $\{Q_n\}$, then $\{f_{h(n)}(t)\}, \{g_{h(n)}(t)\}$ are bounded sequences in E and so, by Lemma 1, $f_{h(n)}(t) - g_{h(n)}(t) \xrightarrow{w} 0$. This is absurd since $\langle h(t), f_{h(n)}(t) - g_{h(n)}(t)\rangle \geq \frac{\sigma}{2\mu(\tilde{S})}$ for each $n \in N$.

The proof is complete.

The proof of Theorem 1 can be adapted to show the following

THEOREM 2. *$(L^p(\mu, E))^*$ is $(W^* UR)$ if and only if E^* is.*

In a similar way, the proofs of the remaining results concerning (WLUR) and (W*LUR) are quite analogous, so it will be enough to display only one of them. To show that our techniques work in the case of a conjugate Banach space, we will prove Theorem 4 concerning (W*LUR).

THEOREM 3. $L^p(\mu, E)$ *is* (WLUR) *if and only if* E is.

THEOREM 4. $(L^p(\mu, E))^*$ *is* (W*LUR) *if and only if* E^* *is.*

PROOF. Clearly, if $(L^p(\mu, E))^*$ is (W*LUR), then E^* is. Vice-versa, suppose that E^* is (W*LUR). We show that jjj) of Lemma 4 is true for $L^p(\mu, E))^*$, by taking $G = G_q$, $1/p + 1/q = 1$, G_q given by $(*)$. Proceeding by contradiction, we assume the existence of an $L \in (L^p(\mu, E))^*$, a bounded sequence $\{L_n\} \subset (L^p(\mu, E))^*$, a $\sigma > 0$ and an $f \in L^p(\mu, E)$, for which $G_q(|\,||L|\,||_*, |\,||L_n|\,||_*, |\,||L + L_n|\,||_*) \to 0$ and $(L_n - L)(f) \geq \sigma$.

According to Lemma 0, $L, L_1, L_2, \ldots$ have integral representations by means of weak* measurable $g, g_1, g_2, \ldots$.

By (d) of Lemma 0,

$$\int_S G_q(||g(s)||_*, ||g_n(s)||_*, ||g(s) + g_n(s)||_*)d\mu =$$

$$G_q\left(\left(\int_S ||g(s)||_*^q d\mu\right)^{1/q}, \left(\int_S ||g_n(s)||_*^q d\mu\right)^{1/q},\right.$$

$$\left.\left(\int_S ||g(s) + g_n(s)||_*^q d\mu\right)^{1/q}\right) \to C$$

and, by (b) of Lemma 0,

$$\int_S \langle g_n(s) - g(s), f(s)\rangle d\mu \geq \sigma.$$

As in Theorem 1 we can suppose that

$$G_q(||g(s)||_* ||g_n(s)||_*, ||g(s) + g_n(s)||_*) \to 0 \text{ a.e. on } S;$$

hence, by properties i) and iii) of the class $\mathcal{G}$, we obtain

$$G_q(||g(s)||_*, ||g_n(s)||_*, ||g(s)||_* + ||g_n(s)||_*) \to 0 \text{ a.e. on } S.$$

Since $\lim_{v\to\infty} G_q(u,v,u+v) = \infty$, we have that $\{g_n(s)\}$ is a bounded sequence in E^*, a.e. on S. Using Lemma 4, $g_n(s) \xrightarrow{w^*} g(s)$ a.e. on S. It follows that $\langle g_n(s) - g(s), f(s)\rangle \to 0$ a.e. on S. On the other hand, for any $X \in \sum$ and any $n \in N$, we have

$$\int_X |\langle g_n(s) - g(s), f(s)\rangle| d\mu \leq \text{const} \left(\int_X ||f(s)||^p d\mu \right)^{1/q}.$$

As a consequence of this, if we define, for $s \in S$ and $n \in N$,

$$h_n(s) = |\langle g_n(s) - g(s), f(s)\rangle|,$$

then we have that $\{h_n\}$ is a sequence in $L^1(\mu\mathbf{R})$ for which all the assumptions of Vitali Convergence Theorem [**6**, Theorem III.6.15] are satisfied. It follows that $h_n \to 0$ in $L^1(\mu, \mathbf{R})$, whence

$$\int_S \langle g_n(s) - g(s), f(s)\rangle d\mu \to 0,$$

a contradiction. The proof is complete.

REMARK 1. To extend Theorem 1 to the case of an arbitrary measure space $(S, \sum, \mu)$ it is enough to notic that, by Lemma III.8.5 of [**6**], for any two sequences $\{f_n\}, \{g_n\} \subset L^p(\mu, E)$ there exist a σ-finite measure space $(S_1, \sum_1, \mu_1)$ and a closed separable subspace E_1 of E such that $L^p(\mu_1, E_1)$ is isometrically isomorphic to a (closed) subspace M of $L^p(\mu, E)$ and $\{f_n\}, \{g_n\} \subset M$. Since $L^p(\mu_1, E_1)$ is (WUR) if E_1 is (this has already been shown), it is clear that condition jjj) of Lemma 1 is verified for $L^p(\mu, E)$ if E is (WUR).

A similar argument shows that Theorem 3 holds for an arbitrary measure space.

REMARK 2. Also the extension of Theorem 2 and Theorem 4 to the case of an arbitrary measure space $(S, \sum, \mu)$ is achieved by means of a suitable application of Lemma III.8.5 of [**6**]. Indeed, according to that, for any sequence $\{H_n\} \subset (L^p(\mu, E))^*$ and any $f \in L^p(\mu, E)$, there are a σ-finite measure space $(S_1, \sum_1, \mu_1)$ and a closed separable subspace E_1 of E such that:

h) there exists an isometric isomorphism i from $L^p(\mu_1, E_1)$ into $L^p(\mu, E)$;

hh) $f \in i(L^p(\mu_1, E_1))$; and

hhh) for each $n \in N$, the norm of the element $H_n \circ i$ of $(L^p(\mu_1, E_1))^*$ is equal to $| \, ||H_n|| \, ||_*$.

From this remark it is clear that condition jjj) of Lemma 2 (resp. Lemma 4) is verified if E^* is (W*UR) (resp. if E^* is W*LUR)).

REMARK 3. The techniques used in this paper allow us to extend the lifting results concerning *local uniform rotundity* and *uniform rotundity in every direction* due to Smith and Turett [**16**, Theorem 2 and Theorem 6] to the case of an arbitrary (not necessarily finite) measure space $(S, \sum, \mu)$. Moreover they can be used to prove that the conjugate space $(L^p(\mu, E))^*$ is *strictly rotund* or *uniformly rotund in every direction* whenever E^* is. We leave the details to the reader.

4. Addendum: two examples. To finish we would like to display some examples showing that no implication in the chart (C) can be reversed.

EXAMPLE 1. (*a* (W*UR) *conjugate norm in* l^∞ *that is not* (WLUR)). By [**19**; Theorem 5, p. 427]) it is possible to introduce in c_0 an equivalent norm which is (WUR). The corresponding dual norm in l^1 is uniformly Gateaux differentiable and so it determines a dual norm in l^∞ that is (W*UR) (for these implications see Cudia [**2**; Corollary 3.14, p. 295]). On the other hand l^∞ cannot be equivalently renormed (WLUR) (see [**12**, Theorem 5.3, p. 261]).

EXAMPLE 2 (*a* (WLUR) *conjugate norm in* $l^1 \times l^2$ *that is not* (W*UR)). In [**15**] Smith gives an example (Example 6) of a conjugate norm on l^1 which is (LUR), denoting it by $|| \cdot ||_E$, and an example (Example 1) of a conjugate norm on l^2 which is (LUR) but not (W*UR), denoting it by $|| \cdot ||_L$. Then, the norm $||(x, y)||_{E \times L} = (||x||_E^2 + ||y||_L^2)^{\frac{1}{2}}$ on $l^1 \times l^2$ is an equivalent conjugate norm that is ((LUR) and hence) (WLUR) but not (W*UR).

REMARK 4. We observe that Example 1 furnishes an example of conjugate norm in l^∞ that is (W*UR), whereas l^∞ does not satisfy the Radon–Nikodym property; in the same way, Example 2 furnishes an example of a norm in $l^1 \times l^2$ that is (WLUR), whereas $(l^1 \times l^2)^* = l^\infty \times l^2$ does not satisfy the Radon–Nikodym property.

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