ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 17, Number 3, Summer 1987

REARRANGEMENT INVARIANT SUBSPACES OF LORENTZ FUNCTION SPACES II

N.L. CAROTHERS

ABSTRACT. For $1 \leq q \leq p < \infty$ and $p > 2$, it is shown that the only subspaces of the Lorentz function space $L_{p,q}[0,1]$ which are isomorphic to r.i. function spaces on $[0,1]$ are $L_2[0,1]$ and $L_{p,q}[0,1]$, up to equivalent renormings. If $p < 2$ or if $1 < p < q < \infty$, then $L_{p,q}[0,1]$ has an r.i. subspace which is not isomorphic to either $L_2[0,1]$ or $L_{p,q}[0,1]$.

1. Introduction. This note is an addendum to a previous paper by the author [**5**] in which it is shown that for $2 \leq q < p < \infty$ the only rearrangement invariant function spaces on $[0,1]$ that embed isomorphically into the Lorentz function space $L_{p,q} = L_{p,q}[0,1]$ are, up to equivalent renormings, L_2 and $L_{p,q}$. In the present note we consider the remaining values of p and q. Now the case $p = q$ (i.e., L_p) is treated in the Memoir of Johnson, Maurey, Schechtman and Tzafriri [**11**]; and since the non-separable, non-reflexive space $L_{p,\infty}$ contains a sublattice isomorphic to ℓ_∞ (hence L_∞), we will be concerned primarily with $p \neq q < \infty$.

In §2 we show that the main result of [**5**], stated above, also holds for $1 \leq q < 2 < p < \infty$. This is an unexpected extension of the results in [**11**], since $L_{p,q}$ is not 2-convex when $q < 2 < p$.

In §3 we give examples to show that in either of the cases $p < 2$ or $1 < p < q < \infty$ there are r.i. subspaces of $L_{p,q}$ that are not isomorphic to either L_2 or $L_{p,q}$. This is also surprising, as $L_{p,q}$ is 2-convex and q-concave when $2 < p < q < \infty$.

For the sake of brevity we will not repeat the arguments from [**5**] in their entirety, but rather simply indicate the necessary additions and alterations. The reader is referred to [**5**] and its references (especially [**11**] and [**13**]) for any unexplained terminology.

For $1 < p < \infty$ and $1 \leq q < \infty$ the Lorentz function space

---

Supported in part by a Faculty Research Grant from Wayne State University.
Mathematics Subject Classification. 46B99
Key words and phrases. Rearrangement invariant spaces, Lorentz function spaces.
Received by the editors on October 10, 1984, and in revised form on June 7, 1985.

$L_{p,q} = L_{p,q}[0,1]$ is the Banach space (of equivalence classes) of all measurable functions f on $[0,1]$ for which $||f|| = ||f||_{p,q} < \infty$, where

$$(1)\qquad ||f|| = \Big(\int_0^1 f^*(t)^q d(t^{q/p})\Big)^{1/q},$$

and where f^* is the decreasing rearrangement of $|f|$. It is well–known that for $1 \le q \le p < \infty$, (1) defines a norm on $L_{p,q}$ under which it is a separable r.i. space on $[0,1]$. Of course $L_{p,p} = L_p$ for any p. Notice also that $L_{p,q}$ is of the form $L_{w,q}$ treated in [**5**] exactly when $1 \le q \le p < \infty$ (see also [**13**, p. 142]). Now when $1 < p < q < \infty$ we could use the duality $L_{p,q} = (L_{p',q'})^*$, $(\frac{1}{p}) + (\frac{1}{p'}) = 1 = (\frac{1}{q}) + (\frac{1}{q'})$, to define the norm on $L_{p,q}$; but for simplicity we will instead observe that (1) defines a quasi–norm on $L_{p,q}$ which is known to be equivalent to the norm, say $|||\cdot|||$, obtained via this duality (see O'Neil [**15**] for a detailed proof). In particular, we will use the fact that for $1 < p < q < \infty$ there is a constant C, depending only on p and q, such that

$$(2)\qquad C^{-1}||f|| \le |||f||| \le C||f||.$$

for all $f \in L_{p,q}$. Throughout we will simply refer to the expression in (1) as the "norm" on $L_{p,q}$, and we will use C (or C_1, C_2, etc.) as a generic symbol representing a positive, finite constant that depends only on p and q.

Now the critical step in any of our attempts to classify the r.i. subspaces of $L_{p,q}$ will be an application of the Classification Theorem of Johnson, Maurey, Schechtman and Tzafriri [**11**, Theorem 6.1] (cf. also [**13**, Theorem 2.e.13]). In order to take full advantage of this deep theorem we will need to catalogue several properties of the $L_{p,q}$–spaces

THEOREM 1. *Let $1 < p < \infty$ and $1 \le q < \infty$. Then:*

(i) *the Haar system is an unconditional basis for $L_{p,q}$;*

(ii) *$L_{p,q}$ satisfies an upper r–estimate and a lower s–estimate for disjoint elements where $r = \min(p,q)$ and $s = \max(p,q)$;*

(iii) *if (f_n) is a disjointly supported sequence of norm–one elements in $L_{p,q}$, then there is a subsequence of (f_n) which is equivalent to the unit vector basis of ℓ_q.*

PROOF. (i). follows from [**13**, Theorem 2.c.6] and the fact that the Boyd indices for $X = L_{p,q}$ satisfy $p_X = q_X = p$ [**3, 4**]. (ii). is due to J. Creekmore [**8**]; in the case $q \leq p$ both of the constants involved may be taken to be 1. (iii). is due to Figiel, Johnson and Tzafriri [**9**, Theorem 5.1] in the case $q < p$. The case $p < q$ is very similar; because the actual details will be needed later, we include a proof. First notice that because $t^{q/p-1}$ is increasing we may re–write (1) as:

$$||f|| = \inf_{\tau} \left(\int_0^1 |f(\tau(t))|^q d(t^{q/p}) \right)^{1/q}, \tag{3}$$

where the infimum is taken over all measure–preserving automorphisms τ from $[0,1]$ onto $[0,1]$. Thus if τ is any automorphism of $[0,1]$, then we always have:

$$\int_0^1 |\sum_n a_n f_n(\tau(t))|^q d(t^{q/p})$$
$$= \sum_n |a_n|^q \int_0^1 |f_n(\tau(t))|^q d(t^{q/p}) \geq \sum_n |a_n|^q,$$

and so:

$$||\sum a_n f_n|| \geq \left(\sum_n |a_n|^q \right)^{1/q}.$$

To prove the other inequality, let $\varepsilon > o$ be given and let $|A|$ denote the Lebesgue measure of a measurable set $A \subset [0,1]$. For each n set $A_n = \sup p\ f_n$, and choose an automorphism $\tau_n : [0, |A_n|] \to A_n$ such that:

$$\int_0^{|A_n|} |f_n(\tau_n(t))|^q d(t^{q/p}) \leq (1+\varepsilon)^q.$$

Now for each n there exists $0 < \varepsilon_n < |A_n|$ such that $||f_n \chi_B|| \leq \varepsilon \cdot 2^{-n/q'}$ whenever $|B| < \varepsilon_n$. By passing to a subsequence if necessary we may suppose that $|A_{n+1}| < \varepsilon_n$ for all n. Let τ be any automorphism of $[0,1]$ such that $\tau = \tau_n$ on $[|A_{n+1}|, |A_n|]$ for every n. Then setting $E_n = \tau([0,\ |A_{n+1}|])$ and $F_n = \tau([|A_{n+1}|,\ |A_n|])$ we have (using (2)):

$$||\sum_n a_n f_n|| \leq C\Big(\sum_n |a_n| ||f_n \chi_{E_n}|| + ||\sum_n a_n f_n \chi_{F_n}|| \Big)$$
$$\leq C\left\{ \varepsilon \cdot \sum_n |a_n| \cdot 2^{-n/q'} + \left(\sum_n |a_n|^q \int_{|A_{n+1}|}^{|A_n|} |f_n(\tau_n(t))|^q d(t^{q/p}) \right)^{1/q} \right\}$$
$$\leq C(1+2\varepsilon) \cdot (\sum_n |a_n|^q)^{1/q}).$$

Let $1 < p < \infty$, $1 \le q < \infty$ and suppose that X is an r.i. function space on $[0,1]$, that $X \neq L_2$ even up to an equivalent norm, and that X is isomorphic to a subspace of $L_{p,q}$. Then by Theorem 1(i). and [**13**, Corollary 2.c.11] the Haar system is an unconditional basis for X. Further, Theorem 1(iii). implies that the Haar basis in X cannot be equivalent to a disjointly supported sequence in $L_{p,q}$. For $q \neq 2$ this is immediate, since $L_{p,q}$ cannot contain a disjointly supported sequence equivalent to the unit vector basis of ℓ_2. When $q = 2 \neq p$ we need only repeat the argument given in [**5**, Lemma 1] (slightly modified when $p < q$). That is, if the Haar basis $(h_{n,i})_{n=0}^{\infty}{}_{,i=1}^{2^n}$ in X is equivalent to a disjointly supported sequence in $L_{p,2}$, then there is an infinite subset $M \subset N$ such that

$$(4) \qquad \|\sum_{n\in M}\sum_{i=1}^{2^n} a_{n,i}h_{h,i}\|_X \overset{C}{\approx} \Big(\sum_{n\in M}\|\sum_{i=1}^{2^n} a_{n,i}h_{n,i}\|_X^2\Big)^{1/2},$$

for any scalars $(a_{n,i})$. From (4) it would then follow that $X = L_2$ up to an equivalent norm. We omit the details.

Finally, these observations and Theorem 6.12 of [**11**] yield

COROLLARY 1. *Let $1 < p, q < \infty$ and let X be an r.i. function space on $[0,1]$ that is isomorphic to a complemented subspace of $L_{p,q}$. Then eigher $X = L_2$ or $X = L_{p,q}$, up to an equivalent norm.*

2. The case $1 \le q < 2 < p < \infty$. An examination of the ingredients in the proof of Theorem 1 of [**5**] reveals that only Lemma 5 of [**5**] appears to require 2–convexity. In fact, as we shall see, the only real use of 2–convexity in [**5**] occurs in an appeal to Corollary 7.3 of [**11**]. However, at least in the case of $L_{p,q}$, $1 \le q < 2 < p < \infty$, it is possible to modify the argument given in Lemma 5 of [**5**] and to circumvent this apparent need for 2–convexity. We begin by giving a modified version of Corollary 7.3 of [**11**]. We will use d_f to denote the distribution function of $|f|$ (i.e., the right–inverse of f^*). Also recall that a sequence $(f_i)_{i=1}^n$ is called symmetrically exchangeable if for any permutation π of $\{1,\dots,n\}$ and any signs $\varepsilon_i = \pm 1, i = 1,\dots,n$, the sequence $(\varepsilon_i f_{\pi(i)})_{i=1}^n$ has the same (probability) distribution as $(f_i)_{i=1}^n$ Note is particular that in this case the f_i's all have the same distribution.

LEMMA 1. *Let $1 \le q \le p < \infty$ and $p > 2$. There is a constant C, depending only on p and q, such that if $(y_i)_{i=1}^n$ is a symmetrically* *exchangeable sequence in* $L_{p,q}$, *and if* $(y_i)_{i=1}^n$ *is a disjointly supported sequence in* $L_{p,q}(0,\infty)$ *with* $d_{y_i} = d_{\tilde{y}_i}$, $i = 1,\dots,n$, *then*

$$\|\sum_{i=1}^n \tilde{y}_i\| \le C\|\sum_{i=1}^n y_i\|. \tag{5}$$

PROOF. Recall from Theorem 1(ii). that $L_{p,q}$ satisfies a lower p-estimate (since $q \le p$). For $p > 2$ it then follows from a result of Maurey [**14**] that $L_{p,q}$ is cotype p; i.e., there is a constant C such that

$$\int_0^1 \|\sum_{i=1}^k r_i(t) f_i\| dt \ge C^{-1}(\sum_{i=1}^k \|f_i\|^p)^{1/p}, \tag{6}$$

for any $(f_i)_{i=1}^k$ in $L_{p,q}$, where $(r_i)_{i=1}^\infty$ is the sequence of Rademacher functions on $[0,1]$.

Now, since $(y_i)_{i=1}^n$ is symmetrically exchangeable, we get from (6) that

$$\|\sum_{i=1}^n y_i\| = \int_0^1 \|\sum_{i=1}^n r_i(t) y_i\| dt \ge C^{-1} n^{1/p} \|y_1\|.$$

But, as in the proof of Lemma 2 of [**5**], we also have

$$\begin{aligned}\|\sum_{i=1}^n \tilde{y}_i\|^q &= \int_0^\infty \Big(\sum_{i=1}^n d_{\tilde{y}_i}(t)\Big)^{q/p} d(t^q)\\ &= n^{q/p}\int_0^\infty (d_{y_1}(t))^{q/p} d(t^q) = n^{q/p}\|y_1\|^q.\end{aligned}$$

Thus

$$\|\sum_{i=1}^n \tilde{y}_i\| = n^{1/p}\|y_1\| \le C\|\sum_{i=1}^n y_i\|.$$

Now we may repeat the proof of [**5**, Lemma 5] in this special case. As in [**5**] we write $z_{n,i}$ for the indicator function of the interval $[(i-1)/n, i/n)$.

LEMMA 2. *Let* $1 \le q < p < \infty$ *and* $p > 2$. *There is a constant* C, *depending only on* p *and* q, *such that if* $(y_i)_{i=1}^n$ *is a symmetrically* *exchangeable sequence in* $L_{p,q}$*, then*

$$||\sum_{i=1}^{n} a_i y_i|| \leq C||\sum_{i=1}^{n} y_i|| \cdot ||\sum_{i=1}^{n} a_i z_{n,i}||,$$

for every choice of scalars $(a_i)_{i=1}^n$.

PROOF. Let $(\tilde{y}_i)_{i=1}^n$ be a disjointly supported sequence in $L_{p,q}(0,\infty)$ with $d_{\tilde{y}_i} = d_{y_i}, i = 1,\ldots,n$. Then, by Lemma 2 of [**5**] and Lemma 1 we have

$$\begin{aligned} ||\sum_{i=1}^{n} a_i \tilde{y}_i|| &\leq ||\sum_{i=1}^{n} \tilde{y}_i|| \cdot ||\sum_{i=1}^{n} a_i z_{n,i}|| \\ &\leq C_1||\sum_{i=1}^{n} y_i|| \cdot ||\sum_{i=1}^{n} a_i z_{n,i}||. \end{aligned}$$

Now, just as in [**5**, Lemma 5], we want to apply the left–hand side of the Classification Formula [**11** Theorem 2.1]; and by [**11**; Remark 1, p. 63] this half of the inequality is valid in any Banach lattice which is s–concave for some $s < \infty$. Thus there is a constant C_2 such that

$$\begin{aligned} ||\sum_{i=1}^{n} a_i y_i|| &\leq C_2 \max\{||\max_{1<i<n} |a_i y_i|||, ||\sum_{i=1}^{n} y_i|| \cdot (\sum_{i=1}^{n} |a_i|^2/n)^{1/2}\} \\ &\leq C_2 \max\{||\sum_{i=1}^{n} a_i \tilde{y}_i||, ||\sum_{i=1}^{n} y_i|| \cdot (\sum_{i=1}^{n} |a_i|^2/n)^{1/2}\} \\ &\leq C_1 C_2||\sum_{i=1}^{n} y_i|| \max\{||\sum_{i=1}^{n} a_i z_{n,i}||, ||\sum_{i=1}^{n} a_i z_{n,i}||_{L_2}\}. \end{aligned}$$

But $||f|| \geq ||f||_{L_2}$ for $f \in L_{p,q}$. Indeed, $||f|| \geq ||f||_{L_p}$ when $1 \leq q \leq p < \infty$, and so $||f|| \geq ||f||_{L_2}$ when $p > 2$ (see [**10**] or [**13**, Proposition 2.b.9]). Thus,

$$||\sum_{i=1}^{n} a_i y_i|| \leq C_1 C_2||\sum_{i=1}^{n} y_i|| \cdot ||\sum_{i=1}^{n} a_i z_{n,i}||.$$

Finally, by incorporating these observations into the proof of theorem 1 of [**5**] we have

THEOREM 2. *Let $1 \leq q \leq p < \infty$ and $p > 2$. Let X be an r.i. function space on $[0,1]$ that is isomorphic to a subspace of $L_{p,q}$. Then, up to an equivalent norm, $X = L_2$ or $X = L_{p,q}$.*

REMARK. It is known that $L_{2,q}$ is not of cotype 2 for $1 \leq q < 2$, and so our proof of Lemma 2 fails in this case. In fact, even the conclusion of Lemma 1 cannot hold in this case (this follows from an example due to Pisier [**13**, Example 1.f.19], [**8**]; but see also [**6**]). We have been unable to determine whether the conclusion of Theorem 2 holds in this remaining case.

3. The cases $\mathbf{p < 2}$ and $\mathbf{1 < p < q < \infty}$. We first remark that the conclusion of Theorem 2 cannot hold for $p < 2$, since it is known not to hold even $L_p, p < 2$. The easiest way to see this is via Proposition 8.9 of [**11**] which states that if for some $1 < r < 2$ an r.i. function space X on $[0,1]$ contains the function $g(t) = t^{-1/r}, 0 < t \leq 1$, then L_r embeds isometrically into X (cf. [**13**, Theorem 2.f.4]). Consequently, given $1 < p < r < 2$ and $1 \leq q < \infty$, $L_{p,q}$ contains an isometric copy of L_r.

Now the technique employed in proving [**11**, Proposition 8.9] supplies a general method for constructing sublattices of an r.i. function space which are themselves isometric to r.i. function spaces on $[0,1]$. Given an r.i. function space X on $[0,1]$ and a positive, decreasing, norm–one $g \in X$ we define the space X_g to be the completion of the simple, integrable functions on $[0,1]$ under the norm

$$||f||_{X_g} = ||f \otimes g||_{X([0,1]^2)}, \tag{7}$$

where $(f \otimes g)(s,t) = f(s)g(t)$. Since the square $[0,1]^2$ is measure-equivalent to the interval $[0,1]$, it is easy to see that X_g is isometric and lattice–isomorphic to a sublattice of X, and further, that (7) defines an r.i. norm on X_g. Henceforth we will identify X with $X([0,1]^2)$ and simply write $||f||_{X_g} = ||f \otimes g||_X$.

It is easy to see that if $X = L_{p,q}, 1 \leq q \leq p < \infty$, then each of the spaces X_g must be isomorphic to $L_{p,q}$. In fact, in this case we have

$$||f||\ ||g||_{L_p} \leq ||f \otimes g|| \leq ||f||\ ||g||, \tag{8}$$

for any $f, g \in L_{p,q}$. To see this, fix $f \in L_{p,q}$ and suppose that g is a step function $g = \sum_{i=1}^{n} a_i z_{n,i}$. Write $f_{n,i} = f \otimes z_{n,i}$ for each $n = 1, 2, \ldots$

and $i = 1, \dots, n$. Then $f \otimes g = \sum_{i=1}^n a_i f_{n,i}$, where the $f_{n,i}$'s are disjoint and all have the same distribution. Thus $||f_{n,1}|| = n^{-1/p}||f||$, and so, by Theorem 1(ii),

$$\begin{aligned}||f \otimes g|| &= ||\sum_{i=1}^n a_i f_{n,i}|| \geq ||f_{n,1}|| \cdot \Big(\sum_{i=1}^n |a_i|^p\Big)^{1/p} \\ &= n^{-1/p}||f|| \cdot \Big(\sum_{i=1}^n |a_i|^p\Big)^{1/p} = ||f||\ ||g||_{L_p}.\end{aligned}$$

The other inequality is given in [**5**, Lemma 2] (also [**15**, Theorem 7.4]). When $1 < p < q < \infty$, the inequalities in (8) are reversed and we reach a much different conclusion:

PROPOSITION 1. *Let $1 < p < q < \infty$. Then there exists g in $X = L_{p,q}$ such that X_g is not isomorphic to either L_2 or $L_{p,q}$.*

PROOF. We use an example given in [**15**]: let $f(t) = t^{-1/p}(1 - \log t)^{-\alpha/p}, 0 < t \leq 1$, where α is chosen to satisfy $2\alpha - 1 < p/q < \alpha < 1$. Then $f \in L_{p,q}$, but, as shown in [**15**, Theorem 7.7], $f \otimes f \notin L_{p,q}$. That is, if $g = f/||f||$, then $g \notin X_g$. Thus, by Corollary 1, X_g cannot be isomorphic to $L_{p,q}$ (for otherwise, $X_g = L_{p,q}$). Finally, X_g cannot be isomorphic to L_2. For $q \neq 2$ this follows from Theorem 1(iii). For $p < 2 = q$ we need only observe that for each n the 1–unconditional basic sequence $(z_{n,i})_{i=1}^n$ in X_g satisfies $||\sum_{i=1}^n z_{n,i}||_{X_g} = n^{1/p}||z_{n,1}||_{X_g}$, and so

$$\frac{\Big(\sum_{i=1}^n ||z_{n,i}||_{X_g}^2\Big)^{1/2}}{||\sum_{i=1}^n z_{n,i}||_{X_g}} = n^{1/2-1/p},$$

which cannot be bounded from below independent of n.

Finally, it should be pointed–out that a subspace X_g of $X = L_{p,q}$ is isomorphic to $L_{p,q}$ precisely when it is complemented in $L_{p,q}$. This follows from Corollary 1 and the following observation (suggested by a similar result due to Casazza and Lin [**7**] for spaces with symmetric basis):

PROPOSITION 2. *Let X be a separable r.i. function space on $[0,1]$ which has unique r.i. structure on $[0,1]$, and which is q-concave for some $q < \infty$. If X_g is isomorphic to X, then X_g is complemented in X.*

PROOF. The assumption of unique r.i. structure implies that $X_g = X$, up to an equivalent norm; in particular, there is a constant $M < \infty$ such that $||f||_{X_g} < M||f||_X$ for all $f \in X$.

Now, in order to fix X_g, let $\sigma : [0,1] \to [0,1]^2$ be a measure equivalence. For each $n = 1,2,\dots$ and $i = 1,\dots,n$, let $g_{n,i} = (z_{n,i} \otimes g) \circ \sigma$, let $A_{n,i} = \sup p\, g_{n,i}$, and let $x_{n,i}$ be the indicator function of $A_{n,i}$. Then X_g is isometric to $[g_{n,i}]_{n=1,\, i=1}^{\infty\;\;\; n}$ in X, and for any n and any scalars $(a_i)_{i=1}^n$ we have

$$||\sum_{i=1}^n a_i g_{n,i}||_X = ||\sum_{i=1}^n a_i z_{n,i}||_{X_g} \leq M||\sum_{i=1}^n a_i z_{n,i}||_X$$
$$= M||\sum_{i=1}^n a_i x_{n,i}||_X.$$

Next, we show that for each n, $[g_{n,i}]_{i=1}^n$ is complemented by a projection of norm at most $M/||g||_{L_1}$. To see this, define $P_n : X \to X$ by

$$P_n f = ||g||_{L_1}^{-1} \cdot \sum_{i=1}^n \left(n \int f x_{n,i}\right) g_{n,i}.$$

Then P_n is a projection onto $[g_{n,i}]_{i=1}^n$ since $||g||_{L_1} = n||g_{n,i}||_{L_1}$ for any $i = 1,\dots,n$ (recall that g is positive), and for $f \in X$ we have

$$||P_n f||_X = ||g||_{L_1}^{-1} \cdot ||\sum_{i=1}^n \left(n \int f x_{n,i}\right) g_{n,i}||_X$$
$$\leq M||g||_{L_1}^{-1} \cdot ||\sum_{i=1}^n \left(n \int f x_{n,i}\right) x_{n,i}||_X \leq M||g||_{L_1}^{-1} \cdot ||f||_X,$$

since conditional expectation is a contradiction on X.

Finally, since X is q–concave, X is a projection band in X^{**} and a standard argument finishes the proof. Let $J : X \to X^{**}$ be the canonical inclusion, and let $Q : X^{**} \to X$ be the canonical projection. Then, if R is a limit point for (P_n^{**}) in the w^*–operator topology, $P = QRJ$ is a projection onto X_g of norm at most $M||g||_{L_1}^{-1}$.

REFERENCES

1. Z. Altshuler, *The modulus of convexity of Lorentz and Orlicz sequence spaces,* Notes in Banach Spaces (H.E. Lacey, ed.) Univ. of Texas Press 1980.

2. ———, P.G. Casazza, B.L. Lin, *On symmetric basic sequences in Lorentz sequence spaces,* Israel J. Math. **15** (1973), 140-155.

3. D.W. Boyd, *The Hilbert transform on rearrangement–invariant spaces,* Can. J. Math. **19** (1967), 599-616.

4. ———, *Indices of function spaces and their relationship to interpolation,* Can. J. Math. **21** (1969), 1245-1254.

5. N.L. Carothers, *Rearrangement invariant subspaces of Lorentz function spaces,* Israel J. Math. **40** (1981), 217-228.

6. N.L. Carothers, P.H. Flinn, *Embedding* $\ell_p^{n^\alpha}$ *in* $\ell_{p,q}^n$, Proc. Amer. Math. Soc. **88** (1983), 523-526.

7. P.G. Casazza, B.L. Lin, *On symmetric basic sequences in Lorentz sequence spaces II,* Israel J. Math. **17** (1974), 191-218.

8. J. Creekmore, *Type and cotype in Lorentz* $L_{p,q}$ *spaces,* Indag. Math. **43** (1981), 145-152.

9. T. Figiel, W.B. Johnson, L. Tzafriri, *On Banach lattices and spaces having local unconditional structure with applications to Lorentz function spaces,* J. Approx. Theory **13** (1975), 297-312.

10. R.A. Hunt, *On* $L(p,q)$ *spaces,* L'Enseignement Math. **12** (1966), 249-274.

11. W.B. Johnson, B. Mauery, G. Sehechtman, L. Tzafriri, *Symmetric structures in Banach spaces,* Memoirs. Amer. Math. Soc. **217** (1979).

12. J. Lindenstrauss, L. Tzafriri, *On Orlicz sequence spaces II,* Israel J. Math. **11** (1970), 355-379.

13. ———, ———, *Classical Banach Spaces II. Function Spaces,* Ergebnisse Math. Grenzgebiete, Bd. **97**, Springer–Verlag, Berlin, 1979.

14. B. Maurey, *Type et cotype dans les espaces munis de structures locales inconditionelles,* Seminaire Maurey–Schwartz, 1973-1974, Exposes 24-25, Ecole Polytechnique, Paris.

15. R. O'Neil, *Integral transforms and tensor products on Orlicz spaces and* $L(p,q)$ *spaces,* J. D'Analyse Math. **21** (1968), 1-176.

WAYNE STATE UNIVERSITY, DETROIT, MI 48202
BOWLING GREEN STATE UNIVERSITY, BOWLING GREEN, OH 43403.