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ON THE SPACE ℓ/c_o

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ABSTRACT. In this paper we correct a mistake contained in [**3**] and we improve and give simpler proofs of some of the results contained there. We also give a very simple proof of the fact (included in Theorem 5.6 of [**5**]) that the dual of every complemented subspace of ℓ^∞/c_o is isomorphic to $(\ell^\infty)'$.

Introduction and notation. If T is a completely regular topological space, βT is its Stone-Cech compactification; if S is a locally compact topological space, αS is its one-point compactification. We recall some facts about ℓ^∞ and ℓ^∞/c_o.

ℓ^∞ is isometric to $C(\beta N)$ and ℓ^∞/c_o is isometric to $C(\beta N\backslash N)$ (cf. [**3**]).

ℓ^∞ is a $\mathcal{P}_1$-space, that is, it is complemented in every Banach space which contains it with a norm-one projection; $(\ell^\infty)' = \ell^1 \oplus c_o^\perp$ (cf. [**2**]).

We use $=$ for "isomorphic to" and $\equiv$ for "isometric to".

If $E_n, n \in N$, are Banach spaces, then

$$(\oplus_n E_n)_p = \{(x_n) | x_n \in E_n \text{ and } \\ ||(x_n)||_p = (\sum_n ||x_n||^p)^{1/p} < \infty\}, \quad 1 \le p \le \infty,$$

$$(\oplus_n E_n)_\infty = \{(x_n)|x_n \in E_n \text{ and } ||(x_n)||_\infty = \sup_n ||x_n|| < \infty\}$$

and $(\oplus_n E_n)_{c_o}$ is the closed subspace of $(\oplus_n E_n)_\infty$ formed by the sequences (x_n) such that $\lim_n ||x_n|| = 0$.

It is easy to show that $(\oplus_n E_n)'_p = (\oplus_n E'_n)_{p'}$ if $1 \le p < \infty$ and $\frac{1}{p} + \frac{1}{p'} = 1$ and $(\oplus_n E_n)'_{c_o} = (\oplus_n E'_n)_1$, but it is false that $(\oplus_n E_n)'_\infty = (\oplus_n E'_n)_1$ in general (for example, consider the case when the E'_ns are Banach spaces with separable dual).

If Γ is a set of indices let $c_o(\Gamma) = \{(x_\alpha)_{\alpha\in\Gamma} | x_\alpha \in C$ and for any $\varepsilon > 0 |x_\alpha| > \varepsilon$ only for a finite number of indices $\}$.

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If $\Gamma = N, c_o(\Gamma)$ is, of course, c_o. $F < E$ means that F is isomorphic to a complemented subspace of E.

Finally if E is a Banach space, $\chi(E)$ is its *density character,* that is, the smallest cardinality of a dense subset of E.

RESULTS. Theorem (5.4) in [**3**] purports to prove that $\ell^\infty/c_o = \ell^\infty \oplus (\ell^\infty/c_o)$ and for this the authors use the following result, which they attribute to Goodner:

(*) If ℓ^∞ is isometric to a subspace M of $C(T)$, T compact Hausdorff, then any complement of M in $C(T)$ is isomorphic to $C(T)$.

First of all, Goodner's result is quite different and is the following (cf. [**1**]):

"If ℓ^∞ is isometric to a subspace of $C(T)$, T compact Hausdorff, then there exist subspaces $M, N \subset C(T)$ such that $C(T) = M \oplus N, M$ is isometric to ℓ^∞ and N is isomorphic to $C(T)$.

This is correct and also holds with *isometric* replaced by *isomorphic* throughout. For an immediate proof it suffices to apply Lemma 1 below, with $E = C(T)$ and $F = \ell^\infty$.

In the second place, (*) is false. Indeed, consider the space $\ell^\infty \oplus c = \ell^\infty \oplus c_o$ and note that $(\ell^\infty \oplus c)_\infty \equiv C(K)$ where $K = (\beta N) \cup (\alpha N)$ (disjoint topological union). It is clear that every complement of ℓ^∞ in $\ell^\infty \oplus c$ is isomorphic to c which, of course, is not isomorphic to $\ell^\infty \oplus c$.

Theorem (5.4) of [**3**] is true even if (*) is false and it is a simple corollary to the following

LEMMA 1. *Let E be a Banach space and let $F < E$ be such that $F^2 = F$. Then $E = E \oplus F$.*

PROOF. In fact, there exists a subspace $F_1 \subset E$ such that $E = F \oplus F_1$ and hence $E = F \oplus F_1 = F \oplus F \oplus F_1 = E \oplus F$.

The isomorphism $\ell^\infty/c_o = \ell^\infty \oplus (\ell^\infty/c_o)$ is now a simple consequence of the fact that $\ell^\infty < \ell^\infty/c_o$ (cf. [**3**]).

In theorem (5.2) of [**3**] the authors prove that ℓ^∞/c_o is isometric to its square by using some topological properties of $\beta N \backslash N$. This is quite unnecessary. In fact, it suffices to observe that, if $T : \ell^\infty \oplus \ell^\infty \to \ell^\infty$ is given by $T((\xi_n), (\eta_n)) = (\xi_1, \eta_1, \xi_2, \eta_2, \ldots)$, then T is an isometry of

$(\ell^\infty \oplus \ell^\infty)_\infty$ onto ℓ^∞ and $T|_{c_o \oplus c_o}$ is an isometry of $c_o \oplus c_o$ onto c_o, so that $\ell^\infty/c_o \equiv (\ell^\infty \oplus \ell^\infty)_\infty/(c_o \oplus c_o) \equiv (\ell^\infty/c_o \oplus \ell^\infty/c_o)_\infty$.

We now improve the result in §3 of [**3**].

THEOREM 1. *ℓ^∞/c_o is not complemented in any dual space.*

PROOF. Let $(A_i)_{i\in I}$ be an uncountable family of pairwise disjoint, open-closed subsets of $\beta N \backslash N$ (cf. [**6**]).

For each $i \in I$ define $f_i \in C(\beta N\backslash N)$ by $f_i(x) = 1$(if $x \in A_i$) and $f_i(x) = 0$ if $x \notin A_i$. It is clear that span $\{f_i\} \equiv c_o(I)$ and hence $c_o(I)$ is a subspace of ℓ^∞/c_o. If ℓ^∞/c_o were complemented in a dual space then, by a theorem of Rosenthal (cf. [**5**]), $\ell^\infty(I) \subset \ell^\infty/c_o$. But this is impossible because $\chi(\ell^\infty/c_o) = c$ and $\chi(\ell^\infty(I)) = 2^{|I|} > c$.

We consider now complemented subspaces of ℓ^∞/c_o.

LEMMA 2. *Let E a Banach space such that $E = (E \oplus E \dots)_\infty$. Then $E' = (E' \oplus E' \dots)_1$.*

PROOF. Let $G = (E' \oplus E' \dots)_1$. Clearly G is isomorphic to its square and $E' < G$. If we prove that $G < E'$ we conclude, by Lemma 1, that $E' = G$ (note that E' is isomorphic to its square too).

Let $F = (E \oplus E \dots)_{c_o}$; then $F' \equiv G$ and F is a closed subspace of $(E \oplus E \dots)_\infty$. Let $\alpha : F \to (E \oplus E \dots)_\infty$ be the isometric inclusion and let $T : G \to (E \oplus E \dots)'_\infty$ be the inclusion map, so that, if $x = (x_n) \in G$ and $z = (z_n) \in (E \oplus E \dots)_\infty$, we have $\langle Tx, z\rangle = \sum_n \langle x_n, z_n\rangle$. Then $|\langle Tx, z\rangle| \le \sum_n ||x_n||\ ||z_n|| \le\ ||x||\ ||z||$, that is, $||T|| \le 1$.

Moreover, we have that $Tx|_F = x$ for every $x \in G$, i.e.,

$$\langle Tx, y\rangle = \langle x, y\rangle \quad V y \in F.$$

Consider the diagram $G \xrightarrow{T} (E \oplus E \dots)'_\infty \xrightarrow{\alpha'} G$ and let $x \in G$ and $y \in F$.

Then $\langle \alpha' Tx, y\rangle = \langle Tx, y\rangle = \langle x, y\rangle$.

This means that $\alpha' T = I_G$; i.e., I_G factors through $(E \oplus E \dots)'_\infty$ and hence $G < E'$.

COROLLARY. $(\ell^\infty)' = ((\ell^\infty)' \oplus (\ell^\infty)' \dots)_1$.

THEOREM 2. *Let M be a complemented subspace of ℓ^∞/c_o. Then $M' = (\ell^\infty)'$.*

PROOF. By Theorem (5.1) in [**3**], $\ell^\infty < M$. Hence $(\ell^\infty)' < M'$ and $M' < (\ell^\infty/c_o)'$. But $(\ell^\infty/c_o)' = c_o^\perp < (\ell^\infty)'$ [**2**] and therefore we have $(\ell^\infty)' < M' < (\ell^\infty)'$. An application of Pelczynski's decomposition method (cf. [**4**]) together with the Corollary to Lemma 2 concludes the proof.

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