ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 17, Number 3, Summer 1987

PERIODIC SOLUTIONS OF DIFFERENTIAL-DELAY EQUATIONS WITH MORE THAN ONE DELAY

STEVEN CHAPIN

Introduction. In this paper we prove the existence of nontrivial periodic solutions of certain differential-delay equations with more than one delay. The method of proof involves techniques which have been used to study differential-delay equations with a single delay, and part of our motivation is to show how these techniques can be generalized. Our results also imply a nonuniqueness result for periodic solutions of some differential-delay equations with more than one delay which have been studied by R.D. Nussbaum.

In [**5**] Nussbaum studies the differential-delay equation

$$x'(t) = -\alpha f(x(t-1)), \tag{0.1}$$

where α is a positive parameter and f is an odd function ($f(-x) = -f(x), \forall x$) which decays like x^{-r} at infinity and satisfies $xf(x) \geq 0$ for all x. Nussbaum's original motivation for studying (0.1) was the case $f(x) = x(1+|x|^{r+1})^{-1}$ for which (0.1) has been suggested as a model for a somewhat more complicated equation which was introduced in a study of physiological control systems [**2,3**]. By now there is a good deal of evidence to suggest that for such f the dynamics of (0.1) are quite complex [**7,8**]. Nussbaum proved (with some additional hypotheses on f, which, nonetheless, included the case $f(x) = x(1+|x|^{r+1})^{-1}$) that for α large enough (0.1) has a periodic solution the minimal period of which tends to infinity as α tends to infinity. These periodic solutions also have special symmetry properties. The proof involves a careful asymptotic analysis of some of the solutions of (0.1), and while the analysis depends on certain special features of the function f, it appears that the techniques involved can be applied to a much larger class of functions. In fact, this author has been able to use these general methods to study (0.1) for the case in which f decays exponentially at

Received by the editors on June 15, 1985, and in revised form on September 25, 1985.
Key words and phrases: Differential-delay equations, slowly oscillating periodic solutions, asymptotic analysis.

infinity, e.g., $f(x) = x\exp(-bx^2), b > 0$ [**1**].

In this paper we shall apply the same general techniques used in [**5**] to study the differential-delay equation

$$x'(t) = -\alpha \sum_{i=0}^{N} \lambda_i f(x(t - p_i)), \tag{0.2}$$

where α is a positive parameter and $\lambda_0, \ldots, \lambda_N$, $p_0, \ldots, p_N$ are all positive constants. We consider the same class of functions as in [**5**], and prove an analogous result. That is, for α large enough (0.2) has a periodic solution the minimal period of which tends to infinity as α tends to infinity. As in the case of (0.1) these solutions have special symmetry properties. These results directly imply a nonuniqueness result for so-called "slowly oscillating" periodic solutions of the equation

$$x'(t) = -\alpha \sum_{i=1}^{N} f(x(t - i)) \tag{0.3}$$

which is studied by Nussbaum in [**6**].

1. Consider the differential-delay equation

$$x'(t) = -\alpha \sum_{i=0}^{N} \lambda_i f(x(t - p_i)), \tag{1.1}$$

where α is a positive parameter, and $\lambda_0, \ldots, \lambda_N$, $p_0, \ldots, p_N$ are all positive constants. Assume that we have arranged the terms in (1.1) so that $p_0 < p_1 < \cdots < p_N$.

A C^1 function $x(t)$ will be called a *slowly oscillating periodic solution* of (1.1) if $x(t)$ solves (1.1), and there are numbers $q_1 > p_N$ and $q_2 > q_1 + p_N$ such that $x(t) > 0$ for $0 < t < q_1, x(t) < 0$ for $q_1 < t < q_2$, and $x(t + q_2) = x(t)$ for all t. (The word "slowly" refers to the fact that the separation between the zeros of $x(t)$ is greater than the largest delay, that is, p_N.)

We want to consider the case where f is odd and decays like x^{-r} at infinity. Therefore, we will always assume that f satisfies the following (see [**5**]):

H1. $f : \mathbf{R} \to \mathbf{R}$ is an odd, continuous map. There exists a number $x_* > 0$ such that $f|[0, x_*]$ is nondecreasing and $f|[x_*, \infty)$ is nonincreasing. There exists positive constants a, d, r and σ and a constant x_0 such that

$$(1.2) \qquad (a - dx^{-\sigma})x^{-r} \leq f(x) \leq (a + dx^{-\sigma})x^{-r}, \quad x \geq x_0.$$

Our goal is to prove the following.

THEOREM 1.1. *Suppose that f satisfies* H1.) *and that $r > 2$ and $\sigma > r/(r-1)$(r and σ as in* H1.) *Then there exists an $\alpha_0 > 0$ such that*

(i) *Equation* (1.1) *has a slowly oscillating periodic solution $x_\alpha(t)$ for all $\alpha \geq \alpha_0$.*

(ii) *$x_\alpha(t) > 0$ on an interval $(0, q_\alpha)$,*

(iii) *$x_\alpha(t + q_\alpha) = -x_\alpha(t)$ for all t, and*

(iv) $\lim_{\alpha \to \infty} q_\alpha = \infty$.

Furthermore, there exists a constant $\beta = \beta(r)$ such that one has

$$q_\alpha \geq \beta \alpha^{r-2}, \alpha \geq \alpha_0.$$

REMARK 1.1. It is easy to check that if $\gamma > 0$ and $r > 2$, then the function

$$f(x) = (\operatorname{sgn}(x))|x|^\gamma (1 = |x|^{\gamma + r})^{-1}$$

satisfies H1. and gives a class of examples for Theorem 1.1.

Before beginning the proof of Theorem 1.1, let us establish some notation which will remain constant throughout. First, note that by changing the timescale and multiplying α by a positive constant we can always assume that $\lambda_0 = p_0 = 1$. Therefore, we will always assume that (1.1) is of the form

$$(1.3) \qquad x'(t) = -\alpha[f(x(t-1)) + \sum_{i=1}^{N} \lambda_i f(x(t - p_i))],$$

where $1 < p_1 < p_2 < \cdots < p_N$.

Also, since we can replace $f(x)$ by $a^{-1} f(x)$, we will always assume that $a = 1$ in H1.

From now on, unless stated otherwise, we will assume that f satisfies

H1.; the letters r, x_*, d, σ and x_0 will always be used as in H1. Let a_1 and a_2 denote fixed positive constants which satisfy

$$a_1 x^{-r} \leq f(x) \leq a_2 x^{-r}, x \geq x_*,$$

and define functions $b_1(x)$ and $b_2(x)$ by

$$b_j(x) = 1 + (-1)^j d x^{-\sigma}, j = 1, 2,$$

so that one has

$$b_1(x) x^{-r} \leq f(x) \leq b_2(x) x^{-r}, x \geq x_0.$$

Let $C[0, p_N]$ denote the Banach space of continuous real-valued functions on $[0, p_N]$ with the usual sup-norm, and recall that the "initial-value" problem

$$\begin{aligned} x'(t) &= -\alpha \sum_{i=0}^{N} \lambda_i f(x(t - p_i)), t \geq p_N \\ x|[0, p_N] &= \phi \end{aligned}$$

has a unique solution, which will be denoted by $x(t) = x(t; \phi, \alpha)$. Finally, if α and k are positive constants define the closed, bounded, convex subset $K_{\alpha,k} \subseteq C[0, p_N]$ by

$$\begin{aligned} K_{\alpha,k} = \{\phi \in C[0, p_N] | \phi(0) \leq p_N (N+1) \lambda_{\max} \alpha f(x_*) \\ \phi \text{ is nonincreasing}, \phi(p_N) = k \alpha^{\varepsilon}, \varepsilon = (r+1)^{-1}\}, \end{aligned}$$

where $\lambda_{\max} = \max\{1, \lambda_1 \lambda_2, \ldots, \lambda_N\}$.

In addition, we will always take $\varepsilon = (r+1)^{-1}$.

The main idea in the proof of Theorem 1.1 is to obtain various estimates on $x(t; \phi, \alpha)$ for $\phi \in K_{\alpha,k}$ and α large. As a first step we prove two lemmas which are variants of Lemmas 1.1 and 1.2 in [**5**].

LEMMA 1.1. *Suppose that f satisfies* H1. *and $\phi \in K_{\alpha,k}$. Assume that*

$$p_N (N+1) \lambda_{\max} \alpha f(x_*) > k \alpha^{\varepsilon},$$

so $K_{\alpha,k}$ is not the empty set. Define

$$z_1 = z_1(\phi, \alpha) = \inf\{t \geq 1 | x(t) = 0\}.$$

If $k \geq 2(a_2\lambda_{\max}(N+1))^{\varepsilon}$ *and* $(1/2)k\alpha^{\varepsilon} > x_*$, *then* $z_1 \geq p_N + 2$.
If we have $\alpha > (x_*)^2(\int_{x_*}^{2x_*} f(s)ds^{-1})$, *then* $x(z_1 - 1) \geq x_*$.

PROOF. It follows from the form of f that

$$\begin{aligned} x'(t) &\geq -\alpha[f(\phi(p_N)) + \sum_{i=1}^{N} \lambda_i f(\phi(p_N - p_1 + 1))] \\ &\geq -\alpha[\lambda_{\max}(N+1)f(\phi(p_N))] \\ &\geq -\alpha\lambda_{\max}(N+1)a_2(k\alpha^{\varepsilon})^{-r}, p_N \leq t \leq p_N + 1. \end{aligned}$$

Thus

$$\begin{aligned} x(p_N + 1) &\geq k\alpha^{\varepsilon} - a_2\lambda_{\max}(N+1)k^{-r}\alpha^{\varepsilon} \\ &\geq (1 - (1/2)^{r+1})k\alpha^{\varepsilon} \geq (1/2)k\alpha^{\varepsilon}. \end{aligned}$$

Since, we are assuming that $(1/2)k\alpha^{\varepsilon} > x_*$ we can repeat the argument on the interval $[p_N + 1, p_N + 2]$ to obtain

$$x(p_N + 2) \geq k_1\alpha^{\varepsilon} - a_2\lambda_{\max}(N+1)k_1^{-r}\alpha^{\varepsilon},$$

where $k_1 = k - a_2\lambda_{\max}(N+1)k^{-r}$. If $k \geq 2(a_2\lambda_{\max}(N+1))^{\varepsilon}$, then one can easily check that $k_1 \geq (a_2\lambda_{\max}(N+1))^{\varepsilon}$; and from this one deduces that $x(p_N + 2) \geq 0$. This proves that $z_1 \geq p_N + 2$.

To prove that $x(z_1 - 1) \geq x_*$, suppose not, and let t_1 be the first time $t > 0$ such that $x(t) = x_*$ (so, by assumption, $z_1 - t_1 \geq 1$). $x(t)$ is decreasing and concave down on $[p_N, t_1 + 1]$. In particular, this implies that $|x'(t_1)| \leq x(t_1) - x(t_1 + 1) \leq x_*$. Using the concavity one sees that

$$x(t_1 - 1 + s) \leq x_* + (1 - s)x_*, \quad 0 \leq s \leq 1.$$

It follows that

$$\begin{aligned} x(t_1 + 1) &= x_* - \alpha\int_0^1 f(x(t_1 - 1 + s))ds \\ &\quad -\alpha\sum_{i=1}^{N} \lambda_i \int_0^1 f(x(t_1 - p_i + s))ds \\ &\leq x_* - \alpha\int_0^1 f(x(t_1 - 1 + s))ds \qquad (1.5) \\ &\leq x_* - \alpha\int_0^1 f(x_* - (1 - s)x_*)ds \\ &= x_* - \alpha(x_*)^{-1}\int_{x_*}^{2x_*} f(u)du. \end{aligned}$$

The assumption that $\alpha > (x_*)^2[\int_{x_*}^{2x_*} f(u)du]^{-1}$ implies that the right-hand side of (1.5) is negative while we are assuming that $x(t_1+1) \geq 0$. Thus, we obtain a contradiction which implies that $x(z_1 - 1) \geq x_*$.

For notational convenience set $K_\alpha = k_{\alpha,k}$, where $k = 2(a_2\lambda_{\max}(N+1))^\varepsilon$. Also, given $\phi \in K_\alpha$ define z_1, m, δ, and δ_1 by

$$\begin{aligned} z_1 &= z_1(\phi,\alpha) = \inf\{t \geq 1 | x(t) = 0\}, \\ m &= m(\phi,\alpha) = x(z_1 - 1), \\ \delta &= \delta(\phi,\alpha) = \inf\{t > 0 | x(z_1 - t) = x_*\}, \\ \delta_1 &= \delta_1(\phi,\alpha) = \inf\{t > 0 | x(z_1 + t) = -x_*\}. \end{aligned}$$

LEMMA 1.2. *Suppose that f satisfies* H1., $r > 1$, *and* $\phi \in K_\alpha$. *Assume that α is large enough so that the conclusions of Lemma 1.1 hold. Then there exists positive constants c_1 and c_2, given by*

$$\begin{aligned} c_1 &= a_1^\varepsilon\{(r-1)^{-1}(1-2^{1-r})^\varepsilon\}, \\ c_2 &= (a_2\lambda_{\max}(N+1))^\varepsilon, \end{aligned}$$

such that

$$c_1\alpha^\varepsilon \leq m \leq c_2\alpha^\varepsilon. \tag{1.6}$$

Also, if α is sufficiently large, then there exists a constant c_3 independent of α and $\phi \in K_\alpha$ such that one has

$$c_3 m^{-1} \leq \delta_1 < \delta \leq x_* m^{-1}, \tag{1.7}$$

and

$$\begin{aligned} \alpha\delta f(m + m\delta) &\leq x_* \leq \alpha\delta\lambda_{\max}(N+1)f(m), \\ \alpha\delta_1 f(m) &\leq x_* \leq \alpha\delta_1\lambda_{\max}(N+1)f(m - m\delta_1). \end{aligned} \tag{1.8}$$

PROOF. We know that $x(z_1 - 1) \geq x_*$, so $x(t)$ is concave down on the interval $[z_1 - 2, z_1]$ and $m \geq x'(z_1 - 1)$. It follows that

$$\begin{aligned} m &= \alpha\int_0^1 f(x(z_1 - 2 + t))dt + \alpha\sum_{i=1}^N \lambda_i \int_0^1 f(x(z_1 - p_i - 1 + t))dt \\ &\geq \alpha\int_0^1 f(x(z_1 - 2 + t))dt \geq \alpha\int_0^1 f(m + (1-t)m)dt \\ &= \frac{\alpha}{m}\int_m^{2m} f(u)du. \end{aligned}$$

This implies that $m \geq c_1\alpha^{\varepsilon}$, where c_1 is given in the statement of the lemma.

One also has

$$\begin{aligned} m &= \alpha \int_{z_2-2}^{z_1-1} f(x(t))dt + \alpha\lambda_1 \int_{z_1-p_1-1}^{z_1-p_1} f(x(t))dt \\ &+ \cdots + \alpha\lambda_N \int_{z_1-p_N-1}^{z_1-p_N} f(x(t))dt \\ &\leq \alpha\lambda_{\max}(N+1)f(m) \leq \alpha\lambda_{\max}(N+1)a_2 m^{-r}, \end{aligned}$$

so that $m \leq (\lambda_{\max}(N+1)a_2)^{\varepsilon}\alpha^{\varepsilon}$.

This proves (1.6).

Lemma 1.1 implies that $\delta \leq 1$. $x(t)$ is concave down on the interval $[p_N, z_1]$, so one has

$$\begin{aligned} x'(t) &\leq -\alpha f(x(z_1-1-\delta)) - \alpha\sum_{i=1}^{N}\lambda_i f(x(z_1-p_i-\delta)) \\ &\leq -\alpha f(x(z_1-1-\delta)), z_1-\delta \leq t \leq z_1. \end{aligned} \tag{1.9}$$

Similarly, one has

$$\begin{aligned} x'(t) &\geq -\alpha f(m) - \alpha\sum_{i=1}^{N}\lambda_i f(x(z_1-p_i)) \\ &\geq -\alpha\lambda_{\max}(N+1)f(m), z_1-\delta \leq t \leq z_1. \end{aligned} \tag{1.10}$$

(1.9) and (1.10) imply that

(1.11)

$$\begin{aligned} -\alpha f(m+\delta m) &\geq -\alpha f(x(z_1-1-\delta)) \\ &\geq x'(t) \geq -\alpha\lambda_{\max}(N+1)f(m), z_1-\delta \leq t \leq z_1. \end{aligned}$$

The inequality (1.11) together with the mean-value theorem gives the first part of (1.8).

It remains to prove (1.7) and the second part of (1.8). The argument used in Lemma 1.1 shows that if

$$\alpha > 2x_*^2\left(\int_{2x_*}^{3x_*} f(u)du\right)^{-1},$$

then $\delta < 1/2$. Since we know that $x(t)$ is concave down on $[z_1 - 1, z_1 - \delta + 1]$, this implies that

$$\delta_1 < \delta \leq x_* m^{-1}. \tag{1.12}$$

Now the same type of argument as above gives

$$-\alpha\lambda_{\max}(N+1)f(m-m\delta_1) \leq x'(t) \leq -\alpha f(m), z_1 \leq t \leq z_1+\delta_1, \tag{1.13}$$

provided that $(1-\delta_1)m \geq x_*$.

Since $\delta_1 < 1/2$, we can use (1.6) to choose α large enough so that $(1-\delta_1)m \geq x_*$, and, hence, (1.13) holds. The inequality (1.13) and the mean-value theorem imply the second part of (1.8). If we also choose α large enough so that $m/2 \geq x_*$ (a computation shows that it suffices to choose $\alpha > (2x_*/c_1)^{r+1}$), then one obtains

$$\begin{aligned} x_* &\leq \alpha f(m - m\delta_1)\delta_1\lambda_{\max}(N+1) \\ &\leq \alpha f(\frac{m}{2})\delta_1\lambda_{\max}(N+1). \end{aligned} \tag{1.14}$$

Using (1.14) and (1.6) a computation gives

$$\delta_1 \geq \{x_*(\frac{1}{2})^r c_1^{r+1}(\lambda_{\max})^{-1}(N+1)^{-1}a_2^{-1}\}m^{-1}, \tag{1.15}$$

which completes the proof.

For notational convenience, given $\phi \in K_\alpha$, define functions ψ_0, ψ_1, ψ_1^*, and ψ_2^* as follows:

$$\begin{aligned} \psi_0(t) &= x(z_1 - 1 + t), \quad 0 \leq t \leq 1, \\ \psi_1(t) &= \alpha\int_0^t f(\psi_0(s))ds, \quad 0 \leq t \leq 1, \\ \psi_1^*(t) &= -x(z_1 + t), \quad 0 \leq t \leq 1, \\ \psi_2^*(t) &= \alpha\int_0^t f(\psi_1^*(s))ds, \quad 0 \leq t \leq 1. \end{aligned}$$

It turns out that the crux of the proof of Theorem 1.1 lies in estimating $x(z_1 + 2p_N)$ in terms of powers of m for $\phi \in K_\alpha$ and α

large. To do this we need to first estimate $\psi_1(1)$ and $\psi_2^*(1)$; this is accomplished by Lemmas 1.3 and 1.4.

LEMMA 1.3. *Suppose that f satisfies* H1., $r > 1$, *and $\phi \in K_\alpha$. If α is large enough, then there exists a positive constant d_1, independent of α and $\phi \in K_\alpha$, such that*

$$\psi_1(1) \geq \left(\frac{\alpha\delta}{x*}\right) \int_0^{x_*} f(\xi)d\xi + (f(m))^{-1} \int_{x_*}^{\infty} f(\xi)d\xi - d_1 m.$$

PROOF. This is essentially Lemma 1.3 of [**5**]. We give the proof only for the sake of completeness.

Because ψ_0 is concave down on the interval $[1-\delta, 1]$ and $f|[0, x_*]$ is nondecreasing we have

$$\begin{aligned} \alpha \int_{1-\delta}^{1} f(\psi_0(s))ds &\geq \alpha \int_0^{\delta} f(s\delta^{-1}x_*)ds \\ &= \left(\frac{\alpha\delta}{x_*}\right) \int_0^{x_*} f(u)du. \end{aligned} \tag{1.16}$$

It also follows by concavity that for $0 \leq s \leq 1-\delta$ one has

$$\begin{aligned} \psi_0(s) &\leq x_* + (1-\delta-s)|\psi_0'(0) \\ &\leq x_* + (1-\delta-s)\alpha f(m) \equiv \theta_0(s), \end{aligned}$$

and using this estimate gives for $\delta_1 \leq t \leq 1-\delta$

$$\begin{aligned} \psi_1(t) &\geq x_* + \alpha \int_{\delta_1}^{t} f(\theta_0(s))ds \\ &\geq x_* + \frac{1}{f(m)} \int_{(1-\delta-t)\alpha f(m)+x_*}^{(1-\delta-\delta_1)\alpha f(m)+x_*} f(\xi)d\xi. \end{aligned} \tag{1.17}$$

Using the fact that $\delta + \delta_1 \leq 2x_*/m$ and (1.6), we can find a positive constant k (independent of α and $\phi \in K_\alpha$) such that

$$(1-\delta-\delta_1)\alpha f(m) + x_* \geq \left(1 - \frac{2x_*}{m}\right)\alpha f(m) + x_* \geq km + x_*. \tag{1.18}$$

If we put $t = 1-\delta$ in (1.17), use (1.18) and the assumption $r > 1$, we obtain

$$\begin{aligned} \psi_1(1-\delta) &\geq x_* + \frac{1}{f(m)} \int_{x_*}^{\infty} f(\xi)d\xi - \frac{1}{f(m)} \int_{km}^{\infty} f(\xi)d\xi \\ &\geq x_* + \frac{1}{f(m)} \int_{x_*}^{\infty} f(\xi)d\xi - m\left(\frac{k^{1-r}}{r-1}\right)\frac{b_2(km)}{b_1(m)}, \end{aligned} \tag{1.19}$$

where b_1 and b_2 are given by (1.4).

The lemma now follows easily from (1.16) and (1.19).

LEMMA 1.4. *Suppose that f satisfies* H1., $r > 1, \sigma > r(r-1)^{-1}$, *and $\phi \in K_\alpha$. If α is large enough, then there exists a positive constant d_2, independent of α and $\phi \in K_\alpha$, such that*

$$\psi_2^*(1) \le (\frac{\alpha\delta}{x_*}) \int_0^{x_*} f(\xi)d\xi + (f(m))^{-1} \int_{x_*}^{\infty} f(\xi)d\xi - d_2 m^{r-1}$$

PROOF. It is not difficult to see that $\psi_1^*(t)$ is concave up on the interval $[0, \delta_1]$, so we have

$$\begin{aligned} \alpha \int_0^{\delta_1} f(\psi_1^*(s))ds &\le \alpha \int_0^{\delta_1} f(\frac{s}{\delta_1} x_*)ds \\ &\le (\frac{\alpha\delta_1}{x_*}) \int_0^{x_*} f(u)du \le (\frac{\alpha\delta}{x_*}) \int_0^{x_*} f(u)du. \end{aligned} \tag{1.20}$$

Note that

$$\begin{aligned} \psi_1^*(1-\delta) &= \psi_1(1-\delta) + \alpha \sum_{i=1}^{N} \lambda_i \int_0^{1-\delta} f(x(z_1 - p_i + s))ds \\ &\ge \psi_1(1-\delta). \end{aligned}$$

It follows from Lemma 1.3 that if α is sufficiently large, then we can find a positive constant c_4 (independently of α and $\phi \in K_\alpha$) such that

$$\psi_1^*(1-\delta) \ge c_4 m^r. \tag{1.21}$$

Now using (1.21), the fact that $\psi_1^*(1-\delta) \ge x_*$, and the estimates of Lemma 1.2 we see that

$$\begin{aligned} \alpha \int_{1-\delta}^{1} f(\psi_1^*(s))ds &\le \alpha\delta f(\psi_1^*(1-\delta)) \\ &\le \alpha\delta f(c_4 m^r) \le c_5, \end{aligned} \tag{1.22}$$

where c_5 is a constant independent of α and $\phi \in K_\alpha$.

It remains to estimate

$$\alpha \int_{\delta_1}^{1-\delta} f(\psi_1^*(s))ds.$$

For notational convenience set $\mu = \psi_0(\delta_1)$ and $\nu = \psi_0'(0)$. $\psi_0(t)$ is concave down so it follows that

$$\psi_0(t) \leq \mu + \nu(t - \delta_1), \ \delta_1 \leq t \leq 1.$$

Using the fact that $f|[x_*, \infty)$ is nondecreasing we obtain

$$\begin{aligned}\psi_1^*(t) =& x_* + \alpha \int_{\delta_1}^{t} f(\psi_0(s))ds + \alpha \sum_{i=1}^{N} \lambda_i \int_{\delta_1}^{t} f(x(z_1 - p_i + s))ds \\ &\geq x_* + \alpha \int_{\delta_1}^{t} f(\psi_0(s)) \geq x_* + \alpha \int_{\delta}^{t} f(\mu + \nu(s - \delta_1))ds \equiv \theta_1(t).\end{aligned}$$

Thus,

$$\alpha \int_{\delta_1}^{1-\delta} f(\psi_1^*(t))dt \leq \alpha \int_{\delta_1}^{1-\delta} f(\theta_1(t))dt. \tag{1.23}$$

To estimate the right-hand side of (1.23) we need to estimate μ and ν. We claim that if α is large enough, then there exist positive constants c_6, c_7, and c_8 (independent of α and $\phi \in K_\alpha$) such that

$$\begin{aligned} c_6 m &\leq \mu \leq m, \\ c_7 m &\leq |\nu| \leq c_8 m. \end{aligned} \tag{1.24}$$

First note that by concavity of $x(t)$ on the interval $[z_1 - 2, z_1]$ one obtains $x(z_1 - 2) \leq 2m$. It follows from this, and Lemma 1.2, that

$$\begin{aligned}|\psi_0'(0)| &= \alpha f(x(z_1 - 2)) + \alpha \sum_{i=1}^{N} \lambda_i f(x(z_1 - p_i - 1)) \\ &\geq \alpha f(x(z_1 - 2)) \geq \alpha f(2m) \geq c_7 m,\end{aligned}$$

where c_7 is a fixed positive constant.

Similarly, concavity of ψ_0 implies that

$$\begin{aligned}|\psi_0'(0)| \leq |\psi_0'(1)| &= \alpha f(m) + \alpha \sum_{i=1}^{N} \lambda_i f(x(z_1 - p_i)) \\ &\leq \alpha(N+1)\lambda_{\max} f(m) \leq c_8 m\end{aligned}$$

for a fixed positive constant c_8. This gives the second part of (1.24).

Again using the concavity of ψ_0, it is easy to see that

$$m \geq \psi_0(\delta_1) \geq m(1-\delta_1).$$

But $\delta_1 < 1/2$, so this implies the first part of (1.24).

Next we claim that if α is large enough, then there exists a positive constant c_9 (independent of α and $\phi \in K_\alpha$) such that

$$\alpha \int_\delta^{1-\delta} f(\theta_1(t))dt \leq \frac{1}{f(m)} \int_{x_*}^{\infty} f(\xi)d\xi - c_9 m^{r-1}. \tag{1.25}$$

But this is essentially the content of Lemma 1.4 of [**5**]. (It is at this point that the assumption $\sigma > r(r-1)^{-1}$ comes in.) We note that our f satisfies the same hypotheses; that θ_1 is defined the same way in terms of μ, ν and δ_1; and that μ, ν, δ, and δ_1 satisfy the same estimates in terms of m. It follows that the proof of Lemma 1.4 of [**5**] implies that (1.25) holds under our hypotheses. We leave it to the reader to check the exact details.

The lemma now follows by combining (1.20) and (1.25).

We are now in a position to prove our key lemma, which gives the desired estimate of $x(z_1 + 2p_N)$ for $\phi \in K_\alpha$ and α sufficiently large.

LEMMA 1.5. *Suppose that f satisfies* H1., $r > 2, \sigma > r(r-1)^{-1}$, *and $\phi \in K_\alpha$. If α is large enough, then there exists a positive constant d_3, independent of α and $\phi \in K_\alpha$, such that*

$$x(z_1 + 2p_N) \leq -d_3 m^{r-1}.$$

PROOF. Assume that α is large enough so that the conclusions of

Lemmas 1.3 and 1.4 hold. For $t \geq 1$, define functions I_1, I_2, and I_3 by

$$\begin{aligned} I_1(t) &= -\alpha \sum_{p_i \leq t+1} \lambda_i \int_{z_1 - p_i}^{z_1 - 1} f(x(s))ds \\ &\quad -\alpha \sum_{p_i > t+1} \lambda_i \int_{z_1 - p_i}^{z_1 - p_i + t} f(x(s))ds, \\ I_2(t) &= -\alpha \sum_{p_i \leq t-1} \lambda_i \int_{z_1 - 1}^{z_1 + 1} f(x(s))ds \\ &\quad -\alpha \sum_{t-1 < p_i < t+1} \lambda_i \int_{z_1 - 1}^{z_1 - p_i + t} f(x(s))ds, \\ I_3(t) &= -\alpha \sum_{p_i < t-1} \lambda_i \int_{z_1 + 1}^{z_1 - p_i + t} f(x(s))ds. \end{aligned}$$

One can check that

$$x(z_1 + t) = I_1(t) + I_2(t) + I_3(t), \quad t \geq 1.$$

And it is easy to see that

$$I_1(t) \leq 0, \quad t \geq 1. \tag{1.26}$$

Next, if $1 \leq t \leq 2$, then one has

$$\begin{aligned} I_2(t) &= -\alpha \int_{z_1 - 1}^{z_1 - 1 + t} f(x(s))ds \\ &\quad -\alpha \sum_{\substack{t-1 < p_i < t+1 \\ p_i \neq 1}} \lambda_i \int_{z_1 - 1}^{z_1 - p_i + t} f(x(s))ds \equiv J_1(t) + J_2(t). \end{aligned} \tag{1.27}$$

We can estimate $J_1(t)$ as follows.

$$\begin{aligned} J_1(t) &= -\alpha \int_{z_1 - 1}^{z_1} f(x(s))ds - \alpha \int_{z_1}^{z_1 - 1 + t} f(x(s))ds \\ &= -\psi_1(1) + \psi_2^*(t-1) \leq -\psi_1(1) + \psi_2^*(1). \end{aligned} \tag{1.28}$$

The second equality follows from the definition of ψ_1 and ψ_2^* and the oddness of f; the inequality since ψ_2^* is increasing.

Since we are assuming that $r > 2$, Lemmas 1.3 and 1.4 imply that, perhaps by increasing α, one can find a positive constant c_{10} (independent of α and $\phi \in K_\alpha$) such that

$$-\psi_1(1) + \psi_2^*(1) \leq -c_{10}m^{r-1}. \tag{1.29}$$

Also, using Lemmas 1.3 and 1.4, it is not difficult to see that

$$J_2(t) \leq 0, \; t \geq 1. \tag{1.30}$$

Therefore, using (1.27)-(1.30) we obtain

$$I_2(t) \leq -c_{10}m^{r-1}, 1 \leq t \leq 2. \tag{1.31}$$

If $t \geq 2$, then

$$I_2(t) = -\alpha \sum_{p_i \leq t-1} \lambda_i \int_{z_1-1}^{z_1+1} f(x(s))ds + J_2(t), \tag{1.32}$$

where $J_2(t)$ is defined as above.

Now, by exactly the previous argument we obtain

$$\begin{aligned} I_2(t) &\leq -\sum_{p_i \leq t-1} \lambda_i c_{10} m^{r-1} + J_2(t) \\ &\leq -\sum_{p_i \leq t-1} \lambda_i c_{10} m^{r-1} \leq -c_{10}m^{r-1}, \; t \geq 2, \end{aligned} \tag{1.33}$$

since $\sum_{p_i \leq t-1} \lambda_i \geq 1$.

Thus, we have shown that

$$I_2(t) \leq -c_{10}m^{r-1}, \; t \geq 1. \tag{1.34}$$

Finally, we make the following

CLAIM. If α is large enough, then there exists a constant c_{11} such that

$$I_3(t) \leq c_{11}m, 1 \leq t \leq 2p_N. \tag{1.35}$$

c_{11} can be chosen independent of $\alpha, \phi \in K_\alpha$, and $t \in [1, 2p_N]$.

PROOF OF CLAIM. For $1 \le t \le 2$, since $p_i \ge 1$ for all i, it follows that $I_3(t) = 0$. Now suppose that there exists an α_J such that for $\alpha \ge \alpha_J$,

$$I_3(t) \le c_J\ m \le t \le J, \tag{1.36}$$

where J is an integer, $J \ge 2$, and c_J is a positive constant.

Then (1.26), (1.34), and (1.36) imply that for α large enough one has

$$x(z_1 + t) \le -c_{10}m^{r-1} + c_J m,\ \ 1 \le t \le J. \tag{1.37}$$

It follows (using the assumption $r > 2$ and perhaps increasing α again) that

$$x(z_1 + t) \le -c'_J m^{r-1},\ \ 1 \le t \le J, \tag{1.38}$$

where c'_J is a positive constant. Using (1.38), the definition of I_3, and the properties of f, it is easy to see that for α large enough, say $\alpha \ge \alpha_{J+1}$,

$$\begin{aligned} I_3(t) &\le \alpha \sum_{p_i < t-1} \lambda_i f(c'_J m^{r-1})(J-1) \\ &\le c_{J+1} m,\ \ 1 \le t \le J+1, \end{aligned}$$

C_{J+1} a positive constant. It follows by induction that (1.36) holds for every integer $J \ge 2$. The claim follows by taking $J \ge 2p_N$ in (1.36).

The lemma is now proved by combining (1.26), (1.34), and (1.35), and setting $t = 2p_N$.

PROOF OF THEOREM 1.1. (Cf. the proof of Theorem 1.1 of [**5**]) If f satisfies the hypotheses of Theorem 1.1, $\phi \in K_\alpha$, and α is large enough, then by Lemma 1.5 we can find a positive constant d_3, independent of α and ϕ in K_α, such that

$$x(z_1 + 2p_N) \le -d_3 m^{r-1}.$$

Thus, using the estimates of Lemma 1.2, and, perhaps, increasing α if necessary, we obtain

$$x(z_1 + 2p_N) \le -k\alpha^{\varepsilon}.$$

It follows from the proof of Lemma 1.5 that $x(t) \le 0$ for $z_1 \le t \le z_1 + p_N$, and so $x(t)$ is nondecreasing on the interval $[z_1 + p_N, z_2]$, where z_2 is the second zero of $x(t)$.

Now, set

$$\tau = \tau(\phi, \alpha) = \inf\{t \geq z_1 + 2p_N | x(t) = -k\alpha^\varepsilon\}$$

and define the map $S_\alpha : K_\alpha \to C[0, p_N]$ by

$$S_\alpha \phi = \psi \text{ where } \psi(t) = -x(\tau - p_N + t), 0 \leq t \leq p_N.$$

It is easy to see that S_α is a continuous, compact map, and the remarks above show that $S_\alpha(K_\alpha) \subseteq K_\alpha$. Therefore, the Schauder fixed point theorem implies that S_α has a fixed point ϕ_α. If $x_1(t) = x(t; \phi_\alpha, \alpha)$ is the corresponding solution, then $x_1(t)$ is periodic of (minimal) period $2(\tau - 1)$ and can be extended to all of $\mathbf{R}$ by periodicity. If z_2 is the second (strictly positive) zero of $x_1(t)$, then $x_\alpha(t) = x_1(t + z_2)$ is the periodic solution of Theorem 1.1 with $q_\alpha = \tau - 1$.

It still remains to estimate the (minimal) period of $x_\alpha(t)$. We use a trick that is used in [**4**]. Since $x_\alpha(t)$ is decreasing on the interval $[p_N + 1, q_\alpha]$, it follows that

$$x_\alpha(t-1) \geq x_\alpha(t), \quad p_N + 2 \leq t \leq q_\alpha - 1.$$

f is decreasing on the interval $[x_*, \infty)$, so this implies that

$$\alpha f(x_\alpha(t-1)) \leq \alpha f(x_\alpha(t)), \; p_N + 2 \leq t \leq q_\alpha - 1. \tag{1.39}$$

Using (1.3) to solve for $\alpha f(x_\alpha(t-1))$ and (1.39) one obtains

$$\begin{aligned} -x_\alpha'(t) &\leq \alpha f(x_\alpha(t)) - \alpha \sum_{i=1}^{N} \lambda_i f(x_\alpha(t - p_i)) \\ &\leq \alpha f(x_\alpha(t)) \leq a_2 \alpha (x_\alpha(t))^{-r}, \; p_N + 2 \leq t \leq q_\alpha - 1. \end{aligned} \tag{1.40}$$

The inequality (1.40) implies that

$$-(x_\alpha(t))^r x_\alpha'(t) \leq a_2 \alpha, \; p_N + 2 \leq t \leq q_\alpha - 1, \tag{1.41}$$

and integrating (1.41) from $p_N + 2$ to $q_\alpha - 1$ gives

$$(r+1)^{-1}[(x_\alpha(p_N + 2))^{r+1} - m^{r+1}] \leq a_2 \alpha [q_\alpha - p_N - 3]. \tag{1.42}$$

It follows from Lemma 1.5 that $x_\alpha(p_N + 2)$ can be bounded below by a positive constant times m^{r-1}. If we use this fact and the estimates of Lemma 1.2, then the estimate on q_α follows easily from (1.31), and this completes the proof of the theorem.

REMARK 1.2. Consider the equation

$$x'(t) = -\alpha \sum_{i=1}^{N} f(x(t-i)). \tag{1.43}$$

This is, of course, a special case of equation (1.1) with $\lambda_i = 1$ and $p_i = i + 1$. Equation (1.43) is studied by Nussbaum in [**6**] where it is proved (as a special case of a more general theorem) that, for f a suitable odd function and α sufficiently large, (1.43) has periodic solutions of minimal period $2(N + 1)$. In particular, from [**6**, Cor. 1] we have the following. Suppose that $f : \mathbf{R} \to \mathbf{R}$ is a continuous, odd function such that $xf(x) \geq 0$ for all x. Assume that $f'(0) > 0$ and $\lim_{x\to 0} f(x)x^{-1} = 0$. Then if $\alpha > \mu_0 =: (\frac{\pi}{f'(0)(N+1)}) \tan (\frac{\pi}{2(N+1)})$, (1.43) has a nonconstant periodic solution $y_\alpha(t)$ such that $y_\alpha(t) \geq 0$ for $0 \leq t \leq N + 1, y_\alpha(-t) = -y_\alpha(t)$ and $y_\alpha(t + N + 1) = -y_\alpha(t)$ for all t. If, in addition, $xf(x) > 0$ for $x \neq 0$ it is easy to show that $y_\alpha(t) > 0$ for $0 < t < N + 1$.

Thus, if f satisfies H1. and $f'(0) > 0$, Theorem 1.1 implies that, for α sufficiently large, (1.43) has at least two distinct slowly oscillating periodic solutions. (Choose α large enough so that q_α in Theorem 1.1 is greater than $N + 1$.) The functions $f(x) = x(1 + |x|^{r+1})^{-1}, r > 2$, provide examples.

REMARK 1.3. The number μ_0 is Remark 1.2 comes from looking at the spectrum of (the complexification of) the operator

$$(Lx)(t) = -\alpha f'(0) \int_0^t \sum_{j=1}^{N} x(s-j)ds$$

which results from considering the linearization of (1.43) at 0. (Here L is considered as a map of the real Banach space of real-valued continuous functions satisfying $x(t + N + 1) = -x(t)$ and $x(-t) = -x(t)$ for all t into itself. See [**6**, Lemma 3 and Corollary 1] for details.) The solutions $y_\alpha(t)$ described in Remark 1.2 actually bifurcate from the constant solution $x \equiv 0$ at $\alpha = \mu_0$ [**6**. Theorem 5].

The constant α_0 in Theorem 1.1 essentially must be chosen large enough so that (in our previous notation)

$$x(z_1 + 2p_N) \leq -(\text{pos. const})m^{r-1} \leq -k\alpha^{\varepsilon}, \ \alpha > \alpha_0.$$

By a more careful analysis one could obtain an explicit upper bound for α_0. This would depend on the size of $f(x)$ for large x and so, in the case of (1.43), there would be no relationship to μ_0; moreover, the computations involved would be very unpleasant.

Acknowledgements. This work includes part of the author's Ph.D. thesis at Rutgers University. The author would like to thank Prof. Roger Nussbaum for many enlightening discussions concerning this problem and in related areas.

References

1. S. Chapin, *Periodic solutions of some nonlinear differential-delay equations*, Ph.D. dissertation, Rutgers University, 1983.
2. L. Glass and M. Mackey, *Oscillation and chaos in physiological control systems*, Science **197** (1977), 287-289.
3. L. Glass and M. Mackey, *Pathological conditions resulting from instabilities in physiological control systems*, in Bifurcation Theory and Applications in Scientific Disciplines, edited by 0. Gurel and O.E. Rossler, Annals of the New York Academy of Sciences.
4. R.D. Nussbaum and S. Chapin, *Asymptotic estimates for the periods of periodic solutions of a differential-delay equation*, Mich. Math. Journal **31** (1984), 215-229.
5. R.D. Nussbaum, *Asymptotic analysis of functional differential equations and solutions of long period*, Arch. Rat. Mech. Anal. **81** (1983), 373-397.
6. R.D. Nussbaum, *Periodic solutions of special differential equations: an example in non-linear functional analysis*, Proc. Royal Soc. Edinburgh **81A** (1978), 131-151.
7. D. Saupe, *Global bifurcation of periodic solutions of some autonomous differential delay equations*, Forschungsschwerpukt Dynamische Systems, Report Nr. **71**, Univ. Bremen, Bremen, July, 1982.
8. D. Saupe, *Accelerated PL-continuation methods and periodic solutions of parametrized differential-delay equations*, Ph.D. dissertation (in German), Univ. Bremen, Bremen, 1982.

MATHEMATICS DEPARTMENT, MICHIGAN STATE UNIVERSITY, EAST LANSING, MI 48824.

MATHEMATICS DEPARTMENT, OHIO UNIVERSITY, ATHENS, OH 45701