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REFINED TAUBERIAN GAP THEOREMS FOR POWER SERIES METHODS OF SUMMABILITY

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1. Introduction. Besides the usual Tauberian gap theorems there are refined forms which, e.g., deal with mixed conditions or lead to summability instead of convergence. Here we are concerned with an instance of the latter kind. Let us assume that $\sum_0^\infty a_n$ is a gap series:

$$a_n = 0, \text{ for } n \neq k_0, k_1, \cdots \tag{1.1}$$

($\{k_n\}$ a given sequence of integers, $0 \leq k_0 < k_1 < \cdots$). Hardy-Littlewood's classical high indices theorem for A_0 (Abel's method of summability) states that

$$A_0 - \sum_0^\infty a_n = s \text{ implies } \sum_0^\infty a_n = s$$

if $k_{n+1} \geq ck_n$ for a constant $c > 1$. Now let p be a non-negative integer. Then, according to a special case of a theorem of Korenblyum (see [**11**]),

$$A_0 - \sum_0^\infty a_n = s \text{ implies } C_p - \sum_0^\infty a_n = s \tag{1.2}$$

if

$$k_{n+p+1} \geq ck_n \text{ for a constant } c > 1 \tag{1.3}$$

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(C_p =Cesaro's method of order p). Results of the same type as Korenblyum's theorem were given by Borwein-Cass [**1**] and Jakimovski-Russell ([**7**; p. 116], [**8**; p. 135 (condition (H_p), p. 136]).

One of the main purposes of the present paper is to show (see §8) that a gap series which is summable by a power series method Q of type PTR (see §4) is also summable by some generalized Cesaro method (see §3) if a gap condition of type (1.3) is fulfilled. The above result follows from some general theorems of independent interest.

A method of proving Tauberian gap theorems, depending on the concept of gap-perfectness (of a summability method, or an FK-space), was developed in [**16**] and [**17**]. This method is extended in §6 so as to be of use for our present purpose (see Theorem 6.3). After formulating and proving Theorem 5.1, on the iteration product summability of a PTR method and certain generalized Cesaro methods, and Theorem 7.1, which is a refined Tabuerian gap theorem for generalized Cesaro methods, we come to our main theorem in §8: Theorem 8.1.

2. Notations. We are dealing with sequences, series, d by $x = \{x_m\} = \{x_0, x_1, \cdots\}$, with $x_{-1} = 0$ in formulas. If nothing is said about n, then $n = 0, 1, \cdots$ (and $n \to \infty$ if a limit process is involved), and likewise for m. We also write $x_n = (x)_n$. Given an (infinite) series $\sum_n a_n$ (or $\sum a_n$ shortly), and also, more generally, given an a, then $s_n = a_0 + \cdots + a_n$; if conversely, some argumentation starts with an s, then automatically $a_n = s_n - s_{n-1}$. Sometimes we start a priori with a couple (a, s). And Q is always a sequence of real positive numbers:

$$q_n > 0, \text{ and } \bar{q}_n = q_0 + \cdots + q_n. \tag{2.1}$$

Special sequences are $e = \{1, 1, 1, \cdots\}$, $e_0 = \{1, 0, 0, \cdots\}$, $e_1 = \{0, 1, 0, \cdots\}$, $\cdots$. A matrix, say B, is an (infinite) matrix with complex elements $b_{nm} = (B)_{n,m}$. The B-transform Bs of a sequence s is said to exist if, for each fixed n, the series $(Bs)_n = \sum_m b_{nm} s_m$ converges. A normal matrix is a lower triangular matrix with non-vanishing elements in the main diagonal. The identity matrix is denoted by I, and S is the normal matrix with the elements $(S)_{nm} = 1 (m = 0, \cdots n)$. For a pair (a, s) $s = Sa, a = S^{-1}s$. The diagonal matrix with the elements c_n in the main diagonal is denoted by diagc_n. Given the pair (a, s), then $B - \sum a_n = s$, (or: $\sum a_n$ is summable B to the value s) as well as the equivalent statement $B - \lim s_n = s$(or: s is limitable B to the value s means that B_s exists and that $\lim S_n = s$. The series-convergence-domain B of B is the set of all a for which $\sum a_n$ is summable B, the sequence-convergence-domain B is the set of all s limitable B; B and$\mathbf{\overline{B}}$ are (isomorphic) FK-spaces. By replacing n with x $(0 \leq x < 1)$ and $n \to \infty$ by $x \to 1^-$, one obtains a semi-continuous sequence-to-function summability method.

3. Generalized Cesàro methods. There is a long history concerning methods of Cesàro type (variants, generalizations, comparison theorems, etc.). Many references can be found in [**19**; Ch. VI]; here we mention Faber 1913, Dobrowolski 1926, Jurkat 1953, Burkill 1961, and for new developments Borwein, Meir, Russell (see [**2**]), and Kuttner [**14**]. We repeat now the definition of the generalized Cesaro methods of orders $p = 0, 1, \cdots$ (cf., e.g., Russell [**18**; p. 419]). We assume that q is a given sequence (see (2.1) and observe that $q_{-1} = \overline{q}_{-1} = 0$). For $p = 1, 2, \cdots$, let $S_p = S_p(\mathbf{q})$ be the normal matrix with the elements

$$(S_p)_{n,m} = \overline{q}_{m+p-1} - \overline{q}_{m-1} = q_m + \cdots + q_{m+p-1} \qquad (m = 0, \cdots, n; p = 1, 2, \cdots).$$

Then (proof by induction) $S_p \cdots S_1 S$ has the elements

$$(S_p \cdots S_1 S)_{n,m} = (\overline{\mathbf{q}}_n - \overline{\mathbf{q}}_{m-1}) \cdots (\overline{q}_{n+p-1} - \overline{\mathbf{q}}_{m-1}) \qquad (m = 0, \cdots, n; p = 1, 2, \cdots). \tag{3.1}$$

Now we define the normal matrices $(C, q, 0), (C, q, 1), \cdots$:

$$(C, q, 0) = I, (C, q, p) = (\operatorname{diag}(\overline{q}_n \cdots \overline{q}_{n+p-1})^{-1}) S_p \cdots S_1 \quad (p = 1, 2, \cdots).$$

The generalized Cesàro methods (C, q, p) are the summability methods connected with these matrices. It follows from (3.1) that (for $p = 1, 2 \cdots$)

$$(S_p \cdots S_1 \mathbf{e})_n = (S_p \cdots S_1 S \mathbf{e}_0)_n = \overline{\mathbf{q}}_n \cdots \overline{q}_{n+p-1}$$

(so the row-sums of (C, q, p) equal one) and, given a couple (a, s), that

$$((C, q, p)s)_n = \sum_{m=0}^{n} (1 - \frac{\overline{q}_{m-1}}{\overline{q}_n}) \cdots (1 - \frac{\overline{q}_{m-1}}{\overline{q}_{n+p-1}}) a_m.$$

Together with q we consider the sequences $q^{(p)}$:

$$q_n^{(p)} = (q_n + \cdots + q_{n+p}) \overline{q}_n \cdots \overline{q}_{n+p} / \overline{q}_{n+p} > 0 \quad (p = 0, 1, \cdots);$$

then $q^{(0)} = q$ and (proof by induction)

$$(3.2) \qquad \overline{q}_n^{(p)} = q_0^{(p)} + \cdots + q_n^{(p)} = \overline{q}_n \cdots \overline{q}_{n+p} > 0 \quad (p = 0, 1, \cdots).$$

We define the normal matrices $M_p = M_p(q)$:

$$(M_p)_{n,m} = q_m^{(p)} / \overline{q}_n^{(p)} \ (m = 1, \cdots, n; p = 0, 1 \cdots).$$

The summability method connected with M_p is the method of weighted means with weights $q_n^{(p)}$. For $p = 1, 2 \cdots$, we have

$$(C, q, p{+}1)(C, q, p)^{-1} = (\operatorname{diag}(\overline{q}_n \cdots \overline{q}_{n+p})^{-1}) S_{p+1} (\operatorname{diag}(\overline{q}_n \cdots \overline{q}_{n+p-1}))$$

and therefore (for $m = 0, \cdots, n$;cf. (3.2))

$$\begin{aligned}((C, q, p + 1)(C, q, p)^{-1})_{n,m} &= \overline{q}_m \cdots \overline{q}_{m+p-1} (\overline{q}_{m+p} - \overline{q}_{m-1}) / \overline{q}_n \cdots \overline{q}_{n+p} \\ &= (M_p)_{n,m}.\end{aligned}$$

Since $(C, q, 1) = M_0$ it follows that

$$(C, q, p + 1) = M_p(C, q, p) = M_p \cdots M_1 M_0 \ (p = 0, 1, \cdots).$$

From now on, in dealing with the methods (C, q, p) and $M_p(q)$ we shall always assume that $\sum q_k = \infty$. Then all these methods are regular, and we have

$$(C, q, p) \subset (C, q, p + 1) \text{ with consistency } \quad (p = 0, 1, ..).$$

(C, e, p) is the ordinary Cesaro method C_p. If we put $\lambda_0 = 0, \lambda_{n+1} = \overline{q}_n$ then (C, q, p) coincides with the method (C, λ, p) in [**18**, p. 419]; $\lambda_0 > 0$ is allowed there, too.

In §7 we shall use the connection between (C, q, p) and the Riesz methods (R, λ, p). Given a sequence $\boldsymbol{\lambda}(0 = \lambda_0 < \lambda_1 < \cdots, \lambda_n \to \infty)$ and an integer $p(p = 0, 1, \cdots)$, then $(R, \lambda, 0)$ means convergence, and for $p = 1, 2, \cdots$ we have

$$(R, \lambda, p) - \sum a_n = (R, \lambda, p) - \lim s_n = \lim_{w \to \infty} \sum_n (1 - \frac{\lambda_n}{w})_+^p a_n$$

($w > 0; x_+^p = x^p$ for $x > 0, x_+^p = 0$ for $x \leq 0$) if the latter limit exists. One knows (Russell [**18**], Meir [**15**]; cf. Borwein-Russell [**2**]):

LEMMA 3.1. *For each* $p = 0, 1, \cdots$, *the methods* (C, q, r) *and* (R, λ, p) *are equivalent if* $\lambda_0 = 0$ *and* $\lambda_{n+1} = \overline{q}_n$.

4. The class PTR. This class of methods was introduced in [**10**]; the three letters refer to Power series, Totally monotone, Regular. We recall:

Let q (for the moment not necessarily with $q_n > 0$) be a real totally monotone sequence, i.e., a sequence admitting a representation

$$q_n = \int_0^1 v^n d\alpha(v) \geq 0,$$

$\alpha(v)$ real, increasing (wide sense) and bounded in $0 \leq v \leq 1$, and furthermore assume that $\sum a_n = \infty$. It follows that $q_n \geq q_{n+1} > 0$, so we are in harmony with our agreement (see §2) that q always is a sequence of positive numbers. We put $q(x) = \sum q_n x^n$ for $0 \leq x < 1$. Then the power series method $Q = Q(q)$ of type PTR is defined by means of the (sequence-to-function) matrix Q having the elements

$$(Q)_{x,m} = (1/q(x))q_m x^m \quad (0 \leq x < 1; m = 0, 1, \cdots x \to 1-).$$

The convergence-domain ($\mathbf{Q}$ as well as $\overline{\mathbf{Q}}$) is an FK-space ([**10**; Lemma 3]) and we have ([**10**; Theorem 4])

LEMMA 4.1. *If* Q *is of type* PTR, *then the* **FK**-*space* $\mathbf{Q}$ *is gap-perfect.*

More about gap-perfectness is said in §6.

5. Interrelation between C and Q. We introduce the methods $Q_0(q)$, $Q_1(q)$, $\cdots$. Let us assume that a q is given and that

$$\sum q_n = \infty, q(x) = \sum q_n x^n \text{ exists for } 0 \leq x < 1. \tag{5.1}$$

It is easy to see that, for each $p = 0, 1, \cdots$, (5.1) is equivalent to

$$\sum q_n^{(p)} n = \infty, \; q^{(p)}(x) = \sum q_n^{(p)} x^n \text{ exists for } 0 \leq x < 1. \tag{5.2}$$

Now we define the method $Q_p = Q_p(q)$ by means of the (sequence-to-function) matrix $Q_p (p = 0, 1, \cdots)$ with the elements

$$(1/q^{(p)}(x))q_m^{(p)}x^m \quad (0 \le x < 1; m = 0, 1, \cdots, ; x \to 1-).$$

Q_p is regular.

If $Q(q)$ is any method of type PTR, then (5.1) and (5.2) are fulfilled, i.e., $Q_0, Q_1, \cdots$ are defined (with $Q_0 = Q$). We announce Theorem 5.1 which is essential to the proof of Theorem 8.1.

THEOREM 5.1. *Let $Q(q)$ be of type* PTR *and p be an integer* $(p = 0, 1, \cdots)$. *If s if limitable $Q(q)$, then $(C, \mathbf{q}, p)\mathbf{s}$ is limitable $Q_p(q)$ with consistency.*

The proof follows from Lemma 5.2. We prepare the formulation of this lemma. To begin with we introduce some notations. Firstly, given a q and a series $\sum a_n$, we put

$$u^{(0)} = \mathbf{s},\ u^{(p)} = S_p u^{(p-1)} \quad (p = 1, 2, \cdots),$$

$$t^{(p)} = (C, q, p)s \quad (p = 0, 1, \cdots);$$

then we have

$$t^{(0)} = u^{(0)} = s,\ t_n^{(p)} = (\bar{q}_n \cdots \bar{q})_{n+p-1})^{-1} u_n^{(p)} \ (p = 1, 2, \cdots).$$

Secondly, given a q and any sequence $\mathbf{t}$, we define formally

$$Q^*(\mathbf{t}, x) = \sum q_n t_n x^n,\ Q_p^*(t, x) = \sum q_n^{(p)} t_n x^n \ (p = 0, 1, \cdots).$$

Now we assume that $Q(q)$ is of type PTR and that $\sum a_n$ is such that $Q^*(s, x)$ exists for $0 \le x < 1$. Since $q_n = \int_0^1 v_1^n d\alpha(v_1)$ we have (for $0 \le x < 1$)

$$\begin{aligned}
\int_0^1 d\alpha(v_1)\frac{1+v_1}{1-xv_1}Q^*(s, xv_1) &= \int_0^1 d\alpha(v_1)(1+v_1)\sum u_n^{(1)}(xv_1)^n \\
&= \sum \int_0^1 d\alpha(v_1)(v_1^n + v_1^{n+1})u_n^{(1)}x^n \\
&= \sum (q_n + q_{n+1})u_n^{(1)}x^n \\
&= \sum q_n^{(1)} t_n^{(1)} x^n = Q_1^*(t^{(1)}, x).
\end{aligned}$$

The existence of the first of these expressions, even if s would be replaced by $\{|s_0|, |s_1|, \cdots\}$, is clear; therefore all our expressions exist. Thus we have proved the case $p = 1$ of

LEMMA 5.2. *Let $Q = Q(q)$ be of type* PTR *and the series $\sum a_n$ be such that $Q^*(s, x)$ exists for $0 \leq x < 1$. Then, for $p = 1, 2, \cdots$ and $0 \leq x < 1$,*

$$Q_p^*(t^{(p)}, x) = \int_0^1 d\alpha(v_p) \cdots \int_0^1 d\alpha(v_1)$$
$$\{\prod_{j=1}^{p} (1 + v_j + \cdots + v_j^j)(1 - xv_j \cdots v_p)^{-1}\} Q^*(s, xv_1 \cdots v_p).$$

The existence of both sides of this equation is guaranteed by the hypotheses.

In order to prove the general case we start with the right-hand side of our formula: Instead of Q^* we introduce $\mathbf{u}^{(1)}$ (see above); then $u^{(2)}, \cdots, u^{(p)}$, and finally $t^{(p)}$. The procedure is seen clearly enough if we execute the case $p = 2$;

$$\int_0^1 d\alpha(v_2) \frac{1 + v_2 + v_2^2}{1 - xv_2} \sum (q_n + q_{n+1}) u_n^{(1)} (xv_2)^n$$
$$= \int_0^1 d\alpha(v_2)(v_2^n + v_2^{n+1} + v_2^{n+2}) \sum (S_2 u^{(1)})_n x^n$$
$$= \sum (q_n + q_{n+1} + q_{n+2}) u_n^{(2)} x^n = \sum q_n^{(2)} t_n^{(2)} x^n.$$

We state the following important consequence of Lemma 5.2. If

$$Q^*(s_1, x) \leq Q^*(s_2, x) \text{ in } 0 \leq x < 1,$$

for two real series $\sum a_n^{(i)} (i = 1, 2)$, and if $t_i^{(p)} = (C, q, p)\mathbf{s}_i$, then

$$Q_p^*(t_1^{(p)}, x) \leq Q_p^*(t_2^{(p)}, x) \text{ in } o \leq x < 1.$$

This is the basis of the

PROOF OF THEOREM 5.1. The case $p = 0$ is trivial, therefore let p be a positive integer. We assume that $Q - \lim s_n = s$ and furthermore, without loss of generality, that $\mathbf{s}$ is real and that s equals 0. Then, given any $\varepsilon > 0$, there exists $\rho = \rho(\varepsilon) > 0$ such that

$$(Qs)_x \leq \varepsilon + \rho q_0/q(x) = (Q(\varepsilon e + \rho e_0))_x, \quad \text{for } 0 \leq x < 1.$$

With $t = (C, q, p)s$ our consequence of Lemma 5.2 yields

$$(Q_p t)_x \leq \varepsilon + \rho (Q_p w)_x, \text{ for } 0 \leq x < 1,$$

where $w = (C, q, p)e_0$ is a null-sequence; it follows that

$$(Q_p t)_x \leq 2\varepsilon \text{ for } 0 \leq \delta(\varepsilon) < 1.$$

Together with the opposite consideration we get the assertion.

An immediate consequence of Theorem 5.1 is the following consistency theorem.

THEOREM 5.3. *Let $Q(q)$ be of type* PTR *and p be an integer* $(p = 0, 1, \cdots)$. *Then the methods $Q(q)$ and (C, q, p) are consistent.*

6. Gap-perfectness. If k is a sequence of integers with $0 \leq k_0 < k_1 < \cdots$, then we say that the sequence x satisfies the gap condition

$$G(\mathbf{k}) \text{ if } x_n = 0, \quad \text{for } n \neq k_0, k_1, \cdots,$$

$G(k)$ shall also denote the set of all x fulfilling this condition. A series $\sum a_n$ is said to satisfy the gap-condition $G(k)$ if $a \in G(\mathbf{k})$. Given an FK-space E and a k, the k-gap-perfectness of E is defined as follows: Each $x \in E$ satisfying $G(k)$ can be approximated with arbitrary accuracy by elements of E which are finite (coordinates ultimately zero) and satisfy $G(k)$. E is called gap-perfect if it is k-gap perfect for all k. We need the following result (see [**17** Satz 1] and the references in [**10**] to Kolodziej and Mazur-Sternbach).

LEMMA 6.1. *If the set of all convergent sequences in an* FK-*space E is not closed in E, then E contains bounded divergent sequences as well as unbounded sequences.*

Important for us is a corresponding statement where the convergent and the bounded sequences are replaced by certain other sets of sequences:

LEMMA 6.2. *Let F be an* FK-*space, B be a normal matrix, and assume that the set of all those elements x of F, for which Bx is a convergent sequence, is not closed in F. Then F contains sequences y for which By is bounded and divergent, as well as sequences z for which Bz is unbounded.*

There are other modifications of this type (more general B, spaces of series, etc.). For the proof we apply Lemma 6.1 to the FK-space E (isomorphic to F) consisting of all sequences Bx with $x \in F$. Essential now for our purposes is the following theorem which will be proved by means of the first part of the assertion of Lemma 6.2.

THEOREM 6.3. *Suppose that E is an* FK-*space which, for a given* $\mathbf{k}$, *is k-gap-perfect, and that B is a normal matrix for which Be and all Be_n are convergent sequences. Suppose furthermore that, for each $x \in E \bigcap G(k)$, for which BSx is bounded, BSx is convergent. Then even $x \in E \bigcap G(k)$ alone implies that BSx is convergent.*

PROOF OF THEOREM 6.3. The FK-space $E_1 = E \bigcap G(k)$ is a closed subspace of E and is isomorphic to the FK-space F consisting of all sequences Sx with $x \in E_1$. By gap-perfectness each element of F can be approximated by ultimately constant sequences, hence (because of our hypothesis about Be, Be_n) by sequences having a convergent B-transform. If now there would exist an element of F with a divergent B-transform, the hypothesis of Lemma 6.2 would be fulfilled and there would exist an element $y \in F$ for which By is bounded and divergent; but this is not possible according to the last hypothesis of Theorem 6.3.

7. A refined gap theorem for C. By combining some known results we obtain the following theorem about the generalized Cesaro methods. It will be needed in §8, but is of independent interest, too. Now and afterwards, instead of $\overline{q}_n$, we also write $\overline{q}(n)$.

THEOREM 7.1. *A series $\sum a_n$ summable $(\mathbf{C}, q, r)$ is also summable (C, q, p), if it satisfies a gap condition $G(k)$ for which*

$$\liminf_{j \to \infty} \overline{q}(k_{j+p+1})/\overline{q}(k_j) > 1 \tag{7.1}$$

holds (q fixed, $\overline{q}_n \to \infty$; r and p integers, $0 \leq p < r$). Here, as is true generally, (C, q, p) and (C, q, r) are consistent.

PROOF. The proof is described by the following chain (always with consistency, if one observes that (q_n) is bounded and $\overline{q}_k \to \infty$)

$$\begin{aligned}(C,q,r)-\sum a_n &\to (R,\lambda,r)-\sum a_n\\ &\to (R,\lambda^*,r)-\sum a_n^* \to (R,\lambda^*,p)-\sum a_n^*\\ &\to (R,\lambda,p)-\sum a_n \to (C,q,p)-\sum a_n,\end{aligned}$$

where $\lambda_0 = 0$, $\lambda_{n+1} = \overline{q}_n$, and (with $j = k_n$) $\lambda_n^* = \lambda_j$, $a_n^* = a_j$. The first step is justified by Lemma 3.1, and likewise the last step. The second and fourth step are immediately clear. The step in the middle, leading from r to p, follows from [**1**, ; p. 205, Theorem], cf. [7; p. 116, Theorem A].

8. A refined gap theorem for Q. Our final result is Theorem 8.1.

THEOREM 8.1. *Let $Q= Q(q)$ be of type PTR and p be an integer* $(p = 0, 1, \cdots)$. *Suppose that each series $\sum a_n$, which is summable $Q_p(q)$ and for which s is bounded, is also summable $M_p(q)$ with consistency. Then each series $\sum a_n$ which is summable Q is also summable* (C, q, p) *with consistency, if a satisfies a gap condition $G(k)$ for which* (7.1) *holds.*

PROOF. Let c (respectively m) be the set of all convergent (resp. bounded) sequences. We assume that the supposition about $Q_p = Q_p(q)$ and $M_p = M_p(q)$ is fulfilled, and that a k is given for which (7.1) holds. Let the FK-space Q be denoted by E and the matrix (C, q, p) by B. We want to show:

$$a \in E \bigcap G(k) \text{ implies } B\mathbf{s} \in c.$$

Because of Theorem 6.3, since E is k-gap-perfect (Lemma 4.1), it is sufficient to show that

$$\{a \in E \cap G(k), Bs \in m\} \text{ implies } Bs \in c$$

Now let a fulfill the hypothesis to the left. It follows from Theorem 5.1 that $Bs \in \overline{Q}_p$, and from the supposition of our theorem that $M_pBs \in c$.

Since $M_pB = (C,q,p+1)$, Theorem 7.1 yields $(C,q,p)s \in c$. The assertion about consistency is guaranteed by Theorem 5.3.

The case $p=0$ of Theorem 8.1 was treated in [**10**].

The transition from Q_p to M_p (cf. the assumption in Theorem 8.1) is freuently possible, and certainly if, roughly speaking, the qn_n behave regularly. This is stated more exactly in the following lemma which is based on a theorem of Jakimovski-Tietz ([**9**; Theorem 4.1]).

LEMMA 8.2. *Given $Q(q)$ of type PTR and $p (p=0,1,\cdots)$, suppose that $q_n = R(n)$, where $R(x) > 0$ is a function which is continuous for $x \geq 0$ and, furthermore, is regular in the sense that there exists a constant $\rho \geq -1$ such that $R(\lambda x)/R(x) \to \lambda^\rho$ for each $\lambda > 0$ as $x \to \infty$. Then, if s is bounded, summability $Q_p(q)$ of $\sum a_n$ implies summability $M_p(q)$ of $\sum a_n$ with consistency.*

PROOF. We need some preparations. The fact that $R(x)$ is a function of the said type shall be expressed by writing $R(x) \in V_\rho$. From $(R(x+1)/R(x)) \to 1$ as $x \to \infty$ (see [**9**; Lemma 3.1]), we deduce

$$\frac{R(x+2)}{R(x)} = \frac{R(x+2)}{R(x+1)}\frac{R(x+1)}{R(x)} \to 1, \quad \frac{R(x+3)}{R(x)} \to 1, \cdots .$$

Furthermore, we have (see [**9**; Lemma 3.2])

$$R^*(x) = \int_0^x R(t)dt \in V_{\rho+1}, \overline{q}_n/R^*(n) \to 1,$$

and (again see [**9**; Lemma 3.1])

$$R^*([x])/R^*(x) \to 1, \; R^*([x]+1)/R^*(x) \to 1.$$

Let the function $\overline{R}(x)$ for $x \geq 0$ be defined in the following way:

$$\overline{R}(n) = \overline{q}_n, \; \overline{R}(x) \text{ linear in each interval } n \leq x \leq n+1.$$

Since, with $n = [x]$,

$$\frac{\overline{q}_n}{R^*(n)}\frac{R^*(n)}{R^*(x)} \leq \frac{\overline{R}(x)}{R^*(x)} \leq \frac{\overline{q}_{n+1}}{R^*(n+1)}\frac{R^*(n+1)}{R^*(x)},$$

it follows that $(\overline{R}(x)/R^*(x)) \to 1$ as $x \to \infty$, and this, for each $\lambda > 0$, yields

$$\frac{\overline{R}(\lambda x)}{\overline{R}(x)} = \frac{\overline{R}(\lambda x)}{R^*(\lambda x)} \frac{R^*(\lambda x)}{R^*(x)} \frac{R^*(x)}{\overline{R}(x)} \to \lambda^{\rho+1}, \text{ i.e., } \overline{R}(x) \in V_{\rho+1}.$$

The case $p = 0$ of our lemma is an immediate consequence of [**9**; Theorem 4.1](cf. [**10**; Lemma 13]); therefore, we assume now that $p = 1, 2, \cdots$. We put

$$R^{(p)}(x) = (R(x) + \cdots + R(x+p))\overline{R}(x) \cdots \overline{R}(x+p-1), \text{ for } x \geq 0;$$

then $R^{(p)}(n) = q_n^{(p)}$ and, for each $\lambda > 0$,

$$R^{(p)}(\lambda x)/R^{(p)}(x) = R_1(x)R_2(x)$$

with

$$R_1(x) = \frac{R(\lambda x) + \cdots + R(\lambda x + p)}{R(x) + \cdots + R(x+p)}, \quad R_2(x) = \frac{\overline{R}(\lambda x)}{\overline{R}(x)} \cdots \frac{\overline{R}(\lambda x + p - 1)}{\overline{R}(x+p-1)}.$$

In dealing with $R_1(x)$, dividing each term of the numerator and also of the denominator by $R(x)$, and writing, e.g.,

$$\frac{R(\lambda x + 1)}{R(x)} \text{ in the form } \frac{R(\lambda x+1)}{R(\lambda x)} \frac{R(\lambda x)}{R(x)},$$

we get

$$R_1(x) \to \frac{(p+1)\lambda^\rho}{p+1} = \lambda^\rho.$$

In dealing with $R_2(x)$ and writing, e.g.,

$$\frac{\overline{R}(\lambda x + 1)}{\overline{R}(x+1)} \text{ in the form } \frac{\overline{R}(\lambda x+1)}{\overline{R}(\lambda x)} \frac{\overline{R}(\lambda x)}{\overline{R}(x)} \frac{\overline{R}(x)}{\overline{R}(x+1)}$$

we get $R_2(x) \to (\lambda^{\rho+1})^p$. It follows that

$$R^{(p)}(\lambda x)/R^{(p)}(x) \to \lambda^{\rho+(\rho+1)p}, \text{ i.e., } R^{(p)}(x) \in V_{p\rho+p+\rho}.$$

Application of [**9**; Theorem 4.1] now completes the proof of our lemma.

9. Examples. The following three examples are in correspondence to the three examples considered in [**10**;§8]. The former examples coincide with the case $p = 0$ of the present ones. Each of our examples is characterized by a certain q; as to the appertaining methods Q, the sequences $\overline{q}$, and the functions $\alpha(v)$ with $q_n = \int_0^1 v^n d\alpha(v)$ we refer the reader to [**10**]. The case $p = 0$ of Example 1 yields Hardy-Littlewood's theorem for A_0 whereas the general case is due to Korenblyum (see §1). The results of our Examples 2 and 3b, in the case $p = 0$, are due to Krishnan [**12**], [**13**]. In connection with example 3b and [**13**], the paper [**4**] is of interest.

Given two methods of summability C and D we say that a gap condition $G(k)$ is of type (C, D) if each series $\sum a_n$ summable C and satisfying $G(k)$ is necessarily summable D with consistency.

EXAMPLE 1. $q_n = 1, Q = A_0$, p fixed $(p = 0, 1, \cdots)$, $(C, q, p) = C_p$. Theorem 8.1 together with Lemma 8.2 (with $R(x) = 1$) yields $G(\mathbf{k})$ is of type (A_0, C_p) if

$$\liminf_{j\to\infty} \frac{k_{j+p+1}}{k_j} > 1. \tag{9.1}$$

EXAMPLE 2. $q_n = \binom{n+\beta}{n}, \beta$ real and fixed $(-1 < \beta < 0)$, $Q = A_\beta(A_\beta$ the generalized Abel method, cf. [**6**; Theorems (8.3) and (8.4)], [**5**; p. 18], [**19**; p. 186]), p fixed $(p = 0, 1, \cdots)$. Writing $\overline{q}_n$ in the form

$$\overline{q}_n = \mu_n(n+1)^{\beta+1} \text{ with } \mu_n \to \frac{1}{\Gamma(\beta+2)},$$

we see that in our present case (7.1) coincides with (9.1). Therefore Theorem 8.1 together with Lemma 8.2 (with $R(x) = \Gamma(x+\beta+1)/(\Gamma(x+1)\Gamma(\beta+1))$ yields: $G(k)$ is of type $(A_\beta,\ (C, q, p))$ if (9.1) is fulfilled. Here (C, q, p) can be replaced by (C, p).

EXAMPLE 3.1. $q_n = (n+1)^\gamma$, γ real and fixed $(-1 \le \gamma < 0)$, p fixed $(p = 0, 1, \cdots)$.

(a) *Case* $-1 < \gamma < 0$. Using Lemma 8.2 with $R(x) = (x+1)^\gamma$ we obtain: $G(k)$ is of type $(Q, (C, q, p))$ if (9.1) is fulfilled. Since (C, q, p) is equivalent to $(R, \{0, \overline{q}_0, \overline{q}_1, \cdots\}, p)$ (see Lemma 3.1), since $(R, \{0, \overline{q}_0, \overline{q}_1, \cdots\}, p)$ is equivalent to $(R, \{0, 1, 2, \cdots\}, p)$ (see [**3**; p. 35,

H_2]), and since $(R, \{0, 1, 2, \cdots\}, p)$ is equivalent to $(C, e, p) = C_p$ (again see Lemma 3.1), we have the following result: $G(k)$ is of type (Q, C_p) if (9.1) is fulfilled.

(b) *Case* $\gamma = -1$. Q is the logarithmic method L, $(C, q, 1) = M_0(q)$ is the logarithmic method ℓ, and (C, q, p) is equivalent to the method ℓ^p (Kuttner [**14**; Theorem 2]; ℓ^p the p-th power of the matrix ℓ). Writing $\overline{q}_n($ for $n = 1, 2, \cdots)$ in the form $\overline{q}_n = \xi_n \log(n+1)$ with $\xi_n \to 1$, we see that now (7.1) coincides with the condition

$$\liminf_{j\to\infty} \frac{\log k_{j+p+1}}{\log k_j} > 1. \tag{9.2}$$

Therefore (with $R(x) = (x+1)^{-1}$ in Lemma 8.2) we can state: $G(k)$ is of type (L, ℓ^p) if (9.2) is fulfilled.

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