# TOPOLOGICAL GAMES: ON THE 50TH ANNIVERSARY OF THE BANACH–MAZUR GAME

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This is an expository paper on infinite positional games of perfect information with special emphasis on their applications to set–theoretic topology. The oldest game of this kind is the Banach–Mazur game still inspiring new results, even after 50 years from its invention.

## Contents

**1. Introduction.** A combinatorial game in a mathematical form was described likely for the first time at the beginning of the XVII century. Bachet de Meziriac [1612] gave the following game: two players alternately choose numbers between 1 and 10; the player, on whose move the sum attains 100, is the winner. This kind of game, called Nim, was studied by Bouton [1901-2], and it has an extensive literature. For a bibliography on combinatorial games the reader is referred to Fraenkel [1983, 198?] and also to Berlekamp, Conway and Guy

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[1982]. For a comprehensive work on the history of game theory we refer to Worobjow [1975].

Zermelo [1913] investigated the game of chess and demonstrated that this (finite) game is determined; more precisely, either both players have drawing strategies, or else one player has a winning strategy. This result was extended by König [1927] and further by Kalmár [1929] for games with plays of finite length. Sprague [1935-36] and independently Grundy [1939] gave a complete analysis of finite games of two players, called impartial games with normal play, where the same set of options is available to each of the players and the last player wins (if a player cannot move, he loses the play).

Borel [1921, 1924, 1927], von Neumann [1928] and Steinhaus [1925] laid foundations for strategic games and essentially influenced the further study of positional games.

The notion of a positional game of perfect information was introduced in the famous monograph of von Neumann and Morgenstern [1944]. The authors proved that each finite positional game can be reduced to a matrix game, and moreover, if the positional game is one of perfect information, then the corresponding matrix has a saddle point (which is a pair of optimal strategies of the players). Robinson [1951] gave an alternative solution for (finite) matrix games by mean of an expansion of a given game into an infinite positional game of perfect information; instead of using mixed strategies in the matrix game, the players can use pure strategies in its positional expansion. The role of information in positional games was studied by Kuhn [1950, 1953].

Cantor [1883] and Bendixson [1883] proved that a subset $X$ of the real line is the union of a countable scattered set and a self–dense set; moreover, if $X$ is an uncountable closed subset of the real line, then $X$ contains a copy of the Cantor Discontinuum. The last conclusion was extended to uncountable Borel sets by Alexandrov and Hausdorff, and to uncountable analytic sets by Suslin (see Kuratowski [1966]). Sierpiński [1924] proved the Alexandrov–Hausdorff theorem in a framework which makes implicit use of an infinite positional game of perfect information (see Section 3 below).

Baire [1899] proved that the intersection of countably many open dense subsets of the real line is dense in the real line (Baire Density Theorem), and its consequence, that the real line has no countable cover by nowhere dense subsets (Baire Category Theorem). Hausdorff [1914] extended the conclusion to complete metric spaces, and Čech [1937] – to $G_\delta$ sets in compact Hausdorff spaces.

In 1935 Stefan Banach started a notebook, called the Scottish Book, where the mathematicians residing in or visiting Lwów proposed various mathematical problems (or conjectures) and also indicated their partial or complete solutions. In the same year Stanistaw Mazur proposed a game related to the Baire Category Theorem. The game is described in Problem 43 of the Scottish Book; its solution, given by Banach, is dated August 4, 1935. Hence, the game became known as the Banach–Mazur game. Problem 43 also contains some modifications of the Banach–Mazur game proposed by Banach and Ulam (see Sections 5 and 6 below), while Problem 67 deals with two set–theoretical variants of the Banach–Mazur game proposed by Banach (see Section 10 below). (We remark that Ulam [1960 Chapter I, §10, states that Mazur discovered the game about 1928 and Oxtoby [1971] also refers to this year; however, later, Ulam [1976] gives the year 1935 referring to a conversation in the Scottisch Café , where "Mazur proposed the first examples of infinite mathematical games").

Since the beginning of World War II to the middle 50's problems of the Scottish Book were disseminated mainly by oral communication. They became known also in the U.S.A. through Ulam, who prepared an English translation of the Scottish Book; the reprints were circulated since 1957, then followed a monograph edition of Ulam [1977], and now there is available the latest edition, by Mauldin [1981], with up–to–date comments.

Although the infinite positional games of perfect information were discovered and initially studied in Poland, the situation changed rapidly in the early 50's. Gale and Stewart [1953] in the U.S.A., and independently Mycielski and Zięba [1955] in Poland, originated the study of infinite positional games of perfect information in a general setting, where the positions in a game are vertices of a tree. Rabin [1957] considered an effective version of an infinite game; his paper initiated a strong interaction between infinite positional games and recursive functions. Mycielski and Steinhaus [1962] introduced a new axiom of set theory, the axiom of determinacy (AD), and Mycielski [1964a,b, 1966] derived interesting consequences from the axiom, such as the Baire property and Leqesgue measurability of any subset of the real line. Solovay [1967] proved that under AD the cardinal $\aleph_1$ is measurable and Martin [1975] proved (in ZFC) the determinacy of binary Borel games (see 5 below). These results attracted the attention of more set theorists to the techniques related to infinite games. In France, Choqet [1969a,b] introduced a modification of the Banach–Mazur game that initiated

various applications of infinite positional games to functional analysis (see 7 below). Vaught and Schilling [1979] introduced a generalization of the Suslin operation, called the Borel–game operation, which led to a far–reaching extension of the R–sets of Kolmogorov [1928] (see Kechris [1978], Schilling and Vaught [1983], and Burgess [1983b]).

The development of infinite positional games of perfect information can hardly be separated from that of strategic games, pursuit games, geometric games, differential games, stochastic games, statistical games, etc., because of the unconstrained flow of ideas from one branch of mathematics to another (cf. McKinsey [1952], Dresher [1961], Hájek [1975], Isaacs [1965], A. Friedman [1971], Wald [1950], Parthasarathy and Raghavan [1971], and Ruckle [1983]). Also the topic itself is not within one branch of mathematics only. The present author stresses mainly the topological studies arising from the Banach–Mazur game, while the results involving infinitary combinatorics, logic, recursive functions, descriptive set theory, definability and model theory, are mentioned very briefly.

The term "topological game" was introduced by Berge [1957a] (see also Berge [1957b] and Pears [1965]). Following an analogy with topological groups, Berge originated the study of positional games of the form $G(X, \Phi)$, where $X$ is a topological space and $\Phi : X \to \mathcal{P}(X)$ is an upper and/or lower semicontinuous multivalued map assigning to a position $x$ the set $\Phi(x)$ of the next legal positions ($\mathcal{P}(X)$ has the Vietoris topology); if $\Phi(x) = \emptyset$, then $x$ is a terminal position.

A somewhat different meaning for "topological game" was proposed by Telgársky [1975a, 1977b]. This term refers to an analogy with matrix games, differential games, statistical games, etc., so that topological games are defined and studied within topology. In a topological game the players choose some objects related to the topological structure of a space, such as points, closed subsets, open covers, etc., and moreover, the condition on a play to be winning for a player may also involve topological notions such as closure, a convergence, etc.

It turns out that topological games are related to (or, can be used to define) the Baire property, Baire spaces, completeness properties, convergence properties, separation properties, covering and base properties, continuous images, Suslin sets, singular spaces, etc.

We remark that some topological constructions have a natural counterpart (an interpretation) in the framework of infinite positional games of perfect information, while, on the other hand, from some topological properties defined by games, the game–theoretic language can be

eliminated (without leaving a trace).

**2. General framework for games and notation.** There are various frameworks for infinite positional games of perfect information (see Martin [1975], Moschovakis [1974], Mycielski [1964b], Parthasarathy and Raghavan [1971]); unless otherwise stated the notions defined below apply to all games considered in Sections 3–10.

We shall always consider games of two players, called Player I and Player II, where Player I starts the play (that is, he makes the first move). Unless otherwise stated, a play of a game is a sequence of type $\omega$, and the result of a play is either a win or a loss for a player. A strategy of Player II is a function defined for each legal finite sequence of moves of Player I; a strategy for Player I is defined similarly. A stationary strategy is a strategy which depends on the opponent's last move only; a Markov strategy is a strategy which depends only on the ordinal number of the move and the opponent's last move.

If $G$ is a game, then $I \uparrow G(II \uparrow G)$ denotes that Player I (Player II) has a winning strategy in $G$. A game $G$ is said to be determined if either $I \uparrow G$ or $II \uparrow G$. Games $G_1$ and $G_2$ are called equivalent if $I \uparrow G_1 \leftrightarrow I \uparrow G_2$ and $II \uparrow G_1 \leftrightarrow II \uparrow G_2$.

$R$ = the real line, $J$ = the closed unit interval, $\omega = \{0, 1, 2, \cdots\}, 2^\omega = \{0, 1\} \times \{0, 1\} \times \cdots$ ($2^\omega$ is homeomorphic to the Cantor Discontinuum $\omega^\omega = \omega \times \omega \times \cdots$ ($\omega^\omega$ is homeomorphic to the set of irrational numbers), $\mathcal{P}(X) = \{Y : Y \subset X\}$, $|X|$ = the cardinality of $X, \kappa$ = a cardinal number, $[X]^\kappa = \{Y \subset X \mid |Y| = \kappa\}$, $[X]^\kappa = \{Y \subset X : |Y| < \kappa\}$ and $[X]^{>\kappa} = \{Y \subset X : |Y| > \kappa\}$. If $E$ is a subset of a topological space $X$, then $\overline{E}$ denotes the closure of $E$ in $X; \mathcal{F}(X)$ denotes the collection of all closed subsets of $X$.

**3. The Sierpiński game.** Sierpiński [1924] proved the Alexandrov-Hausdorff theorem that each uncountable Borel set on the real line contains a copy of the Cantor Discontinuum by mean of strategies in an infinite positional game of perfect information. However, he used no terms like games, plays, or strategies. The advantage of Sierpiński's approach was recognized by Dellacherie [1969a], who introduced the notion of a scraper (rabotage – in French), and further applied the notion in [1969b,c,1970,1971,1872a,b], in particular, to the capacitability of sets. It turns out that the scraper corresponds to a winning strategy of Player II in a topological game. The explicit definition of the games related to scrapers was given by Telgársky [1977b]. Let $\mathcal{E}$ be a

family of subsets of a space $Y$ such that $(i)E \in \mathcal{E}$ and $E \subset E' \subset Y$ implies $E' \in \mathcal{E}$, and (ii) $\bigcup_{n<\omega} E_n \in \mathcal{E}$ implies $E_n \in E$ for some $m < \omega$. Moreover, let for each decreasing sequence $(A_n : n < \omega)$ of subsets of $Y$ be associated a family $H(A_n : n < \omega)$ of subsets of $Y$ such that (a) if $A \in H(A_n : n < \omega)$ and $A \subset A' \subset Y$, then $A' \in H(A_n : n < \omega)$, (b) if $B_0, B_1, \cdots \in H(A_n : n < \omega)$, then $\bigcap_{n<\omega} B_m \in H(A_n : n < \omega)$. Now, given a subset $X$ of $Y$, the game $S(X, Y, \mathcal{E}, H)$ is played as follows. The players choose alternately elements $E_0, E_1, \cdots$ of $\mathcal{E}$ so that $X \supset E_0 \supset E_1 \cdots$ Player II wins the play if $X \in H(E_n : n < \omega)$. If $II \uparrow S(X, Y, E, H)$, then $X$ is called a smooth set

In particular, if $X$ is an uncountable subset of a Euclidean space $Y$, if $\mathcal{E} = [X]^{>\omega}$, and if $H(A_n : n < w) = \{A \subset X : A \supset \bigcap_{n<\omega} \overline{E}_n\}$, we get the game of Sierpiński, denoted by $S(X, Y)$.

If $I \uparrow S(X, J)$, then $J \backslash X$ contains a copy of the Cantor Discontinuum; if $II \uparrow S(X, J)$, then $X$ contains a copy of the Cantor Discontinuum. If $X$ is an analytic set in $J$, then $II \uparrow S(X, J)$, but it is not known whether the converse implication holds. Kubicki [1982], [1986b] proved that if $J \backslash X$ contains an analytic set which is not Borel separated from $X$, then $I \uparrow S(X, J)$; hence, if $X$ is a coanalytic non-Borel subset of $J$, then $I \uparrow S(X, J)$. However, it is an unsettled question whether the game is determined for every set $X$ in the $\sigma$–algebra generated by analytic subsets of $J$. It turns out, however, that the family $\{X \subset J : II \uparrow S(X, J)\}$ is closed under the Suslin operation (see Kubicki [1986b]).

Kubicki [1986a] studied the modification $S_0(X, J)$ of $S(X, J)$, where the players choose countable selfdense subsets of $X$. He proved that if $X$ is not a strongly Baire subset of $J$, then $I \uparrow S_0(X, J)$; moreover, if $X$ is a $G_\delta$ subset of $J$, or if $J \backslash X$ is a Luzin set in $J$, then $II \uparrow S_0(X, J)$.

We remark that by a theorem of Galvin and Telgársky [1986], winning strategies in $S(X, J)$ or $S_0(X, J)$ can be replaced by stationary winning strategies.

**4. The Banach–Mazur game.** This is the first infinite positional game of perfect information studied by mathematicians. The game was proposed in 1935 by the Polish mathematician Stanislaw Mazur and recorded in the Scottish Book (see Mauldin [1981], Problem 43).

Given a subset $X$ of the unit interval $J$, the players alternately choose subintervals $J_0, J_1, \cdots$ of $J$, where $J_0 \supset J_1 \supset \cdots$; Player I wins the play if and only if $X \cap \bigcap_{n<\omega} J_n \neq \emptyset$. Denote this game by $MB(X, J)$.

Mazur observed that (a) if $X$ is residual in an interval $J_0 \subset J$, then

$I \uparrow MB(X, J)$, and that (b) if $X$ is a set of the first category in $J$, then $II \uparrow MB(X, J)$, and he posed the question whether the converse implications hold in (a) and (b). This question was answered in the affirmative by Stefan Banach (see Mauldin [1981], Problem 43; August 4, 1935), but his proof was never published. The game and its solution, however, became known as the Banach–Mazur game, and its importance went far beyond the Baire category classification.

The theorem on the Banach–Mazur game was published for the first time by Mycielski, Świerczkowski and Zięba [1956], but its proof was postponed to an expanded paper. Soon after, Oxtoby [1957] published the proof of the theorem in the following more general setting.

Let $X$ be a subset of a topological space $Y$, and let $W$ be a family of subsets of $Y$ such that (i) each $W$ in $\mathcal{W}$ contains a nonempty open subset of $Y$, and (ii) each nonempty open subset of $Y$ contains an element $W$ of $\mathcal{W}$. In this game, denoted here by $MB(X, Y, \mathcal{W})$, the players alternately choose elements $W_0, W_1, \cdots$ of $\mathcal{W}$ where $W_0 \supset W_1 \supset \cdots$. Player I wins the play if and only if $X \cap \bigcap_{n < \omega} W_n \neq \emptyset$.

Oxtoby [1957] proved that $II \uparrow MB(X, Y, W)$ if and only if $X$ is of the first category in $Y$, and, when assuming that $Y$ is a complete metric space, that $I \uparrow MB(X, Y, W)$ if and only if X is residual in some nonempty open subset of $Y$. Hence, if $X$ has the Baire property in $Y$, then the game $MB(X, Y, W)$ is determined. Independently, Volkmann [1959] published similar results for the Banach–Mazur game on subsets of the unit ball in a n–dimensional Euclidean space. Morgan [1974] considered games of the form $BM(X, Y, W)$, where $W$ satisfies other conditions that (i) and (ii) above, but is designed for the study of a Baire–type classification and singular sets, both related to the family $W$ (e.g., $Y = R$ and $W$ consists of all subsets of order type $1 + \lambda + 1$). For other studies of singular sets by mean of infinite games the reader is referred to Stockii [1962], Gardner [1979], and Méndez [1981].

Schmidt [1966] introduced a generalization of the Banach–Mazur game, where the choices of both players are constrained, for an application in number theory. He used a particular case of this games to prove that there are continuum many badly approximable numbers (see also Schmidt [1969], 1980], and Freiling [1982, 1984a]). Given two real numbers $a, b \in (0, 1)$ and a subset $X$ of the n–dimensional Euclidean space $R^n$, the players alternately choose closed balls $A_0, B_0, A_1, B_1, \cdots$ in $R^n$ so that $A_n \supset B_n \supset A_{n+1}$, $r(B_n) = a \cdot r(A_n)$ and $r(A_{n+1}) = b \cdot r(B_n)$ for each $n < \omega$, where $r(B)$ is the radius of $B$. Player II wins the play

if and only if $\bigcap_{n<\omega} B_n \subset X$. Moreover, Schmidt [1966] proved that for
the general games, any winning strategy of Player II can be reduced
to a stationary winning strategy. This result was further extended by
Galvin and Telgársky [1986].

Choquet [1958] introduced two generalizations of Čech complete
spaces, the siftable and the strongly siftable spaces (espaces tamisable
et fortement tamisable – in French). It turns out that Čech complete
$\Rightarrow$ strongly siftable $\Rightarrow$ Baire. Moreover, siftable and strongly siftable
spaces can be defined in terms of stationary strategies in suitable mod-
ifications of the Banach–Mazur game.

For, let $BM(X)$ denote the modification of the game $MB(X,Y,W)$,
where $X = Y, \mathcal{W}$ is the family of all nonempty open sets in $X$, and
Player II wins a play $(W_0, W_1, \cdots)$ if and only if $\bigcap_{n<\omega} W_n \neq \emptyset$. Then
$X$ is siftable if and only if Player II has a stationary winning strategy
in $BM(X)$. The strongly siftable spaces are related to the game in-
troduced by Choquet [1969a] (see the game $\mathrm{Ch}(X)$ in 7 below), where
both properties were defined by mean of topological games and both
renamed to $\alpha$–favorable and strongly $\alpha$–favorable respectively. We re-
mark, however, that the lectures of Choquet [1969a,b,c] contain no
reference to any earlier paper on infinite positional games.

White [1974, 1975b] introduced the following notion: $X$ is called
weakly $\alpha$–favorable if Player II has a winning strategy in $BM(X)$.
Krom [1974b] observed that $X$ is a Baire space if and only if Player
I has no winning strategy in $BM(X)$ (see also Haworth and McCoy
[1977], but the result is essentially due to Oxtoby [1957]). Hence it
follows, in particular, that each weakly $\alpha$–favorable space is a Baire
space. On the other hand, every Bernstein set on the real line is a
Baire space which is not weakly $\alpha$–favorable. In fact, from the result
of Oxtoby [1957] it follows that a metric space is (weakly) $\alpha$–favorable
if and only if it has a dense $G_\delta$ subset which is completely metrizable.
White [1974, 1975b] extended the last equivalence for $T_0$ spaces with a
base of countable order. For further extension the reader is referred to
Galvin and Telgársky [1986].

A Baire space $X$ such that $X \times X$ is of the first category, is called
a barely Baire space (Fleissner and Kunen [1978]). Using the contin-
uum hypothesis (CH), Oxtoby [1961] constructed a completely regular
barely Baire space; another construction, also using CH, was given by
White [1975a]. Krom [1974a] defined an ultrametric space $X^{\#}$ asso-
ciated with a topological space $X$, such that $X^{\#} \times Y$ is Baire if and
only if $X \times Y$ is Baire. As a consequence, from the barely Baire space

of Oxtoby [1961], he obtained a (non–separable) metric space which is
a barely Baire space. Using a technique derived from forcing, Cohen
[1976] constructed an absolute example of a barely Baire space (that is,
without the assumption of CH). Fleissner and Kunen [1978] presented
a direct combinatorial construction of a barely Baire space. Moreover,
they showed that for each $m \leq \omega$ there is a (metric) space $X$ such that
$X^n$ is a Baire space for $n < m$, while $X^m$ is of the first category, and
proved also further results on products of Baire spaces.

The above results on Baire spaces are in sharp contrast with those
on weakly $\alpha$–favorable spaces. Choquet [1969a] proved that $II \uparrow$
$BM(P_{t \in T} X_t)$ provided that $II \uparrow BM(X_t)$ for every $t \in T$. White
[1974, 1975b] extended this result to arbitrary box products. More-
over, if $II \uparrow BM(X)$ and if $Y$ is a Baire space, then $X \times Y$ is a Baire
space (Choquet [1969a] and White [1974, 1975b]).

However, a closed subset or a $G_\delta$ subset of a weakly $\alpha$–favorable space
need not be weakly $\alpha$–favorable (for example, let $X = R^2 \backslash \{(p, 0):\ p$
is irrational$\}$). Therefore weakly $\alpha$–favorable spaces do not satisfy the
completeness axioms postulated by Wicke and Worrell [1972, 1974].
However, the property "weakly $\alpha$–favorable" is preserved by the equiv-
alence of topologies on a set $X$ introduced by Todd [1981]. For more in-
formation on Baire spaces and completeness–type properties the reader
is referred to Aarts and Lutzer [1974], Haworth and McCoy [1977], and
Császár [1978].

Although the question whether weakly $\alpha$–favorable implies $\alpha$–favor-
able was known to many authors, it appeared in print for the first time
in the paper of Fleissner and Kunen [1978]. White [1974, 1975b] proved
that if $X$ is a weakly $\alpha$–favorable space with a $\sigma$–disjoint pseudo–base,
then $X$ is $\alpha$–favorable. This result was generalized by Galvin and
Telgársky [1986], where $X$ is assumed to have a pseudo–base $\mathcal{B}$ such
that $\mathcal{B} = \cup_{n < \omega} \mathcal{B}_n$ with $\{B' \in \mathcal{B}_n : B'_n \supset B\}$ finite for every $n < \omega$ and
$B \in \bigcup \mathcal{B}_n$. Debs [1984] proved that the real line with the finer topol-
ogy determined by basic open sets of the form $(a, b) \backslash C$, where $C$ is
countable, provides a weakly $\alpha$–favorable non–regular Hausdorff space
which is not $\alpha$–favorable. Later, in [1985], he constructed a completely
regular space which is weakly $\alpha$–favorable but not $\alpha$–favorable.

A quasi–regular space $X$ is said to be pseudo–complete (Oxtoby
[1961]), if there is a sequence $\mathcal{B}_0, \mathcal{B}_1, \cdots$ of open pseudo–bases in $X$
such that $\bigcap_{n < \omega} B_n \neq \emptyset$ whenever $\overline{B_{n+1}} \subset B_n \in \mathcal{B}_n$ for every $n < \omega$.
White [1974, 1975b] observed that a pseudo–complete space is weakly
$\alpha$–favorable. Galvin and Telgársky [1986] proved that a Markov win-

ning strategy of Player II in $BM(X)$ can be reduced to a station-
ary winning strategy. Hence it follows that pseudo–complete space is
$\alpha$–favorable. However, it is not known whether each $\alpha$–favorable space
is pseudo–complete.

Debs [1985] and independently Galvin and Telgársky [1986] proved
that if Player II has a winning strategy in $BM(X)$, then he has a win-
ning strategy depending only on two preceding moves (that is, on the
last one made by Player I and on the earlier one by Player II). However,
it is an unsettled question (posed by Debs [1985] whether a winning
strategy of Player II in $BM(X)$ can be reduced to a winning strategy
which depends only on last two moves of Player I. (I conjecture that
a winning strategy depending only on the last $k + 1$ moves of Player
I cannot be reduced, in general, to a winning strategy depending only
on his last $k$ moves.)

Galvin and Telgársky [1986] introduced the following equivalent form
of the game $BM(X)$. Given a subset $X$ of a Tihonov cube $J^\kappa$, the
players alternately choose open subsets $W_0, W_1, \cdots$ of $J^\kappa$ such that
$W_{n+1} \subset W_n$ and $W_n \bigcap X \neq \emptyset$ for each $n < \omega$; Player II wins the
play if and only if $X \cap \bigcap_{n<\omega} W_n \neq \emptyset$. It turns out that if $X$ is a
completely regular space such that $II \uparrow BM(X)$, then Player II has
a stationary winning strategy in the modified game. They also de-
fined the dual game $BM^*(X)$ to the game $BM(X)$ as follows: Player I
chooses a pseudo–base $\mathcal{B}_0$ for $X$, Player II chooses a set $B_0 \in \mathcal{B}_0$, Player
I chooses a pseudo–base $\mathcal{B}_1$ for $B_0$, Player II chooses a set $B_1 \in \mathcal{B}_1$, and
so forth; Player I wins the play if and only if $\bigcap_{n<\omega} B_n \neq \emptyset$. It turns out
that $I \uparrow BM^*(X) \leftrightarrow II \uparrow BM(X)$, and $II \uparrow BM^*(X) \leftrightarrow I \uparrow BM(X)$.
Moreover, if $II \uparrow BM(X)$, then Player I has a stationary winning strat-
egy in $BM^*(X)$.

Gerlits and Nagy [1980] studied two modifications of the game
$BM(X)$, where the condition $\bigcap_{n<\omega} W_n \neq \emptyset$ was considerably strength-
ened. Denote their games by $GN_0(X)$ and $GN_1(X)$. Player II
wins the play $(W_0, W_1, \cdots)$ in $GN_0(X)$ if and only if each sequence
$(x_0, x_1, \cdots) \in W_0 \times W_1 \times \cdots$ has a cluster point in $X$; he wins the play
in $GN_1(X)$ if and only if $\{W_n : n < \omega\}$ is a neighborhood base of a
point in $X$. Gerlits and Nagy [1980] showed that if $X$ if Čech complete,
then $II \uparrow GN_0(X)$; moreover, that $II \uparrow GN_1(X)$ if and only if $X$ has
a dense $G_\delta$ subset which is completely metrizable.

Let $BM(K, X)$ denote the following modification of the game $BM(X)$.
Given a space $X$ and a class $K$ of spaces, the game is played as $BM(X)$,
but Player II chooses with $V_n$ also a set $E_n \in \mathcal{P}(X) \bigcap K$; moreover,

Player II wins the play $(U_0, (V_0, E_0), U_1, (V_1, E_1), \cdots)$ if and only if $\bigcap_{n<\omega} V_n \cap \overline{\bigcup_{n<\omega} E_n} \neq \emptyset$. If $K = \{X\}$, then $BM(K, X)$ coincides with $BM(X)$. If $K = \{\{x\} : x \in X\}$, then $BM(K, X)$ coincides with the game $G_\sigma$ of Saint–Raymond [1983]. If $K$ consists of all $K$–analytic subsets of $X$, then $BM(K, X)$ coincides with the game $J_a(X)$ studied by Debs [1986]; he proved that if Player I has no winning strategy in the last game, then $X$ is a Namioka space (see 7 below). He has also shown that if $X$ is a $K$–analytic, regular, and a Baire space, then $II \uparrow BM(\{\{x\} : x \in X\}, X)$.

**5. The Ulam game.** In 1935, Stanisław Ulam proposed the following modification of the Banach–Mazur game: given a subset $X$ of the real line, the players I and II alternately choose binary digits $x_0, x_1, \cdots$; Player I wins the play if and only if $(x_0, x_1, \cdots)_2 \in X$, that is, $\sum_{n=0}^{\infty} \frac{x_n}{2^n} \in X$ (see Mauldin [1981], Problem 43).

This game is also called the binary game and it is equivalent to the following setting: given a subset $X$ of the Cantor Discontinuum $2^\omega$, the players alternately choose $x_0, x_1, \cdots \in \{0,1\}$; Player I wins the play $(x_0, x_1, \cdots)$ if $(x_0, x_1, \cdots) \in X$ (see Gale and Stewart [1953] and Mycielski and Zięba [1955]). Denote this game by $U(X)$.

We remark that the complexity of this game bears more of a combinatorial than a topological character. In fact, each finite positional game of perfect information is isomorphic to a game $U(X)$, where $X$ is a closed–open subset of $2^\omega$. Stein and Ulam [1955] and Walden [1957] studied the f inite version of $U(X)$, where $X \subset \{0,1\}^n$, with the aid of computers. On the other hand, it turned out that the study of the Ulam game with infinite plays requires advanced methods of descriptive set theory and infinitary combinatorics.

Gale and Stewart [1953], and independently Mycielski and Zięba [1955] proved that the Ulam game is undetermined for every Bernstein set in $2^\omega$; they also posed the question whether the game is determined for every Borel set. This problem turned out to be a very difficult one. Mycielski and Zięba [1955] showed that the Ulam game is determined for open subsets and also for closed subsets of $2^\omega$, Gale and Stewart [1953] – for sets in the Boolean algebra generated by open subsets of $2^\omega$, Mycielski, Świerczkowski and Zięba [1956] and Wolfe [1955] – for $F_\sigma$ sets and $G_\delta$ sets, and Davis [1964] – for $F_{\sigma\delta}$ sets and $G_{\delta\sigma}$ sets. Mycielski [1964a] observed that under the axiom of constructability ($V = L$) the Ulam game is undetermined for an analytic set. Martin [1970] proved that the game is determined for Borel or even analytic sets when as-

suming strong combinatorial properties of cardinals; in particular, that
if a measurable cardinal exists, then the game is determined for ana-
lytic sets. After that Paris [1972] proved (even in ZF) that the game is
determined for $F_{\sigma\delta\sigma}$ sets and $G_{\delta\sigma\delta}$ sets. Finally, Martin [1975] proved
(in ZFC) that the Ulam game is determined for any Borel set. His
original proof is rather complex (see also Martin and Kechris [1980]
and Martin [1985, 198?b]), but it is based on a simple idea: the game
on a Borel set is shown to be equivalent to a modified game on an open
set.

In the mean time, however, infinite games of perfect information en-
tered into the foundations of mathematics.

Jan Mycielski and Hugo Steinhaus [1962] introduced a new axiom
of set theory, called the axiom of determinateness (AD for short): the
Ulam game $U(X)$ is determined for every subset $X$ of $2^\omega$.

This statement is incompatible with the well–orderability of the con-
tinuum, so it contradicts the axiom of choice. This new competitive
principle raised again the need of an evaluation of the role of the ax-
iom of choice (see Comfort [1979] and Jech [1973, 1981]). AD destroys
the familiar view of ZFC models by making the cardinal numbers very
unusual, e.g., $\aleph_1$ and $\aleph_2$ become measurable (Solovay [1967]). On the
other hand, AD makes life easier with the analysis involving separable
metric spaces. Also, AD yields the formally stronger statement that
each win–lose two–player positional game on a tree with at most count-
ably many successors and of height $\omega$ is determined (Mycielski [1964b]).
Here are some consequences of this statement: (a) each subset of the
real line is Lebesque measurable, (b) each subset on the real line has
the Baire property, (c) each uncountable subset of the real line contains
a copy of the Cantor Discontinuum, and (d) each strongly measure zero
set of the real line is countable. (Each of these statements is equivalent
to the determinacy of an infinite positional game of perfect informa-
tion.) Already the determinacy of $U(X)$ for low levels of the projective
hierarchy requires strong combinatorial principles (see Martin [1978]).
It follows from the observation of $S$. Aanderaa (see Fenstad [1971])
that AD implies $\beta\omega\backslash\omega = \emptyset$. This consequence of AD has no appeal for
topologists (see Comfort [1979]). Nevertheless, AD admits ultrafilters
on some uncountable well–ordered sets (see Becker [1981a,b], Steel and
Van Wesep [1982], and Mignone [1979, 1981, 1983]).

Consequences of AD were initially studied by Mycielski [1964a,b,
1966], and Mycielski and Świerczkowski [1964], but after the results of
Solovay [1967] and Martin [1975], AD attracted the attention of many

mathematicians involved in logic, set theory, model theory, descriptive set theory and recursive functions.

There are two main streams coming out of the invention of Mycielski and Steinhaus. In the first stream there are studied consequences of AD, statements consistent with AD, and even some strengthenings of AD; there is a hope that the relative consistency of AD will be proved by constructing a suitable model. In the second stream, AD is considered to be an extremely strong postulate, so the determinacy is restricted to projective sets (the axiom of projective determinacy, PD), or even to some classes of projective sets only, like $\Sigma_n^1$ or $\Pi_n^1$. Moreover, in this stream there are preferred definable objects for a mathematical study, in contrast to those objects, whose existence is assured by a new axiom. However, both streams bring valuable results and have a very strong interaction.

Mycielski [1964a] introduced the following generalization of the binary game. There is given a nonempty set $B$, an ordinal $\alpha$, and a subset $X$ of the Cartesian product $B^\alpha$ of $\alpha$ copies of $B$. Let $\xi < \alpha$; if $E$ is even, then Player I chooses an element $b_\xi$ of $B$; if $\xi$ is odd, then Player II chooses an element $b_\xi$ of $B$. Player I wins the play $(b_\xi : \xi < \alpha)$ if and only if $(b_\xi : \xi < \alpha) \in X$. Denote this game by $U_B^\alpha(X)$. Then the game $U(X)$ coincides with $U_2^\omega(X)$, where $2 = \{0, 1\}$.

Let $AD_B^\omega$ denote the statement: the game $U_B^\omega(X)$ is determined for every $X \subset B^\alpha$. Mycielski [1964 a] showed that $AD_2^\omega \Leftrightarrow AD_\omega^\omega$, and moreover, that $AD_{\omega_1}^\omega$ and $AD_2^{\omega_1}$ are false in ZF. Blass [1975], and independently Mycielski in 1967 (unpublished), showed that $AD_R^\omega \Leftrightarrow AD_\omega^{\omega^2}$. For further results on $U_B^\alpha(X)$ and its generalizations the reader is referred to Moschovakis [1981], Hodges and Shelah [1981], and Harrington and Kechris [1981, 1982].

Mycielski [1964a] observed that a strategy of a player in $U_B^\omega(X)$ corresponds to a continuous function from $B^\omega$ to $B^\omega$ satisfying a Lipschitz condition. More precisely, endow $B^\omega$ with the metric defined as follows. If $(a_0, a_1, \cdots) \neq (b_0, b_1, \cdots)$, then the distance is $2^{-n}$, where $n = \min\{k < \omega : a_k \neq b_k\}$, and otherwise it is 0. Since $B^\omega \times B^\omega$ is homeomorphic to $B^\omega$, we may assume that $X \subset B^\omega \times B^\omega$. Then $I \uparrow U_B^\omega(X)$ if and only if $X$ contains the graph of a continuous function satisfying the Lipschitz condition with constant $1/2$, and $II \uparrow U_B^\omega(X)$ if and only if $(B^\omega \times B^\omega) \setminus X$ contains the graph of a continuous function satisfying the Lipschitz condition with constant 1. From this he concluded, that AD implies that for each set $X \subset 2^\omega \times 2^\omega$, either $X$ or

$(2^\omega \times 2^\omega) \backslash X$ contains the graph of a continuous function $f : 2^\omega \to 2^\omega$. Blass [1973] proved the converse implication; moreover, he indicated that the Borel determinacy of the Ulam game is equivalent to the following: for each Borel set $X \subset 2^\omega \times 2^\omega$, either $X$ or $(2^\omega \times 2^\omega) \backslash X$ contains the graph of a continuous function $f : 2^\omega \to 2^\omega$.

Wadge [197?] studied games inducing continuous maps and corresponding quasi–ordering relations on $\mathcal{P}(B^\omega)$. These games are called Wadge games and the equivalence classes determined by the quasi–orderings are called Wadge degrees (they are similar to Turing degrees). For more information the reader is referred to Van Wesep [1978a], Martin and Kechris [1980], Moschvakis [1980], and Kunen and Miller [1983].

Blass [1972b] studied other orderings on $\mathcal{P}(B^\omega)$ for comparing the options of the win for a player on various sets. He proved, e.g., that there are "many" incomparable undetermined games $U_B^\omega(X)$, where $X \subset B^\omega$.

Blackwell [1970] studied the complexity of winning strategies in games $U_\omega^\omega(X)$, where $X$ is a $G_\delta$ set, by mean of ordinal numbers. Büchi [1983] studied winning strategies presented by state–recursion provided that $X$ is an $F_{\sigma\delta} \cap G_{\delta\sigma}$ set (see also Büchi [1970]). Clote [1982, 198?] studied the complexity of winning strategies in $U_\omega^\omega(X)$ for clopen sets $X$. For other results the reader is referred to Moschovakis [1980].

Kechris, Kleinberg, Moschovakis and Woodin [1981] obtained a purely set theoretical characterization of AD within $R^+$, the smallest admissible set containing the continuum; namely, AD holds in $R^+$ if and only if for each ordinal $\alpha$ the power set $\mathcal{P}(\alpha)$ exists and there are arbitrarily large cardinals with the strong partition property. Kechris [1984] proved that if AD holds in $L[R]$, then DC, the principle of dependent choice, holds in $L[R]$ as well. Solovay [1978] proved that DC does not follow from AD; more precisely, that Con $(ZF + AD_R^\omega) \Rightarrow$ Con$(ZF + AD_R^\omega + \neg DC)$. For other results on AD or PD the reader is referred to the papers of Rao [1970], Fenstad [1971], H. M. Friedman [1971], Kechris [1973, 1975, 1979], Martin [1977, 1978, 198?a] and Henle [1981], the books of Kleinberg [1977] and Moschovakis [1980], and to the Cabal Seminar Proceedings edited by Kechris and Moschovakis [1978], and Kechris, Martin and Moschovakis [1981, 1983].

From the results of Harrington [1978] and Steel [1980] it follows that the assumption of the determinacy for all analytic sets in the Ulam game is equivalent both to the proposition that all analytic non–Borel sets on the real line are Borel isomorphic and also to the proposition

that every analytic non–Borel set on the real line is universal, in the generalized sense, for the analytic sets on the real line (see Rogers and Jayne [1980]).

Davis [1964] studied the following modification of the Ulam game, proposed by L. E. Dubins. Given a subset $X$ of $2^\omega$, Player I chooses a finite sequence $s_0$ of binary digits, Player II chooses a binary digit $x_0$, Player I chooses a finite sequence $s_1$ of binary digits, Player II chooses a binary digit $x_1$, and so forth. Player I wins the play $(s_0, x_0, s_1, x_1, \cdots)$ if and only if $s_0 x_0 s_1 x_1 \cdots \in X$. Denote this game by $D(X)$. Davis [1964] proved that $I \uparrow D(X)$ if and only if $X$ contains a copy of $2^\omega$, and, that $II \uparrow D(X)$ if and only if $X$ is countable. If this game is further modified to allow the choice of finite sequences of binary digits for both players, then the resulting game is equivalent to the Banach–Mazur game, more precisely, to $MB(X, 2^\omega)$ (see Mycielski [1964a]).

Mycielski [1966] proposed the following set–theoretical game. Given a set $X$, Player I chooses a subset $A_0$ of $X$, Player II chooses a $B_0 \in \{A_o, X \backslash A_0\}$, Player I chooses an $A_1 \subset B_0$, Player II chooses a $B_1 \in \{A_1, B_0 \backslash A_1\}$, and so forth (i.e., Player I cuts and Player II chooses). Player I wins the play $(A_0, B_0, A_1, B_1, \cdots)$ if and only if $\bigcap_{n < \omega} B_n \neq \emptyset$. Denote this game by $M(X)$. It turns out that this game is similar to the game $D(X)$. Galvin, Mycielski and Solovay [1971, 1974] proved that $I \uparrow M(X)$ if and only if $X$ has the cardinality at least continuum, and $II \uparrow M(X)$ if and only if $X$ is countable; they also studied various set–theoretic modifications of $M(X)$. Ulam [1964] invented a similar game, where both players cut and choose. Telgársky [1977b] proposed several topological modifications of $M(X)$.

Kechris [1977] defined the following modification of $D(X)$. Given $X \subset \omega^\omega$, Player I chooses a finite sequence $s_0$ from $\omega$, Player II chooses a $k_0$ in $\omega$, Player I chooses a nonempty finite sequence $s_1$ from $\omega$, Player II chooses a $k_1$ in $\omega$, Player I chooses a nonempty finite sequence $S_2$ from $\omega$, Player II chooses a $k_2$ in $\omega$, and so forth. Player I wins the play if and only if $s_0 s_1 \cdots \in X$ and if for each $n < \omega$ the first member of $s_{n+1}$ is bigger that $k_n$. Denote this game by $K(X)$. Kechris [1977] proved that $I \uparrow K(X)$ if and only if $X$ contains a closed copy of $\omega^\omega$ (that is, $X$ contains a closed copy of the space of irrational numbers), and $II \uparrow K(X)$ if and only if $X$ is $\sigma$–totally bounded in $\omega^\omega$ (that is, $X$ is contained in a $\sigma$–compact subset of $\omega^\omega$). He also considered various modifications of $K(X)$ related to the notions of small sets.

Kechris [1981] introduced a modification of the game $U_\omega^\omega(X)$, where the players choose binary splitting trees $T_n \subset [\omega]^{<\omega}$ such that $T_0 \supset$

$T_1 \supset \cdots$ and each $T_{n+1}$ has the diameter less that $1/(n+1)$, and he studied the properties of forcing with $\Delta$ perfect trees.

**6. The Banach game.** In 1935, Stefan Banach proposed the following game: given a subset $X$ of the real line, the players alternately choose positive real numbers $x_0, x_1, \cdots$, where $x_{n+1} < x_n$ for every $n < \omega$; Player I wins the play if and only if $\Sigma_{n=0}^{\infty} x_n$ exists and is in $X$ (see Mauldin [1981], Problem 43). Denote this game by $B(X)$.

Turowicz [1955] showed that $I \uparrow B(P)$ where $P$ is the set of all irrational numbers. Hence $II \uparrow B(Q)$ where $Q$ is the set of all rational numbers. Zubrzycki [1957] showed that $II \uparrow B(X)$ if $X$ is any countable subset of the real line. Zięba [1957] defined a set $S \subset (0,1)$ such that $II \uparrow B(S)$, but $I \uparrow B(S \bigcup\{1\})$. Hartman [1957] gave a sufficient condition for $X$ which yields $II \uparrow B(X)$; his observation was improved by Reichbach [1957], who constructed a nowhere dense perfect set $X$ of Lebesgue measure 0 such that $I \uparrow B(X)$.

Hanani [1960], and Hanani and Reichbach [1961] studied various modifications of the Banach game and gave a characterization of compact subsets of $J = [0,1]$ for which Player I has a winning strategy. Mycielski [1964b] introduced further modifications of the Banach game, where the series in the result of a play is replaced by the limit, and he proved that they are determined for $F_\sigma$ sets and $G_\delta$ sets (and some for $F_{\sigma\delta}$ sets and $G_{\delta\sigma}$ sets). Moran [1971] proved that the Banach game and two modifications of the game are undetermined for Bernstein sets.

Let ADB denote the statement: the game $B(X)$ is determined for each $X \subset R$. In 1979, D. A. Martin proved that ADB$\Rightarrow$AD. This and other results are found in the paper of Freiling [1984b], while Becker [1985] proved that AD$\Rightarrow$ADB. Therefore, in ZF, ADB$\Leftrightarrow$AD.

Ehrenfeucht and Moran [1973] introduced several modifications of the Banach game, called size–direction games. Two of these games, denoted by $\Gamma^S(X)$ and $\Gamma^D(X)$, are defined as follows. Given a subset $X$ of the real line, in $\Gamma^S(X)$ Player I chooses two positive real numbers $s_0$ and $x_0$, Player II chooses an $\varepsilon_0 \in \{-1,1\}$, Player I chooses a positive real number $x_1$, Player II chooses an $\varepsilon_1 \in \{-1,1\}$, and so forth. In $\Gamma^D(X)$ Player I chooses a positive real number $s_0$, Player II chooses an $\varepsilon_0 \in \{-1,1\}$, Player I chooses a positive real number $x_0$, and so forth. Put $s_{n+1} = s_n + \varepsilon_n x_n$ for each $n < \omega$. In both games, Player I wins the play if and only if $\lim_{n\to\infty} s_n$ exists and is in $X$. Ehrenfeucht and Moran [1973] proved that $II \uparrow I^D(X)$ if and only if $X$ is discrete, and $I \uparrow I^D(X)$ if and only if $X$ is not discrete; moreover, if $X$ is countable,

then $II \uparrow I^S(X)$. Furthermore, they considered positional strategies in these games, while Moran [1973] considered also continuous strategies and some further modifications of these games. For further results we refer to Moran and Shelah [1973].

Freiling [1984b] introduced and studied various modifications of the Banach game. In particular, in the game $B^*(X)$ the rules are the same as in $B(X)$, but Player I wins the play $(x_0, x_1, \cdots)$ if and only if $\Sigma_{n=0}^{\infty}(-1)^n x_n$ exists and is in $X$. He proved that $I \uparrow B^*(X)$ if and only if there is an $a > 0$ such that $(0, a)\backslash X$ is countable, and $II \uparrow B^*(X)$ if and only if for each $a > 0$ the set $(0, a)\backslash X$ contains a copy of $2^\omega$. In another modification, denoted by $B^{**}(X)$, each player is allowed to make two moves in a row, so that a play is a sequence $((x_0, x_1), (x_2, x_3), \cdots)$, where $x_0 > x_1 > x_2 > x_3 > \cdots$. Player I wins the play if and only if $\Sigma_{n=0}^{\infty} x_n$ exists and is in $X$. He proved that $I \uparrow B^{**}(X)$ if and only if $X$ is residual in some interval $(a, b)$, where $0 < a < b$, and $II \uparrow B^{**}(X)$ if and only if $x \bigcap (0, \infty)$ is of the first category.

**7. The Choquet game.** Čech [1937] showed that the Baire Density Theorem is valid for the absolute $G_\delta$ spaces, later called Čech complete spaces. Choquet [1951] pointed out several difficulties in generalizing completeness so that the Baire Density Theorem would still be valid. In [1958] he introduced the siftable and strongly siftable spaces; the siftable spaces were discussed in Section 4, while the strongly siftable spaces are related to the following game he introduced in [1969a].

Given a topological space $X$, Player I chooses a point $x_0$ in $X$ and its open neighborhood $U_0$, Player II chooses an open neighborhood $V_0$ of $x_0$ with $V_0 \subset U_0$, Player I chooses a point $x_1$ in $V_0$ and its open neighborhood $U_1$ with $U_1 \subset V_0$, Player II chooses an open neighborhood $V_1$ of $x_1$ with $V_1 \subset U_1$, and so forth; Player II wins the play $((x_0, U_0), V_0, (x_1, U_1), V_1, \cdots)$ if and only if $\bigcap_{n<\omega} V_n \neq \emptyset$. Denote this game by $\mathrm{Ch}(X)$.

It turns out that $X$ is strongly siftable if and only if Player II has a stationary winning strategy in $\mathrm{Ch}(X)$. Porada [1979] showed that each Čech complete space is strongly siftable. The converse implication does not hold, because also countably compact spaces and scattered spaces are strongly siftable (see Telgársky [1982]). However, Choquet [1969a] proved that if $X$ is metrizable, then $II \uparrow Ch(X)$ if and only if $X$ is Čech complete (i.e., $X$ is metrizable by a complete metric); moreover, $II \uparrow \mathrm{Ch}(P_{t\in T} X_t)$ provided that $II \uparrow \mathrm{Ch}(X_t)$ for each $t \in T$. Earlier, Frolík [1961] introduced countably complete spaces and proved

that $P_{t \in T} X_t$ is countably complete provided that all $X_t$ are countably complete. It turns out that $II \uparrow \mathrm{Ch}(X)$ if $X$ is countably complete. Telgársky [1986] considered the following property: $X$ is regular and it has a sequence $(\mathcal{B} : n < \omega)$ of open bases such that $\bigcap_{n<\omega} \mathcal{B}_n \neq \emptyset$ whenever $\overline{\mathcal{B}_{n+1}} \subset \mathcal{B}_n \in \mathcal{B}_n$ for each $n < \omega$. This property is weaker than the countable completeness, it is parallel to the pseudo–completeness of Oxtoby [1961], and it implies that Player II has a Markov winning strategy in $\mathrm{Ch}(X)$. However, it is not known whether it implies that $X$ is strongly siftable.

Choquet [1969b] proved that if $X$ is the set of all extreme points of a compact convex set in a locally convex linear space, then $II \uparrow \mathrm{Ch}(X)$. This result initiated applications of topological games in functional analysis. For further results the reader is referred to papers of Talagrand [1979] and Debs [1980].

Using a result of Porada [1979]. Telgársky [1986] proved that if $X$ is a metrizable space, then $I \uparrow \mathrm{Ch}(X)$ if and only if $X$ is not strongly Baire space (or, equivalently, $X$ contains a closed copy of the space $Q$ of rational numbers). For non–metrizable spaces a similar characterization is not known, however, it follows from a general theorem of Galvin and Telgársky [1986], that a winning strategy of Player I in $\mathrm{Ch}(X)$ can be reduced to a stationary winning strategy.

Although the condition $II \uparrow \mathrm{Ch}(X)$ beats many good candidates for a generalized completeness property, the structure of the underlying space $X$ still displays several kinds of singularities. For example: $II \uparrow \mathrm{Ch}(R_Q)$, where $R_Q$ is the Michael line (see Engelking [1977], p. 384), but $I \uparrow \mathrm{Ch}(Q)$, where $Q$, the space of rational numbers, is closed in $R_Q$. Moreover, if $X$ is pseudocompact or $C$–scattered, then $II \uparrow \mathrm{Ch}(X)$ (see Telgársky [1986]). For the discussion of criteria of completeness as well as for various notions of completeness the reader is referred to Aarts and Lutzer [1974], Császár [1978], and Wicke and Worrell [1974].

Porada [1979] introduced the following modification of the game $\mathrm{Ch}(X)$. Given a subset $X$ of a topological space $Y$, the play is again $((x_0, U_0), V_0, (x_1, U_1), V_1, \cdots)$, but $U_n$ and $V_n$ are open in $Y$, $x_n \in X$, and Player II wins the play if and only if $\emptyset \neq \bigcap_{n<\omega} V_n \subset X$. Denote this game by $P(X, Y)$.

Clearly, $P(X, X)$ coincides with $\mathrm{Ch}(X)$. Telgársky [1986] showed that $II \uparrow P(X, Y)$ if and only if $II \uparrow \mathrm{Ch}(X)$ and $X$ is a $W_\delta$ subset of $Y$; the notion of a $W_\delta$ set was introduced by Wicke and Worrell [1971]. Let $WW(X, Y)$ denote the modification of $P(X, Y)$, where the condition

$\bigcap_{n<\omega} V_n \neq \emptyset$ is dropped. Then $X$ is a $W_\delta$ subset of $Y$ if and only if $II \uparrow WW(X, Y)$ (see Telgársky [1986]).

Topsøe [1982] strengthened the condition $\bigcap_{n<\omega} V_n \neq \emptyset$ of the game $Ch(X)$ as follows: if $\mathcal{F}$ is a filter base of subsets of $X$ such that for each $n < \omega$ there is an $F \in \mathcal{F}$ with $F \subset V_n$, then $\mathcal{F}$ clusters (i.e., $\bigcap\{\overline{F} : F \in \mathcal{F}\} \neq \emptyset$). Denote the modified game by $T(X)$.

It turns out that if $X$ is metrizable, then the game $T(X)$ is equivalent to $Ch(X)$. Telgársky [1984] proved that if $X$ is completely regular, then $T(X)$ is equivalent to $P(X, \beta X)$. Topsøe [1982] proved that $II \uparrow T(X)$ if and only if $X$ is sieve–complete. The last term was introduced by Michael [1977], but the spaces were introduced earlier by Wicke and Worrell [1971] (Condition $\mathcal{K}$), and further studied Chaber, Čoban and Nagami [1974] (monotonically Čech complete spaces).

Wicke and Worrell [1971] proved that sieve–completeness is equivalent to each of the following conditions: (a) $X$ is a $W_\delta$ set in $\beta X$, and (b) $X$ is the open continuous image of a Čech complete space. Hence it follows that Čech complete spaces are sieve–complete. However, a paracompact sieve–complete space is also Čech complete (see Topsøe [1982]). Furthermore, sieve–complete spaces are strongly Baire, but, in contrast to strongly siftable spaces, they are only countably productible.

Actually, Topsøe [1982] introduced several modifications of the games $BM(X)$ and $Ch(X)$, and suggested to consider various clustering conditions for the winning plays of Player II. Let us mention the modification $kT(X)$ of $T(X)$, where points $x_n$ are replaced by compact sets $C_n$. It turns out that $II \uparrow kT(X) \Leftrightarrow II \uparrow T(X)$. Telgársky [1984] proved that $kT(X)$ is equivalent to the analogical modification of $P(X, \beta X)$, and also to the game $H(\beta X \backslash X)$ (see 8 below).

Michael [1986] introduced the notion of a complete exhaustive sieve and an associated game which is a modifications of both games $BM(X)$ and $T(X)$. He proved that a space has a complete exhaustive sieve if and only if Player II has a (stationary) winning strategy in this game.

Christensen [1981] introduced two modification of the game $Ch(X)$ for the study of points of continuity of separately continuous functions. In these games, denoted here by $Ch_\sigma(X)$ and $Ch_\tau(X)$, $X$ is a Hausdorff space; Player I chooses a nonempty open set $U_0$, Player II chooses a point $x_0$ in $U_0$ and its open neighborhood $V_0$ with $V_0 \subset U_0$, Player I chooses a nonempty open set $U_1$ with $U_1 \subset V_0$, Player II chooses a point $x_1$ in $U_1$ and its open neighborhood $V_1$ with $V_1 \subset U_1$, and so forth. Player II wins the play $(U_0, (x_0, V_0), U_1, (x_1, V_1), \cdots)$ in $Ch_\sigma(X)$

if and only if each subsequence $(x_{n(k)} : k < \omega)$ of $(x_n : n < \omega)$ has a cluster point in $\bigcap_{n<\omega} V_n$; Player II wins the play in $Ch_\tau(X)$ if and only if each subnet $(x_{n(t)} : t \in T)$ of $(x_n : n < \omega)$ has a cluster point in $\bigcap_{n<\omega} V_n$.

A Hausdorff space $X$ is said to be a Namioka space if for each separately continuous function $f : X \times Y \to Z$ where $Y$ is a compact Hausdorff space and $Z$ is a metrizable space, there is a $G_\delta$ set $A$ dense in $X$ such that $f$ is (jointly) continuous at each point of $A \times X$ (cf. Namioka [1974]). Christensen [1981] proved that if Player II has a stationary winning strategy in $Ch_\tau(X)$, then $X$ is a Namioka space. Saint–Raymond [1983] proved that each Namioka space is a Baire space, and he strengthened the result of Christensen: if Player I has no winning strategy in $G_\sigma(X)$, then $X$ is a Namioka space, where $G_\sigma(X)$ is the modification of $Ch_\sigma(X)$ such that Player II wins the play if and only if $(x_n : n < \omega)$ has at least one cluster point in $\bigcap_{n<\omega} V_n$. Further results were obtained by Debs [1986b] and Talagrand [1985]. Christensen [1982, 1983] also applied the game $Ch_\sigma(X)$ to some sets of functions and to multivalued functions.

**8. The point–open game.** In 1970, Fred Galvin defined the following topological game. Given a topological space $X$, at move $n$, Player I chooses a point $x_n$ in $X$ and Player II chooses an open neighborhood $U_n$ of $x_n$; Player I wins the play $(x_0, U_0, x_1, U_1, \cdots)$ if and only if $\bigcup_{n<\omega} U_n = X$. Denote this game by $G(X)$. Some results on this game, called point–open game, are contained in the unpublished paper of Galvin, Mycielski and Solovay [1974].

In 1972 Telgársky [1974] introduced the following game $G(K, X)$. There are given a space $X$ and a class $K$ of spaces such that $Y \in K \Rightarrow \mathcal{F}(Y) \subset K$. (Recall that $\mathcal{F}(Y)$ denotes the collection of all closed subsets of the space $Y$.) Player I chooses an $H_0 \in \mathcal{F}(X) \bigcap K$, Player II chooses an $E_1 \in \mathcal{F}(X)$ with $E_1 \subset X \backslash H_0$, Player I chooses a $H_1 \in \mathcal{F}(X) \bigcap K$ with $H_1 \subset E_1$, Player II chooses an $E_2 \in \mathcal{F}(X)$ with $E_2 \subset E_1 \backslash H_1$, and so forth. Put $E_0 = X$. Player I wins the play $(E_0, H_0, E_1, H_1, \cdots)$ if and only if $\bigcap_{n<\omega} E_n = \emptyset$.

The game $G(K, X)$ was studied for various classes, e.g., for $\mathbf{1} =$ the class of all at most one–point spaces, $F =$ the class of all finite spaces, $C =$ the class of all compact spaces, $D =$ the class of all discrete spaces, $DC =$ the class of all spaces with a discrete cover by compact sets.

It turns out that the games $G(X)$, $G(1, X)$ and $G(F, X)$ are mutually

equivalent (Galvin [1978] and Telgársky [1975a]). Moreover, the game $G(K, X)$ is equivalent to the $K$–open game: given a space $X$, at move $n$, Player I chooses a $H_n \in \mathcal{F}(X) \cap K$ and then Player II chooses an open neighborhood $U_n$ of $H_n$; Player I wins the play $(H_0, U_0, H_1, U_1, \cdots)$ if and only if $\bigcup_{n < \omega} U_n = X$ (Telgársky [1980, 1983a]). If $K = 1$, $F, C, D$, or $DC$, then the corresponding game is called point–open, finite–open, compact–open, discrete–open, or DC–open, respectively.

Galvin [1978] introduced the dual game $G^*(X)$ to the game $G(X)$: at move $n$ Player I chooses an open cover $\mathcal{U}_n$ of $X$ and Player II chooses a $U_n \in \mathcal{U}_n$; Player II wins the play $(\mathcal{U}_0, U_0, \mathcal{U}_1, U_1, \cdots)$ if and only if $\bigcup_{n < \omega} U_n = X$. Galvin [1978] proved that $I \uparrow G^*(X) \Leftrightarrow II \uparrow G(X)$, and $II \uparrow G^*(X) \Leftrightarrow I \uparrow G(X)$. A similar relation holds for the dual games to the $K$–open games (see Telgársky [1983a]).

Telgársky [1984] defined the game $H(X)$, called the Hurewicz game, which is related to the Hurewicz property $E^*$ (see Hurewicz [1926], Lelek [1969], Telgársky [1975a]). Given topological space $X$, at move $n$, Player I chooses an open cover $\mathcal{U}_n$ of $X$ and Player II chooses a finite subfamily $\mathcal{H}_n$ of $\mathcal{U}_n$; Player II wins the play if and only if $\bigcup_{n < \omega} \bigcup \mathcal{H}_n = X$. Telgársky [1984] proved that $II \uparrow H(X) \Leftrightarrow I \uparrow G(C, X)$, and $II \uparrow H(X_\delta) \Leftrightarrow I \uparrow G(X)$, where $X_\delta$ is the $G_\delta$ modification of $X$.

A space $X$ for which $I \uparrow G(K, X)$ is called $K$–like (Telgársky [1975a]). According to the general theorem of Galvin and Telgársky [1986], a winning strategy of Player I can be reduced to a stationary winning strategy (this result involves the axiom of choice). Therefore, a space $X$ is $K$–like if and only if there is a map $s : \mathcal{F}(X) \to \mathcal{F}(X) \cap K$ such that (i) $s(E) \subset E$ for every $E \in \mathcal{F}(X)$, and (ii) $\bigcap_{n < \omega} E_n = \emptyset$ whenever $E_0 = X, E_{n+1} \in \mathcal{F}(X)$ and $E_{n+1} \subset E_n \backslash s(E_n)$ for every $n < \omega$.

A space $X$ is called $K$–scattered if for every nonempty closed subset $E$ of $X$ there is an open set $U$ such that $\emptyset \neq E \cap \overline{U} \in K$. Telgársky [1975a] proved that a Lindelöf regular space $X$ with a countable cover by $K$–scattered closed subsets is $K$–like; in [1983a] he proved that each $K$–like space has a countable cover by $K$–scattered subsets. Nogura [1983] constructed a completely regular $C$–like space $X$ with no countable cover by $C$–scattered closed sets. Telgársky [1983b] proved that $X$ is sieve–complete if and only if $\beta X \backslash X$ is $C$–like, and from this result he derived the following characterization: $X$ is $C$–like if and only if $\beta(X \times Q) \backslash (X \times Q)$ is sieve–complete, where $Q$ is the space of the rational numbers. In [1983a] he proved that $I \uparrow G(X) \Leftrightarrow I \uparrow G(X_\delta)$, where $X_\delta$ is the $G_\delta$–modification of $X$, and from this result he derived in [1983b] that $I \uparrow G(X)$ if and only if $\beta(X_\delta \times Q) \backslash (X_\delta \times Q)$ is

sieve–complete.

Each $\sigma$–compact space is $C$–like; this implication is reversible under metrizability. Each strongly Baire $C$–like space is $C$–scattered; each $C$–scattered Lindelöf space is $C$–like. A finite product of $C$–like spaces is $C$–like; moreover, if $X$ is $C$–like and $Y$ is a Lindelöf space (resp. $Y$ is a Hurewicz space), then $X \times Y$ is a Lindelöf space (resp. a Hurewicz space).

$DC$–like spaces have similar properties to $C$–like spaces. If $X$ has a closure–preserving closed cover by compact sets, or if $X$ is a paracompact $C$–scattered space, then $X$ is $DC$–like (see Potoczny [1973] and Telgársky [1975a]). Yajima [1976] proved that a $\sigma$–locally compact paracompact space is totally paracompact. This result was generalized by Tamano and Yajima [1979] and further by Yajima [1983a], who proved that a paracompact $DC$–like space is totally paracompact. This result was further refined by Hattori [1980] (the paper of Yajima [1983a] was also submitted in 1980). Telgársky [1975a] proved that if $X$ is a paracompact $DC$–like space and $Y$ is a paracompact, then $X \times Y$ is paracompact. This result was extended to metacompact spaces and to more general situations by Yajima [1981, 1983a,b,1984]; in particular, he studied various kinds of rectangular refinements of open covers of $X \times Y$, where either $X$ or $X \times Y$ is assumed to be $K$–like for some $K$. Let $\dim_n$ denote the class of all normal spaces $X$ with $\dim X \leq n$. Telgársky and Yajima [1980] proved that if $X$ is a normal $\dim_n$–like space, then $X \in \dim_n$ (this result generalizes the sum theorem for the covering dimension dim). Analogical result for the large inductive dimension Ind was proved by Yajima [1979]. Games $G(K,X)$ related to infinite dimensional spaces were studied by Hattori [1983, 1985].

Galvin and Telgársky [1986] proved that $II \uparrow G(X)$ if and only if there is an open cover $\mathcal{U}$ of $X$ such that (i) if $x \in X$ and $U \in \mathcal{U}$, then $U \bigcup \{x\} \subset U'$ for some $U' \in \mathcal{U}$ and (ii) if $U_n \in \mathcal{U}$ and $U_n \subset U_{n+1}$ for each $n < \omega$, then $\bigcup_{n<\omega} U_n \neq X$.

Galvin, Mycielski and Solovay [1971, 1974,1979] considered the following game. Given a subset $X$ of the real line $R$, at move $n$, Player I chooses a positive real number $p_n$, and then Player II chooses a real number $r_n$; Player II wins the play $(p_0, r_0, p_1, r_1, \cdots)$ if and only if $X \subset \bigcup_{n<\omega} [r_n, r_n + p_n]$. Denote this game by $G_0(X)$.

It turns out that $I \uparrow G_0(X)$ if and only if $X$ is not a set of the strongly measure zero, and $II \uparrow G_0(X)$ if and only if $X$ is countable. Thus the determinacy of $G_0(X)$ for each $X \subset R$ is equivalent to the Borel Conjecture (BC for short). Laver [1976] proved the relative consistency of

BC with ZFC. Since the determinacy of $G_0(X)$ implies the determinacy of $G(X)$, it follows that the statement "$G(X)$ is determined for each $X \subset R$" is also consistent with ZFC. On the other hand, assuming CH (or MA), Galvin [1978] constructed a subset $X$ of the real line such that $G(X)$ is undetermined. Telgársky [1983a] showed (in ZFC) that $G(X)$ is undetermined for some Lindelöf $P$–space of cardinality $\aleph_1$.

Martínez [1984] introduced the following refinement of the finite-open game. Given a subset $X$ of a $T_3$–space $Y$, at move $n$, Player I chooses a finite set $H_n$ of points in $X$, and Player II chooses an open neighborhood $U_n$ of $H_n$. Player I wins the play if and only if $X \subset \bigcup_{n<k} U_n$ for some $k < \omega$. If Player I has a winning strategy in the game, then $X$ is called accessible in $Y$. Martinez [1984] used this game to study the expressive power of the language $(L_{\omega_1\omega})_t$ for $T_3$–spaces.

**9. Further topological games.** Blackwell [1967] introduced a topological game related to two continuous functions from $\omega^\omega$ into a Polish space to derive the Kuratowski Coreduction Principle for analytic sets from the determinacy of the Ulam game for open sets. This result inspired Martin [1968], and independently Addison and Moschovakis [1968], who essentially extended its range. They shown that the determinacy of the Ulam game for sets in a projective class has deep consequences for separation and reduction properties of sets in "close" projective classes (see Van Wesep [1978a,b] and Moschovakis [1980]).

Maitra [1971] associated a topological game to two Luzin sieves and proved the reduction theorem for Suslin sets in more general spaces than Polish spaces.

Vaught [1974] introduced the following set–theoretic operation. Given a collection $\{E(k_0, \cdots, k_{2n}) : k_0, \cdots, k_{2n}, n < \omega\}$ of sets, the result of the operation is $E = \bigcap_{i_0} \bigcup_{j_0} \bigcap_{i_1} \bigcup_{j_1} \cdots \bigcap_n E(i_0, j_0, \cdots, i_n, j_n)$. Since the membership relation $x \in E$ can be described in terms of an infinite game, the operation is called a game operation. He proved that $E$ can also be obtained by the Suslin operation from the sets $E(i_0, j_0, \cdots, i_n, j_n)$. Vaught and Schilling [1979] introduced a generalization of the Suslin operation, called the Borel–game operation. Let $B$ be a Borel subset of $\mathcal{P}(\omega)$ and let $\mathcal{U} = \{U(k_0, \cdots k_n) : (k_0, \cdots, k_n) \in [\omega]^{<\omega}\}$ be an indexed family of open subsets of a Polish space $X$. Then the Borel–game operation $\Gamma$ with the target $B$ is defined by $\Gamma_B(\mathcal{U}) = \{x \in X : \forall k_0 \exists k_1 \forall k_2 \exists k_3 \cdots \{n < \omega : x \in U(k_0, \cdots K_n)\} \in B\}$. The membership relation $x \in \Gamma_B(\mathcal{U})$ is decidable since it is equiv-

alent to the Borel determinacy of the game $U_\omega^\omega(\Phi_x(B))$, because
$\Phi_x(B) = \{(k_0, k_1, \cdots) \in \omega^\omega : \{n < \omega : x \in U(k_0, \cdots, k_n)\} \in B\}$ is
a Borel set in $\omega^\omega$ (see Schilling [1980] and Schilling and Vaught [1983]).
However, already in the earlier paper, Vaught [1974] introduced a some-
what similar game operation, where he used a fusion of the games
$BM(X)$ and $U_\omega^\omega(X)$; this operation was useful in invariant descriptive
set theory (see also Burgess and Miller [1975]). Kechris [1978] stud-
ied this fusion for its applications to forcing; he considered the game
$BM(X)$ in a more general framework, where the players choose ele-
ments of a countable partially ordered set. Moreover, he proved the
Game Formula, which states the equivalence of two ways of playing a
game.

It is a classical theorem that the Suslin operation preserves the
Baire property (see Kuratowski [1966]). R. M. Solovay proved that
the Borel–game operation preserves the Baire property in any separa-
ble metric space. By a method of Kechris [1978], Schilling and Vaught
[1983] proved that $\Gamma_B$ preserves the Baire property in an arbitrary
topological space.

Burgess [1983b] showed that if $B$ is in some low Borel classes, then the
Borel–game operations $\Gamma_B$ provide a characterization of some higher
classes in the descriptive hierarchy, namely: clopen–game=Borel,
closed–game=analytic, open game=co–analytic, $(F_\sigma \cap G_\delta)$–game $=C$-
sets of Selivanovskii [1928], and $(F_{\sigma\delta} \cap G_{\delta\sigma})$–game $=R$–sets of Kol-
mogorov [1928]. The proof of the last equivalence involves playing
auxiliary games with the play length $\alpha < \omega_1$.

Telgársky [1977a] introduced a following topological game $G(X, Y)$.
Given a subset $X$ of space $Y$, Player I chooses a partition $(E(0, k) :
k < \omega)$ of $X$, Player II chooses a $k_0 < \omega$, Player I chooses a parti-
tion $(E(1, k) : k < \omega)$ of $E(0, k_0)$, Player II chooses a $k_1 < \omega$, and
so forth. Player I wins the play if and only if $\bigcap_{n<\omega} \overline{E(n, k_n)} \subset X$.
Telgársky [1977a] proved that $I \uparrow G(X, Y)$ if and only if $X$ is a Suslin
set in $Y$. Ostaszewski and Telgársky [1980] proved that if $X$ is a co-
analytic non–Borel set in a Polish space $Y$, then $I \uparrow G(X, Y)$. It is
an open question whether this game is determined for the sets in the
$\sigma$–algebra generated by analytic sets in a Polish space. DeWilde [1967]
introduced a special class of linear topological spaces, called webbed
spaces (l'espaces a reseaux – in French), for proving the Closed Graph
Theorem in a general setting. Telgársky [1977b] gave a game–theoretic
interpretation of webbed spaces, analogous to Suslin sets.

Blackwell [1981] introduced a game related to the classical definition

of Borel sets on the real line. Let $T$ denote the set of all finite sequences of numbers in $\omega$. A subset $S$ of $T$ is called a stop rule if every infinite sequence $(k_0, k_1, \cdots)$ has exactly one segment in $S$. Let $f$ be a function on $S$ whose values are intervals and let $r \in R$. The game $G(S, f, r)$ is defined as follows. The players alternately choose numbers $k_0, k_1, \cdots$ in $\omega$; the play stops when $(k_0, \cdots k_n) \in S$. Player I wins if and only if $r \in f(k_0, \cdots, k_n)$. Let $B(S, f) = \{r \in R : I \uparrow G(S, f, r)\}$. Then the sets of the form $B(S, f)$ are just Borel sets.

The first axiom of countability is a kind of the completeness about a point and it can be modified in many ways (see Siwiec [1975] and Wicke and Worrell [1972, 1976]). Gruenhage [1976] introduced the following game. Let $X$ be a topological space and let $x$ be a point of $X$. At move $n$, Player I chooses an open neighborhood $U_n$ of $x$, and then Player II chooses a $x_n$ in $U_n$; Player I wins the play $(U_0, x_0, U_1, x_1, \cdots)$ if and only if $\lim_{n \to \infty} x_n = x$. Denote this game by $G(X, x)$.

Clearly, if $x$ has a countable neighborhood base in $X$, then $I \uparrow G(X, x)$. Gruenhage [1976] proved, that if $X$ is a Hausdorff space for which $I \uparrow G(X, x)$, then either $\{x\}$ is a $G_\delta$ in $X$ or there is an uncountable discrete set $D$ in $X \backslash \{x\}$ such that $D \bigcup \{x\}$ is compact.

Let $C_\pi(X)$ denote the space of real valued continuous functions with the topology of pointwise convergence, and let $\mathbf{0}$ be the function constantly equal to 0. Recall the $G(X)$ is the point–open game of Section 8. Gerlits and Nagy [1982] proved that $I \uparrow G(C_\pi(X), \mathbf{0}_m) \Leftrightarrow I \uparrow G(X)$. For further results on the Gruenhage game the reader is referred to Sharma [1978], Galvin [1981], Gerlits [1983], Galvin and Miller [1984] and McCoy and Ntantu [1986b].

Gruenhage [1984a] introduced a generalization of the game $G(X, x)$, which we denote by $G(X, H)$. Given a (closed) subset $H$ of a space $X$, at move $n$, Player I chooses an open neighborhood $U_n$ of $H$, and then Player II chooses an $x_n$ in $U_n$. Player I wins the play if and only if for each open neighborhood $U$ of $H$ there is a $n < \omega$ such that $\{x_n, x_{n+1}, \cdots\} \subset U$. Let $\Delta_X = \{(x, x) : x \in X\}$. Gruenhage [1984] proved that for a compact Hausdorff space $X$ the following conditions are equivalent: (a) $I \uparrow G(X \times X, \Delta_X)$, (b) $X$ is a Corson compact, and (c) $(X \times X) \backslash \Delta_X$ is a meta–Lindelöf space.

Lutzer and McCoy [1980] introduced a game, denoted by $\Gamma(X)$, for testing the Baire category of $C_\pi(X)$. A play of $\Gamma(X)$ is a sequence $(F_n : n < \omega)$ of pairwise disjoint finite subsets of $X$, where $F_0$ is a starting set, Player I chooses $F_1, F_3, \cdots$ and Player II chooses $F_2, F_4, \cdots$.

Player II wins the play if and only if $\bigcup_{n<\omega}F_n$ is closed and discrete in $X$. Lutzer and McCoy [1980] proved that if $I \uparrow \Gamma(X)$, then $C_\pi(X)$ is of the first category. Moreover, if $X$ is pseudo–normal and completely regular, then the following conditions are equivalent: (a) $II \uparrow \Gamma(X)$, (b) every countable subset of $X$ is closed, (c) $II \uparrow BM(C_\pi(X))$, and (d) $C_\pi(X)$ is pseudo–complete.

McCoy and Ntantu [1986a] introduced two games, $\Gamma_1(X)$ and $\Gamma_2(X)$, for the study of completeness properties of $C_k(X)$, the space of all real–valued continuous functions with the compact–open topology. In both games the players choose compact subsets $A_0, B_0, A_1, B_1, \cdots$ of $X$. In $\Gamma_1(X)$ the only restriction is on Player I, who must choose $A_n$ disjoint from $\bigcup_{k<n}B_k$ for $n > 0$. Player II wins the play if and only if $\{A_n : n < \omega\}$ is a discrete family in $X$. In $\Gamma_2(X)$ there is no restriction on the choice of compact sets, but Player II wins the play if and only if $\{H_n : n < \omega\}$ is a discrete family in $X$, where $H_n = A_n \backslash \bigcup_{k<n}B_n$ for each $n < \omega$. McCoy and Ntantu [1986] proved that $II \uparrow BM(C_k(X)) \Rightarrow II \uparrow \Gamma_1(X)$, and for a normal space $X$, $II \uparrow \Gamma_2(X) \Rightarrow II \uparrow BM(C_k(X))$. Obviously, $II \uparrow \Gamma_2(X) \Rightarrow II \uparrow \Gamma_1(X)$; the converse implication holds if $X$ is locally compact.

Morita [1963, 1964] introduced the MP–spaces and proved that $X$ is a normal (resp. a paracompact) MP–space if and only if $X \times Y$ is normal (resp. paracompact) for every metric space $Y$. (Following the suggestion of Naber [1977], we change the original term $P$–spaces to MP–spaces to avoid confusion with the widely accepted meaning of $P$–space – the one in which all $G_\delta$ sets are open.) Telgársky [1975b] introduced a game such that Player II has a winning strategy if and only if the space is a MP–space. It turns out that MP–spaces constitute a natural strengthening of countably metacompact spaces. Yajima [1984] applied the game–theoretic characterization of MP–spaces to prove the following result. Let $K$ be a class of spaces such that $Y \in K \Rightarrow \mathcal{F}(Y) \subset K$, and let $K_I = \{X : I \uparrow G(K, X)\}$. (The game $G(K, X)$ is defined in Section 8.) If $X$ is a MP–space and $I \uparrow G(K_I, X)$, then $I \uparrow G(K, X)$.

Malyhin, Rančin, Uljanov and Šapirovskii [1977] studied topological games with transfinite plays defined as follows. Let $X$ be a nonempty set and let $\mathcal{W}$ be a nonempty subfamily of $\mathcal{P}(X)$. At move $\xi < |X|^+$ the players alternately choose points $a_\xi$ and $b_\xi$ of $X$, but no point can be chosen twice, and Player I chooses first. The play is completed if all points of $X$ have been chosen. Let $A_\alpha = \{a_\xi : \xi < \alpha\}$ and $B_\alpha = \{b_\xi : \xi < \alpha\}$. Fix an $\alpha < |X|^+$ such that $A_\alpha$ and $B_\alpha$ are defined.

If $A_\alpha \in \mathcal{W}$ and $B_\alpha \notin \mathcal{W}$, then Player I wins the play; if $A_\alpha \notin \mathcal{W}$ and $B_\alpha \in \mathcal{W}$, then Player II wins the play; if either $A_\alpha \notin \mathcal{W}$ and $B_\alpha \notin \mathcal{W}$, or else if $A_\alpha \in \mathcal{W}$, $B_\alpha \in \mathcal{W}$ and $A_\alpha \bigcup B_\alpha = X$, then the result of the play is a draw. The authors considered the game for topological spaces $X$, where the family $W$ consists of discrete subsets, nowhere dense subsets, closed subsets, or compact subsets of $X$, respectively.

Csirmaz and Nagy [1979] considered a game on a topological space $X$, denoted here by $CN(X)$. The players alternately choose points of $X$, but no point can be chosen twice. Player I starts the play and make moves at limit stages (if any). A play is completed if all points of $X$ have been chosen. Player I wins if and only if he has chosen all elements of a nonempty open set. Csirmaz and Nagy [1979] proved that (a) if $X$ is a self–dense locally compact $T_2$–space then $I \uparrow CN(X)$, (b) there is a countable self–dense zero–dimensional $T_2$–space $X$ such that $I \uparrow CN(X)$, and (c) assuming CH there is a countable zero–dimensional $T_2$–space $X$ such that $CN(X)$ is undetermined.

Isbell [1962] proved that a uniform space $\mu X$ is supercomplete if and only if $X$ is topologically paracompact and the locally fine coreflection of $\mu$ is the fine uniformity of $X$. Hohti [1986] introduced a game $G(\mu X, \mathcal{V})$ such that $\mu X$ is supercomplete if and only if $I \uparrow G(\mu X, \mathcal{V})$ for each open cover $\mathcal{V}$ of $X$.

Fleissner [1986] introduced the following game. Given a space $X$ and a base $\mathcal{B}$, at move $n$, Player I chooses a $B_n$ in $\mathcal{B}$, and then Player II chooses a countable subset $C_n$ of $X$. Player I wins the play if and only if $\{B_n : n < \omega\}$ contains a neighborhood base for a point $x$ in $X \backslash \bigcup_{n<\omega} C_n$. Denote this game by $G(X, \mathcal{B})$. Fleissner [1986] proved that for a $T_1$–space $X$ with a point–countable base $\mathcal{B}$ the following conditions are equivalent: (a) $X$ is left–separated; (b) $X$ is $\sigma$–$T_0$–separated; (c) $X$ has a closure–preserving cover by countable closed sets; (d) $II \uparrow G(X, \mathcal{B})$; (e) Player II has a stationary winning strategy in the modification of the game, where Player II chooses singletons.

Berner and Juhász [1984] introduced the games $G_\alpha^P(X)$, called point–picking games, where $X$ is a space, $P$ is a property of subsets of $X$, and $\alpha$ is an ordinal. At move $\xi < \alpha$, Player I chooses an open set $U_\xi$ and then Player II picks a point $x_\xi$ in $U_\xi$. Player I wins the play if and only if the set $\{x_\xi : \xi < \alpha\}$ has the property $P$. They studied the games for the following properties $P$: dense in $X$, self–dense, somewhere dense, and non–discrete. Berner [1984] studied various types of winning strategies in these games.

**10. Some other infinite games.** The role of the axiom of choice in positional games was apparent from the early beginning of their development. Mycielski [1964a] observed that the axiom of choice is equivalent to the determinacy of a game $G(\mathcal{F})$ for every nonempty family $\mathcal{F}$ of nonempty sets, where the game is defined as follows. Player I chooses an $F$ in $\mathcal{F}$ and the Player II chooses an $x$ in $\bigcup \mathcal{F}$; Player II wins the play $(F, x)$ if and only if $x \in F$. Clearly, Player II can always win; however, his winning strategy is a choice function for $\mathcal{F}$. A similar paradox occurs in a game of Hechler [1974], where a player has a winning pseudo–strategy, but has no winning strategy. Under AD the real line cannot be well ordered. Moschovakis [1970] originated the study of the prewellorderings of the real line, compatible with AD, and he established their close connections with the separation and reduction properties of projective sets (see also Moschovakis [1980]). Some results on the principle of dependent choices (DC) were mentioned in §5, while for the countable axiom of choice the reader is referred to Mycielski [1971] and Jech [1973].

Rabin [1957] considered an effective version of the game $U_\omega^\omega(X)$ and gave an example of recursive game in which a player has an obvious winning strategy, but has no recursive winning strategy. This paper originated various applications of infinite games to the theory of recursive functions and vice versa (see Blass [1972a], Jones [1974, 1982], Kirby and Paris [1982], Kleinberg [1973], Lachlan [1970], and Yates [1974, 1976]).

Ehrenfeucht [1957] introduced a game related to the relative interpretability of one formal system (a theory) in another one; in particular, the game gives a necessary and sufficient condition for two structures to be elementarily equivalent. His paper originated applications of infinite games to mathematical logic and model theory, and was followed by papers of Ehrenfeucht [1961], Vinner [1972], Badger [1977], and others. For more references the reader is referred to Weese [1980].

Henkin [1961] initiated the study of infinitely long formulas and also of their game–theoretic interpretation. He observed that Skolem functors for a formula correspond to strategies in an associated positional game of perfect information, e.g., the formula $\forall x_0 \exists y_0 \forall x_1 \exists y_1 \cdots \pi(x_0, y_0, x_1, y_1, \cdots)$ is valid if and only if Player "$\exists$" has a winning strategy in the associated game; also, he introduced new quantifiers, called dependent quantifiers. Infinite strings of alternating quantifiers $\forall$ and $\exists$ were further studied by Keisler [1965, 1969]. The idea of a quantifier corresponding to the Suslin operation appeared in the paper of Hinman [1969]. The

notion of a generalized quantifier corresponding to a family of sets was introduced by Aczel [1970]. Moschovakis [1972] introduced the notions of Suslin quantifier and game quantifier for eliminating infinite strings of quantifiers $\forall$ and $\exists$ (see also Moschovakis [1974, 1980] and Büchi [1977]). Vaught [1974] gave a characterization of analytic sets by mean of a formula with the infinite string of alternating quantifiers $\forall$ and $\exists$. For further development of these ideas the reader is referred to the papers of Burgess and Miller [1975], Burgess [1983a,b], and Burgess and Lockhart [1983].

Blackwell [1978] introduced Borel–programmable set (BP–sets) and showed that they form a $\sigma$–algebra closed with respect to the Suslin operation and containing all Borel sets. Hence $C$–sets $\subset$ BP–sets. Lockhart [1978] studied iterations of the programming operation on $R$-sets. Burgess and Lockhart [1983] proved that $C$–sets $\subset$ BP–sets $\subset$ $P$–sets $=$ $R$–sets, where the inclusions are proper, and where $P$–sets (or, programmable sets) are obtained from the Borel sets by iterating the programming operation. The proof of the second inclusion employs a game with plays of length $\alpha < \omega_1$. It turns out that BP–sets $\subset$ bi–primitive $R$–sets, and $R$–sets $\subset$ absolutely (PCA$\cap$CPCA)–sets (i.e., absolutely $\Delta_2^1$–sets). Burgess and Lockhart [1983] also introduced Borel–meta–programmable sets (BMP–sets) and stated that the inclusion $P$–sets $\subset$ BMP–sets is proper. Shreve [1981] introduced another generalization of BP–sets, called Borel–approachable sets (BA–sets). For other related results the reader is referred to Moschovakis [1980] (especially 6E and 7C).

Mycielski [1969] defined some ideals of subsets of the real line by means of some infinite games (see also Méndez [1978]). S.Aanderaa proposed a game associated with ultrafilter on $\omega$ and showed that the game is not determined (see Fenstad [1971]). Grigorieff [1971] gave a characterization of some ultrafilters by means of certain conditions involving trees, which are readily seen to be of a game–theoretic nature. F. Galvin invented a variety of games involving ultrafilters, and called ultrafilter games. Some of these games were studied in the unpublished paper of McKenzie [1972], and also many others in the unpublished paper of Galvin, Mycielski and Solovay [1974]. For example, Q–points in $\beta\omega\setminus\omega$ are characterized by mean of a game with transfinite plays, where the players choose finite subsets of $\omega$. In the game of this kind, there is given a non–principal ultrafilter $\mathcal{F}$ on $\omega$ and the players alternately choose elements of $\omega$ but no element is chosen twice. Player I starts the plays and also makes moves at limit stages (if any). A play is com-

pleted when all elements of $\omega$ have been chosen. Player I wins the play
if and only if the set of elements he has chosen is in $\mathcal{F}$. McKenzie [1972]
proved that this game is undetermined. Csirmaz and Nagy [1979] gen-
eralized the scheme of this game so that $\mathcal{F}$ is a family of subsets of a
set $X$ and moreover, the plays have length $< \alpha$ (thus the plays may
terminate before $X$ is partitioned between the players).

Jech [1978] introduced the following Boolean variant of the Banach-
Mazur game, called the Boolean game. Given a Boolean algebra $B$,
the players alternately choose elements $a_0, b_0, a_1, b_1, \cdots$ of $B$ so that
$a_0 \geq b_0 \geq a_1 \geq b_1 \geq \cdots$; Player II wins the play if and only if there
is a nonzero element $b$ in $B$ such that $b_n \geq b$ for each $n < \omega$. He
proved that Player I has a winning strategy for $B$ if and only if $B$ is
not $\aleph_o, \infty)$–distributive. He also considered the game on a partially
ordered set related to the notion of forcing which adjoins a closed un-
bounded subset of a given stationary set (the partially ordered set is
canonically embedded in a complete Boolean algebra). Kechris [1978]
introduced another variant of the Banach–Mazur game $BM(X)$ on a
countable partially ordered set and proved the Banach–Mazur theorem
in this more general setting, where the topology is induced by the par-
tial ordering; this game is also related to the notion of forcing–Galvin,
Jech and Magidor [1978] studied the games, where $B$ is the power set of
a set, but the players make choices from a given $\sigma$–complete ideal. This
study was continued by Jech [1984a,b] and Veličkovič [198?]. Boolean
games with transfinite plays were considered by Foreman [1983] and
further by Vojtáš [1982a,b,1983,198?]. Gray [1982] and independently
Taylor [198?] introduced a game played on a partially ordered set $P$
such that Player II has a winning strategy in the game if and only if
$P$ is proper, that is, forcing with $P$ does not destroy any stationary
subset of $[\kappa]^\omega$ for $\kappa > \omega$. Moreover, Player I has a winning strategy in
the game if and only if forcing with $P$ destroys some unbounded closed
subset of $[\kappa]^\omega$ for some $\kappa > \omega$. This characterization of proper forcing
was used by Fleissner [1986] for a topological result. Another game on
a partially ordered set with transfinite plays for a suitable forcing was
used by Velleman [1983]. For further applications of proper forcing the
reader is referred to Baumgartner [1984].

In 1935, S. Banach proposed two set–theoretic modifications of the
Banach–Mazur game (see Mauldin [1981], Problem 67). Denote the
games by $B_1(X)$ and $B_2(X)$, where $X$ is a given infinite set. In
$B_1(X)$ the players alternately choose subsets $E_0, E_1, \cdots$ of $X$ so that
$E_0 = X, E_{n+1} \subset E_n$ and $|E_{n+1}| = |E_n \backslash E_{n+1}|$ for each $n < \omega$;

Player I wins the play if and only if $\bigcap_{n<\omega} E_n = \emptyset$. In $B_2(X)$, the players again choose subsets $E_0, E_1, \cdots$ of $X$, where $E_0 = X$, but $E_{n+1} \subset X \backslash \bigcup_{k<n} E_k$ and $|E_{n+1}| = |X \backslash \bigcup_{k<n+1} E_k|$ for each $n < \omega$; Player I wins if $\bigcup_{n>0} E_n = X$. Schreier [1938] proved that $I \uparrow B_1(X)$ and $I \uparrow B_2(X)$ (see also Mauldin [1981, p 138]). Many related games were studied in the unpublished paper of Galvin, Mycielski and Solovay [1971, 1974].

Ramsey [1930] proved that if $[S]^m = \bigcup_{k<n} X_k$, where $S$ is an infinite set, then there is a $k < n$ and an infinite subset $H$ of $S$ such that $[H]^m \subset X_k$; $H$ is called a homogeneous set. Erdös and Rado [1952] showed (using the axiom of choice) that the Ramsey theorem does not extend to $[\omega]^\omega = X_0 \bigcup X_1$. Nash–Williams [1965] proved that if $U$ and $V$ are disjoint open subsets of $P(\omega)$, then there is a $H \in [\omega]^\omega$ such that either $[H]^\omega \bigcap U = \emptyset$ or $[H]^\omega \bigcap V = \emptyset$. (Note that $P(\omega)$ is isomorphic to $2^\omega$, so the isomorphism transfers the topology from $2^\omega$ into $P(\omega)$). After this result it became apparent that topological variants of the Ramsey theorem might be available in ZFC.

A subset $X$ of $[\omega]^\omega$ is called Ramsey if there is an $H \in [\omega]^\omega$ such that either $[H]^\omega \subset X$ or else $[H]^\omega \subset [\omega]^\omega \backslash X$ (Galvin and Prikry [1973]). Galvin [1968] extended the result of Nash–Williams [1965]; it was implicitly stated in both references that all open sets in $[\omega]^\omega$ are Ramsey. Mathias [1968] showed that in the model of Solovay [1970] all subsets of $P(\omega)$ are Ramsey (in this model the axiom of choice is false, but all sets of real numbers are Lebesgue measureable). Galvin and Prikry [1973] proved (before 1970) that all Borel sets are Ramsey. Silver [1970] proved that all sets in the $\sigma$–algebra generated by analytic sets are Ramsey, but his proof used deep metamathematical ideas, while the proof of Galvin and Prikry [1973] used only classical set theory. Silver [1970] also proved that if there is measurable cardinal, then all PCA–sets (i.e., $\Sigma_2^1$–sets) are Ramsey. Ellentuck [1974] introduced the notion of completely Ramsey sets and using a topology on $P(\omega)$, finer than the ordinary topology, he proved that all $C$–sets are completely Ramsey; in fact, a completely Ramsey sets coincide with the sets having the Baire property in the modified topology. Prikry [1976] proved that if the Banach–Mazur game $BM(X)$ is determined for each $X \subset R_d^\omega$, where $R_d$ denotes the real line with the discrete topology, then all subsets of $[\omega]^\omega$ are Ramsey. Mansfield [1978] introduced a game which was suitably modified by Kastanas [1983] so that the latter provided a game–theoretic characterization of Ramsey sets.

Given a subset $X$ of $[\omega]^\omega$, Player I chooses an $A_0 \in [\omega]^\omega$, Player II chooses a $(k_0, B_0)$ such that $k_0 \in A_0$, $B_0 \in [A_0]^\omega$ and $\min B_0 > k_0$, Player I chooses an $A_1 \in [B_0]^\omega$, Player II chooses a $(k_1, B_1)$ such that $k_1 \in A_1, B_1 \in [A_1]^\omega$ and $\min B_1 > k_1$, and so forth. Player I wins the play if and only if $\{k_n : n < \omega\} \in X$. Denote this game by $R(X)$. Kastanas [1983] proved that $I \uparrow R(X) \Leftrightarrow [H^\omega]^\omega \subset X$, and $II \uparrow R(X) \Leftrightarrow \forall A \in [\omega]^\omega \exists H \in [A]^\omega : [H]^\omega \subset [\omega]^\omega X$.

The Ramsey theorem also inspired many results in combinatorial set theory (see Williams [1977]) and also in infinite games related to partition calculus of P. Erdös and R. Rado. For some of these games the reader is referred to Galvin, Mycielski and Solovay [1974], Baumgartner, Galvin, Laver and McKenzie [1975], Beck and Csirmaz [1982] Komjath [19??] and Stepráns and Watson [198?].

P. Erdös proposed the following game $G_m$ on the real line, where $m < \omega$. At move $n$, Player I chooses a real number $r_n$ which was not chosen earlier, and Player II chooses a countable set $A_n$ of real numbers which were not chosen earlier. Player I wins the play in $G_m$ if and only if $\{r_n : n < \omega\}$ contains an arithmetic progression consisting of $m + 1$ terms. F. Galvin and independently Zs. Nagy proved that $2^{\aleph_0} \geq \aleph n \Rightarrow I \uparrow G_n$, and $2^{\aleph_0} < \aleph_n \Rightarrow II \uparrow G_n$ (see Beck and Csirmaz [1982]).

Burd and Takeuti [1976] introduced the notion of a weave and an infinite game related to tensor product of a sequence of normal weaves (cf. also Burd [1976]).

Gruenhage [1984b] introduced a game related to the coloring number of an infinite graph and by mean of the underlying property he solved some problems involving metacompact and meta–Lindelöf spaces.

All games considered above are discrete–time games and they are studied by completely different methods than games with continuous time. For the latter the reader is referred to Ruckle [1983] and Mycielski [1986].

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