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POINTWISE CONVERGENCE OF CAUCHY SEQUENCES

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Many problems in analysis require the introduction of a linear space of functions equipped with a norm appropriate to the application. It is often desirable in such situations that convergence in norm imply pointwise convergence on subsets of the common domain.

The purpose of this paper is to extend the results of [6], where pointwise convergence of convergent sequences was studied, to Cauchy sequences. Our primary tool is convergence on a filter. Specifically, we characterize filters which guarantee the desired convergence properties.

Let G be a linear space of functions with a common domain S. Let X denote the formal linear span of S. It was shown in [4] that every seminormed topology on G is equivalent to the topology of convergence on a filter of subsets of X.

In [6], Brace and Thomison introduced the notions of uniform, subuniform, pointwise and subpointwise convergence and investigated conditions under which convergence on a filter implied any of the four. In this paper, we extend these notions to Cauchy sequences of functions in G. We also investigate conditions under which completions of G preserve these properties.

§I contains the basic definitions and discusses the relationship between various notions of convergence. §II investigates certain conditions which guarantee the various convergence properties. §III characterizes conditions under which completions of G retain the convergence properties. §IV is devoted to examples.

We use the notation of [5], [6] and [9]. Proofs of all results are at the end of the sections.

I. Several notions of convergence. In the following, $\mathfrak{F}$ is a filter in a set S. The functions are scalar valued and have S as their common domain; G is a linear space composed of such functions. A filter in a linear space X will always be assumed to possess a basis consisting of balanced, convex sets.

DEFINITION I.1. [6] A sequence $\{f_n\}$ converges to f_0 uniformly (pointwise) on $\mathfrak{F}$ when there is a set F in $\mathfrak{F}$ such that $\{f_n\}$ converges uniformly

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(pointwise) to f_0 on F. A sequence converges subuniformly (subpointwise) on $\mathfrak{F}$ when every subsequence of $\{f_n\}$ has in turn a subsequence converging to f_0 uniformly (pointwise) on $\mathfrak{F}$.

DEFINITION I.2. A sequence $\{f_n\}$ is said to be uniformly (pointwise) Cauchy on $\mathfrak{F}$ when there is a set F in $\mathfrak{F}$ such that $\{f_n\}$ is uniformly (pointwise) Cauchy on F. A sequence $\{f_n\}$ is said to be subuniformly (subpointwise) Cauchy on $\mathfrak{F}$ when every subsequence of $\{f_n\}$ has in turn a subsequence which is uniformly (pointwise) Cauchy on $\mathfrak{F}$.

DEFINITION I.3. [3] The topology on G with subbasis at the zero function given by the sets $u(\varepsilon, \mathfrak{F}) = \{g \in G$: there exists $F_g \in \mathfrak{F}$ such that $|g(x)| < \varepsilon$ if $x \in F_g\}$, will be referred to as the topology of convergence on $\mathfrak{F}$. A sequence $\{f_n\}$ is said to converge to f_0 on the filter $\mathfrak{F}$ if $\{f_n\}$ converges to f_0 in the $\mathfrak{F}$-topology. The space G equipped with the $\mathfrak{F}$-topology will be denoted by $(G, \mathfrak{F})$.

REMARKS I.4. (i) The $\mathfrak{F}$-topology on G is linear if and only if for any g in G there exists F in $\mathfrak{F}$ such that g is bounded on F.

(ii) Every seminormed topology on G can be obtained via convergence on a filter in X, the formal linear span of S [4].

(iii) If the sequence $\{f_n\}$ converges to f_0 uniformly or subuniformly on $\mathfrak{F}$, then it converges to f_0 on $\mathfrak{F}$ [6].

(iv) The $\mathfrak{F}$-topology on G coincides with the topology of uniform convergence on the set $A = \{\overline{e(F)}: F \in \mathfrak{F}\}$. Here $e: X \to G^*$ denotes the natural evaluation and closures are taken with respect to the $\sigma(G^*, G)$-topology, G^* denoting the algebraic dual of G [4].

DEFINITION I.5. [5] Consider filters $\mathfrak{F}_1$ and $\mathfrak{F}_2$ in a linear space X. We write $\mathfrak{F}_1 > \mathfrak{F}_2$ when there exists a number $r \geqq 1$ such that $\mathfrak{F}_1$ is a refinement of $r\,\mathfrak{F}_2 = \{rF: F \in \mathfrak{F}_2\}$. We say $\mathfrak{F}_1$ is equivalent to $\mathfrak{F}_2$, when $\mathfrak{F}_1 > \mathfrak{F}_2$ and $\mathfrak{F}_2 > \mathfrak{F}_1$.

PROPOSITION I.6. [5] *Let (G, ρ) be a seminormed space consisting of linear forms on a linear space X, and let $\mathfrak{F}$ be a filter in X such that $(G, \mathfrak{F}) = (G, \rho)$. The collection of all subsets of X of the form $M = \{x: |f(x)| < a\}$ for some f in G and $a > \rho(f)$ is a subbasis for a filter $\mathfrak{m}$ in X inducing the ρ-topology on G. Moreover, if $\mathfrak{F}$ is any filter in X satisfying $(G, \mathfrak{F}) = (G, \rho)$, then $\mathfrak{F} > \mathfrak{m}$. We refer to $\mathfrak{m}$ as the minimal filter.*

II. Cauchy filter convergence spaces. Throughout this section $(G, \mathfrak{F})$ is a seminormed space of linear forms on a linear space X. The seminorm is ρ; $\mathfrak{m}$ is the minimal filter determined by ρ and G. Convergence on $\mathfrak{F}$ is equivalent to ρ-convergence. All filters are composed of subsets of X. $B(X, \mathfrak{F})$ denotes the collection of all linear forms on X bounded on some element of $\mathfrak{F}$.

DEFINITION II.1. [6] $(G, \mathfrak{F})$ is a uniform (subuniform, pointwise, subpointwise) filter convergence space when convergence on $\mathfrak{F}$ implies uniform (subuniform, pointwise, subpointwise) convergence on $\mathfrak{F}$. These are abbreviated as u.f.c., s.u.f.c., p.f.c., and s.p.f.c. spaces respectively.

DEFINITION II.2. $(G, \mathfrak{F})$ is a uniform (subuniform, pointwise, subpointwise) Cauchy filter convergence space when every Cauchy sequence in the $\mathfrak{F}$-topology is uniformly (subuniformly, pointwise, subpointwise) Cauchy on $\mathfrak{F}$. These are abbreviated u.c.f.c., s.u.c.f.c., p.c.f.c, and s.p.c.f.c. spaces respectively.

PROPOSITION II.3. *Every u.f.c.(p.f.c.) space is a (u.c.f.c.) c.f.c.space*

REMARK II.4. A similar statement cannot seem to be made for s.u.f.c. and s.p.f.c. spaces. In fact, there does not seem to exist elementary necessary and sufficient conditions for a space to be either a s.u.c.f.c. or a s.p.c.f.c. space. Instead of pursuing this type of question, we introduce constructions which guarantee a space to be either a s.u.c.f.c. or a s.p.c.f.c. space. Similar constructions, which we also introduce, provide a complete picture of u.c.f.c. and p.c.f.c. spaces.

DEFINITION II.5. (i) We let $u(\mathfrak{F})$ $(p(\mathfrak{F}))$ denote the collection of all subsets of X containing a subset of the form $\bigcap_{k=0}^{\infty} b_k F_k (F_0 \cap \text{span}\,\{\bigcup_{n=1}^{\infty} \bigcap_{k=n}^{\infty} b_k F_k\})$, where F_k is in $\mathfrak{F}$ and $b_k \geqq 1$ for all k and $\lim_{k\to\infty} b_k = \infty$.

(ii) If $s \geqq 1$, we let su$(\mathfrak{F})$ [sp$(\mathfrak{F})$] denote the collection of all subsets of X containing a subset of the form $\bigcap_{k=1}^{\infty} b_k F_k (F_0 \cap \text{span}\,\{\bigcup_{n=1}^{\infty} \bigcap_{k=n}^{\infty} b_k F_k\})$, where F_k is in $\mathfrak{F}$ for all k and $\sum_{k=1}^{\infty} b_k^{-s} < 1$.

THEOREM II.6. (i) $(B(X, u(\mathfrak{m})), u(\mathfrak{m}))$ *is always a u.c.f.c. space.*

(ii) $(B(X, p(\mathfrak{m}))\, p(\mathfrak{m}))$ *is always a p.c.f.c. space.*

(iii) $(B(X, su(\mathfrak{m})), su(\mathfrak{m}))$ *is always a s.u.c.f.c. space.*

(iv) $(B(X, sp(\mathfrak{m})), sp(\mathfrak{m}))$ *is always a s.p.c.f.c. space.*

REMARK II.7. We note that $G \subset B(X, u(\mathfrak{m}))$ $(B(X, p(\mathfrak{m})), B(X, \text{su}(\mathfrak{m})), B(X, \text{sp}(\mathfrak{m})))$. However, each of the topologies induced on G by II.6 may be finer than the original topology.

THEOREM II.8. [6] *The set of all filters such that* $(G, \mathfrak{F}) = (G, \rho)$ *is a u.c.f.c. (p.c.f.c.) space has, when nonempty, a unique within equivalence small est element. In the u.c.f.c. space this smallest element is* $u(\mathfrak{m})$.

PROPOSITION II.9. *In the p.c.f.c. case of Theorem* II.8, *the smallest element is* $p(\mathfrak{m})$.

THEOREM II.10. $p(\mathfrak{m})$ $(sp(\mathfrak{m}))$ *induces the ρ-topology on G if and only if, whenever g is in G with $g(x) = 0$ for all x in* span $\{\bigcup_{n=1}^{\infty} \bigcap_{k=n}^{\infty} b_k M_k\}$ *where* M_k *are in* $\mathfrak{m}$ *and* $\lim_{k\to\infty} b_k = \infty$ $(\sum_{k=1}^{\infty} b_k^{-s} < 1)$, *then* $\rho(g) = 0$.

II.11. Proof of Theorem II.3. In the u.f.c. case, this is actually a resatement of Theorem IV.1 of **[6]**. The result in the p.f.c. case follows directly from Theorem IV.4 of **[6]**.

II.12. Proof of Theorem II.6. (u.c.f.c. and p.c.f.c. cases). In light of Theorem II.2 of **[6]** and Theorem II.3 above, we need only show that $u(\mathfrak{m}) = u(u(\mathfrak{m}))$ ($p(\mathfrak{m}) = p(p(\mathfrak{m}))$, since if we let $\mathfrak{m}^*$ be the minimal filter for $B(X, u(\mathfrak{m}))$ ($B(X, p(\mathfrak{m}))$), then $\mathfrak{m}^* < u(\mathfrak{m})$ ($\mathfrak{m}^* < p(\mathfrak{m})$) implies $u(\mathfrak{m})^* < u(u(\mathfrak{m}))$ ($p\mathfrak{m}(^*) < p(p(\mathfrak{m}))$). The first was shown in the proof of Theorem II.5 of **[6]**. The second is proved here in a similar fashion. First note that if

$$F = \bigcap_{k=1}^{\infty} \{M_{k,0} \cap \operatorname{span} \{\bigcup_{t=1}^{\infty} \bigcap_{s=t}^{\infty} b_{k,s} M_{k,s}\}\},$$

then F contains the set

$$\bigcap_{k=1}^{n} M_{k,0} \cap \operatorname{span} \{\bigcup_{t=1}^{\infty} \bigcap_{p=t}^{\infty} b_p M_p\},$$

where the index p is derived from ordering the pairs (s, k) in the following manner: $(1, 1), (1, 2), (1, 3) \ldots (1, n), (2, 1), (2, 2) \ldots (2, n), (3, 1), (3, 2) \ldots$, and $\lim_{p\to\infty} b_p = \infty$. Therefore, without loss of generality, we can let $F_k = M_{k,0} \cap \operatorname{span}\{\bigcup_{t=1}^{\infty} \bigcap_{s=t}^{\infty} b_{k,s} M_{k,s}\}$ be in $p(\mathfrak{m})$ and $a_k \geqq 1$ for all k with $\lim_{k\to\infty} a_k = \infty$. Enumerate the set $\{k, s\colon k = 1, 2, 3, \ldots, s = 0, 1, 2, 3, \ldots\}$ as follows

$$\begin{array}{cccccccc}
1,0 & \to & 1,1 & & 1,2 & \to & 1,3 & \cdots \\
 & \swarrow & & \nearrow & & \swarrow & & \\
2,0 & & 2,1 & & 2,2 & & \cdots & \\
\downarrow & \nearrow & & \swarrow & & & & \\
3,0 & & 3,1 & & \cdots & & & \\
 & \swarrow & & & & & & \\
4,0 & & \cdots & & & & &
\end{array}$$

Then the set $\operatorname{span}\{\bigcup_{n=1}^{\infty} \bigcap_{k=n}^{\infty} a_k F_k\}$ contains the set $\operatorname{span}\{\bigcup_{t=1}^{\infty} \bigcap_{p=t}^{\infty} a_k M_{k,0}\} \cap \operatorname{span}\{\bigcup_{t=1}^{\infty} \bigcap_{p=t}^{\infty} a_p b_p M_p\}$ which is in $p(\mathfrak{m})$. The index p is derived from the enumeration of k, s as denoted above.

Lemma II.13. *If H is an element of* $\mathrm{su}(\mathfrak{F})$ [$\mathrm{sp}(\mathfrak{F})$], *then for every $\varepsilon > 0$ there exists F_k in $\mathfrak{F}$ and b_k with $\sum_{k=1}^{\infty} b_k^{-s} < \varepsilon$ such that H contains $F_0 \cap \{\bigcap_{k=1}^{\infty} b_k F_k\}$ ($F_0 \cap \operatorname{span}\{\bigcup_{n=1}^{\infty} \bigcap_{k=n}^{\infty} b_k F_k\}$).*

Proof. First choose H in su $(\mathfrak{F})$ ($\mathrm{sp}(\mathfrak{F})$) containing $H_0 \cap \{\bigcap_{k=1}^{\infty} a_k H_k\}$ ($H_0 \cap \operatorname{span}\{\bigcup_{n=1}^{\infty} \bigcap_{k=n}^{\infty} a_k H_k\}$), where H_k are in $\mathfrak{F}$ and $\sum_{k=1}^{\infty} a_k^{-s} < 1$. Pick k_0 such that $\sum_{k=k_0}^{\infty} a_k^{-s} < \varepsilon$ and let $F_0 = H_0 \cap \{\bigcap_{k=1}^{k_0} a_k H_k\}$, $F_k = H_{k+k_0}$, $b_k = a_{k+k_0}$. It follows that if H in $\mathrm{su}(\mathfrak{F})$ [$sp(\mathfrak{F})$] contains a finite intersection of sets of the above form, H also contains a set of that form.

II.14. PROOF OF THEOREM II.6. (s.u.c.f.c. and s.p.c.f.c. cases). Let $\{g_k\}$ be a Cauchy sequence in $(B(X, \mathrm{su}(\mathfrak{m})), \mathrm{su}(\mathfrak{m}))$ $((B(X, \mathrm{sp}(\mathfrak{m})), \mathrm{sp}(\mathfrak{m})))$ such that $\rho(g_k - g_{k+1}) < 5^{-k/s}$. Then $\{g_k(x)\}$ converges uniformly on $\bigcap_{k=1}^{\infty} 2^{k/s}\{x: |g_k(x) - g_{k+1}(x)| < 4^{-k/s}\}$ (pointwise on span $\{\bigcup_{n=1}^{\infty} \bigcap_{k=n}^{\infty} 2^{k/s}\{x: |g_k(x) - g_{k+1}(x)| < 4^{-k/s}\}$) which is in $\mathrm{su}(\mathfrak{m}^*)$ $(\mathrm{sp}(\mathfrak{m}^*))$, where $\mathfrak{m}^*$ is the minimal filter for $(B(X, \mathrm{su}(\mathfrak{m})), \mathrm{su}(\mathfrak{m}))$ $((B(X, \mathrm{sp}(\mathfrak{m})), \mathrm{sp}(\mathfrak{m})))$. Since $\mathfrak{m}^* < \mathrm{su}(\mathfrak{m})$ $(\mathfrak{m}^* < \mathrm{sp}(\mathfrak{m}))$ if we can show $\mathrm{su}(\mathfrak{m}) = \mathrm{su}(\mathrm{su}(\mathfrak{m}))$ $(\mathrm{sp}(\mathfrak{m}) = \mathrm{sp}(\mathrm{sp}(\mathfrak{m})))$, we will be done. Let $F_k = M_{k,0} \cap \{\bigcap_{s=1}^{\infty} b_{k,s}M_{k,s}\}$ $(M_{k,0} \cap \text{span}\{\bigcup_{t=1}^{\infty} \bigcap_{s=t}^{\infty} b_{k,s}M_{k,s}\})$, where $\sum_{s=1}^{\infty} b_{k,s}^{-s} < 2^{-k}$ and $\sum_{k=1}^{\infty} a_k^{-s} < 1/2$. Let $b_{k,0} = 2^{-k}$ for all k and enumerate the set $\{k, s: k = 1, 2, 3, \ldots, s = 0, 1, 2, 3, \ldots\}$ as in II.12. Then $\sum_{k=1}^{\infty} \sum_{s=0}^{\infty} (a_k b_{k,s})^{-s} < 1$ and $\bigcap_{k=1}^{\infty} a_k F_k$ $(\text{span}\{\bigcup_{n=1}^{\infty} \bigcap_{k=1}^{\infty} a_t F_k\})$ contains $\bigcap_{p=1}^{\infty} a_p b_p M_p$ $(\text{span}\{\bigcup_{n=1}^{\infty} \bigcap_{p=n}^{\infty} a_p b_p M_p\})$ (where p is the enumeration of (k, s) mentioned above) which is in $\mathrm{su}(\mathfrak{m})$ $[\mathrm{sp}(\mathfrak{m})]$.

II.15. PROOF OF PROPOSITION II.9. This follows directly from the fact that $p(\mathfrak{m}) = p(p(\mathfrak{m}))$ as shown in II.12.

II.16. PROOF OF THEOREM II.10. This follows directly from Lemma II.9 of [6].

III. Completion of Cauchy filter convergence spaces.

DEFINITION III.1. [5] $C(X, G, \mathfrak{F}) = \{f \in X^*: \lim_{\mathfrak{H}} f$ exists for each refinement $\mathfrak{H}$ of $\mathfrak{F}$ such that $\lim_{\mathfrak{H}} g$ exists for all g in $G\}$.

THEOREM III.2. [5] *$C(X, G, \mathfrak{F})$ is the closure of G in X^* under the topology of convergence on $\mathfrak{F}$.*

REMARK III.3. The significance of the preceding result should not be understated. It implies that the only linear functions on X which may be added to G in any attempt to complete it under the $\mathfrak{F}$-topology are those in $C(X, G, \mathfrak{F})$. If $C(X, G, \mathfrak{F})$ is not complete, then $(G, \mathfrak{F})$ does not have a completion in X^*. Thus if one wishes to complete G by linear functions, one must refine $\mathfrak{F}$ and risk weakening the topology.

THEOREM III.4. (i) *If $(G, \mathfrak{F})$ is a u.c.:.f.c. spa(ce, then $CX, G, \mathfrak{F})$ is the completion of $(G, \mathfrak{F})$ and every Cauchy sequence in G converges uniformly on $\mathfrak{F}$ to one of its limits* [6]. *In that case, $C(X, G, u(\mathfrak{m}))$ is a complete u.c.f.c. space.*

(ii) *If $(G, \mathfrak{F})$ is a s.u.c.f.c. space, then $C(X, G, \mathfrak{F})$ is a completion of $(G, \mathfrak{F})$ and every Cauchy sequence in G converges subuniformly on $\mathfrak{F}$ to one of its limits. Furthermore, if $\mathfrak{F}$ is a refinement of $su(\mathfrak{m})$ for some $s \geqq 1$, then $su(\mathfrak{m})$ induces the same topology on G as $\mathfrak{F}$ and $C(X, G, su(\mathfrak{m}))$ is a complete s.u.c.f.c.* space.

COROLLARY III.5. *$C(X, G, u(\mathfrak{m}))$ $(C(X, G, \mathrm{su}(\mathfrak{m})))$ is always a complete u.c.f.c (s.u.c.f.c) space.*

THEOREM III.6. (i) *If $(G, \mathfrak{F})$ is a p.c.f.c. space, then $C(X, G, \mathfrak{F})$ is a completion of $(G, \mathfrak{F})$, and every Cauchy sequence in G converges pointwise on $\mathfrak{F}$ to one of its limits if and only if, for every Cauchy sequence $\{g_k\}$ in G and $M > \lim_{k\to\infty} \rho(g_k)$, $\bigcup_{n=1}^{\infty} \bigcap_{k=n}^{\infty} \{x: |g_k(x)| < M\}$ is in $\mathfrak{F}$. Furthermore, in that case there exists a filter $\mathfrak{H}$ in X, with $\mathfrak{H} < \mathfrak{F}$, inducing the same topology on G such that $C(X, G, \mathfrak{H})$ is a complete p.c.f.c. space.*

(ii) *If $(G, \mathfrak{F})$ is a s.p.c.f.c. space, then $C(X, G, \mathfrak{F})$ is a completion of $(G, \mathfrak{F})$ and every Cauchy sequence in G converges subpointwise on $\mathfrak{F}$ to one of its limits if and only if for every Cauchy sequence in G and $M > \lim_{k\to\infty} \rho(g_k)$, there is a subsequence $\{g_{k_t}\}$ such that $\bigcup_{n=1}^{\infty} \bigcap_{t=n}^{\infty} \{x: |g_{k_t}(x)| < M\}$ is in $\mathfrak{F}$. Furthermore, if $\mathfrak{F}$ is a refinement of $\mathrm{sp}(\mathfrak{m})$ for some $s \geqq 1$, then there exists a filter $\mathfrak{H}$ in X, with $\mathfrak{H} < \mathfrak{F}$, inducing the same topology on G as $\mathfrak{F}$, such that $C(X, G, \mathfrak{H})$ is a complete s.p.c.f.c. space.*

III.7 THEOREM. *There exists a filter $\mathfrak{H}$, inducing the same topology on G as $p(\mathfrak{m})$ $(\mathrm{sp}(\mathfrak{m}))$ such that $C(X, G, \mathfrak{H})$ is a complete p.c.f.c. (s.p.c.f.c.) space if and only if every Cauchy sequence which converges pointwise (subpointwise) to 0 on $\mathfrak{F}$ converges to 0 on $\mathfrak{F}$.*

III.8 PROOF OF THEOREM III.4. Part (i) follows from Theorem IV.3 of **[6]** and Theorem II.8.

Part (ii) follows from Theorems IV.1 and IV.3 of **[6]** with the substitution of a subsequence for sequence throughout the proofs and Theorem II.6.

III.9 PROOF OF THEOREM III.6. (i) The condition is clearly necessary since $\bigcup_{n=1}^{\infty} \bigcap_{k=n}^{\infty} \{x: |g_k(x)| < M\}$ contains $F \cap \{x: |g(x)| < M\}$, where F is the element in $\mathfrak{F}$ on which $\{g_k\}$ converges pointwise to its limit g. Conversely, let $\{g_k\}$ be a Cauchy in $(G, \mathfrak{F})$ and F an element in $\mathfrak{F}$ such that $\{g_k(x)\}$ converges for all x in F. Let g be in X^* such that $g(x) = \lim_{k\to\infty} g_k(x)$ for all x in F. Pick $\varepsilon > 0$ and choose N so that $\rho(g_N - g_n) < \varepsilon/2$ for all $n \geqq N$. Then $\{x: |g_N(x) - g(x)| < \varepsilon/2\}$ contains $\bigcup_{n=N}^{\infty} \bigcap_{n=k}^{\infty} \{x: |g_N(x) - g_k(x)| < \varepsilon/2\} \cap F$ and is thus in $\mathfrak{F}$. Therefore $\{g_k\}$ converges to g on $\mathfrak{F}$ and thus $C(X, G, \mathfrak{F})$ is complete and each Cauchy sequence in G converges pointwise to one of its limits.

To show the second part of the result, let $\hat{G} = \{f \in C(X, G, \mathfrak{F})$; there exists a Cauchy sequence in $(G, \mathfrak{F})$ converging to f on $\mathfrak{F}$ which converges pointwise on $p(\mathfrak{m})$ to $f\}$. Let $\hat{\rho}$ be the semi-norm on $\hat{G}$ generated by $\mathfrak{F}$; let $\mathfrak{N}$ be the minimal filter for $(\hat{G}, \hat{\rho})$. We need to show that $p(\mathfrak{N}) < \mathfrak{F}$. In that case the $p(\mathfrak{N})$-topology and the $\mathfrak{F}$-topology will be the same on G and $C(X, G, p(\mathfrak{N}))$ is a p.c.f.c. space by Theorem II.6 and is complete since it contains $\hat{G}$.

Let $F = \bigcup_{n=1}^{\infty} \bigcap_{k=n}^{\infty} b_k\{x: |f_k(x)| < 1\}$, where $\{f_k\}$ is a sequence in $\hat{G}$ with $\hat{\rho}(f_k) < 1$ and $b_k \geqq 1$ with $\lim_{k\to\infty} b_k = \infty$. For each k, pick $\{g_k\}$ in G converging to f on $\mathfrak{F}$ and converging pointwise on $p(\mathfrak{m})$, $\rho(g_{ks}) < 2^{-k}$ and $\rho(g_{k,s} - g_{k,s+1}) < 4^{-(k+s)}$. Order the collection $2^{k+s}\{x: |g_{k,s}(x) - g_{k,s+1}(x)| < 4^{-(k+s)}\}$ in the same manner as in II.12. Then $H_1 = \bigcup \bigcap 2^{k+s}$ $\{x: |g_{k,s}(x) - g_{k,s+1}(x)| < 4^{-(k+s)}\}$ is in $p(\mathfrak{m})$ and thus in $\mathfrak{F}$. Next let $H_2 = \bigcup_{n=1}^{\infty} \bigcap_{k=n}^{\infty} b_k/2$ $\{x: |g_{k,k}(x)| < 1/2^k\}$. H_2 is in $p(\mathfrak{m})$ and thus in $\mathfrak{F}$. Finally, let $\{F_k\}$ be a sequence of subspaces of X, each in $p(\mathfrak{m})$ such that $\{g_{k,s}(x)\}$ converges to $f(x)$ for all x in F_k. Then $H_3 = \bigcup_{n=1}^{\infty} \bigcap_{k=n}^{\infty} kF_k$ is in $\mathfrak{F}$ since $p(\mathfrak{m}) = p(p(\mathfrak{m})) < \mathfrak{F}$. Therefore, since F contains $H_1 \cap H_2 \cap H_3$, F is also in $\mathfrak{F}$ and the result follows.

(ii) This part follows in a similar manner by choosing subsequences in the appropriate places.

III.10. Proof of Theorem III.7. We show only the p.c.f.c. case. The necessity of the condition is obvious. To show that it is sufficient, let $\hat{G} = \{\mathrm{f} \in X^*$: there exists a Cauchy sequence $\{g_k\}$ in $(G, p(\mathfrak{m}))$ which converges pointwise to f on $p(\mathfrak{m})\}$. Define $\hat{\rho}(f) = \lim_{k\to\infty} \rho(g_k)$ for all f in $\hat{G}$. Our hypothesis implies $\hat{\rho}$ is a well defined seminorm on $\hat{G}$ and $\hat{\rho}(g) = \rho(g)$ for all g in G. Let $\mathfrak{N}$ be the minimal filter for $(\hat{G}, \hat{\rho})$ and let $\mathfrak{F} = p(\mathfrak{m}) \vee \mathfrak{N}$. Then $(\hat{G}, \mathfrak{F})$ is complete, $\mathfrak{F}$ induces the same topology on G as $\mathfrak{N}$ (Lemma II.9 of **[6]**), and every Cauchy sequence in $(G, \mathfrak{F})$ converges pointwise on $\mathfrak{F}$ to one of its limits. Apply Theorem III.6 to find $\mathfrak{H}$.

IV. Examples.

IV.1. *L^pS spaces.* Let $G = L^p(S)$, $\rho = \|\cdot\|_p$ and $\mathfrak{m}$ be the minimal filter in the formal linear span of S, which we will denote by X, for (G, ρ). We show that $\mathrm{sp}(\mathfrak{m})$ (with $s = p$) is a s.p.c.f.c. space inducing the ρ-topology on G.

Let $F = \mathrm{span}\,\{\bigcup_{n=1}^{\infty} \bigcap_{k=n}^{\infty} b_k\{x: |g_k(x)| < 1\}\}$, where $\rho(g_k) < 1$, $\sum_{k=1}^{\infty} b_k^{-s} < 1$. Then since $T = \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} \{s \in S: |g_k(s)| > b_k\}$ is a set of measure zero, and F contains span $\{T^c\}$ in X. Thus $\mathrm{sp}(\mathfrak{m})$ is courser than $\mathfrak{m} \vee \mathfrak{B}$, where $\mathfrak{B}$ is the filter in X having the linear spans of complements of sets of measure zero as a basis. Lemma II.9 of **[6]** implies that $\mathfrak{m} \vee \mathfrak{B}$ gives the ρ-topology on G. Therefore, so does $\mathrm{sp}(\mathfrak{m})$.

IV.2. *M. Riesz Potentials.* Let n be a positive integer greater than 2 and α a positive integer between 0 and n. The Riez kernel of order α on $\mathbf{R}^n$ is given by

$$K_{\alpha,n}(S) = \frac{\Gamma\left(\frac{n-\alpha}{2}\right)}{\Gamma\left(\frac{\alpha}{2}\right)} \pi^{-n/2} 2^{-\alpha} |s|^{\alpha-n}$$

for s in $\mathbf{R}^n$. (We will hereafter not write the subscripts α and n.) Let E^+ be the collection of all positive Radon measures λ on $\mathbf{R}^n$ such that $(\bar{\rho}(\lambda))^2 = \int_{\mathbf{R}^n}\int_{\mathbf{R}^n} K(s-t)d\lambda(t)d\lambda(s) < \infty$. Let $E = E^+ - E^+$. Then $\bar{\rho}$ is a well defined seminorm on E.

For each λ in E, we associate the extended real valued function u^λ, where $u^\lambda(s) = \int_{\mathbf{R}^n} K(s-t)d\lambda(t)$ for all s in $\mathbf{R}^n$. We will call u^λ the potential of the measure λ.

Let $G = \{u^\lambda: \lambda \in E\}$, $\rho(u^\lambda) = \bar{\rho}(\lambda)$ and X the formal linear span of $\mathbf{R}^n$. We will show that $2p(\mathfrak{m})$, where $\mathfrak{m}$ is the minimal filter for (G, ρ), generates the ρ-topology on G. By Theorem II.6, $(G, 2p(\mathfrak{m}))$ is a s.p.c.f.c. space. As before, let $F = \bigcup_{n=1}^\infty \bigcap_{k=1}^\infty b_k\{x: |u_k^\lambda(x)| < 1\}$, where $\rho(u_k^\lambda) < 1$, $\sum_{k=1}^\infty b_k^{-2} < 1$. Let $T = \bigcap_{n=1}^\infty \bigcup_{k=n}^\infty \{s \in \mathbf{R}^n: |u_k^\lambda(s)| < b_k\}$. Then since outer capacity is countably subadditive, the capacity of T is zero. But F contains the span $\{T^c\}$ in X, so $2p(\mathfrak{m})$ is coarser than the filter $\mathfrak{m} \vee \mathfrak{B}$, where $\mathfrak{B}$ has a basis of linear spans of the complements of sets of capacity zero. Therefore Theorem 3 of [**8**] and Lemma II.9 of [**6**] imply that $2p(\mathfrak{m})$ induces the ρ-topology on G.

IV.3 *Bepo Levi Functions.* Let S be a non-empty, connected, open subset of $\mathbf{R}^n$. A continuous vector field $f(s) = (f_1(s), f_2(s), f_3(s), \ldots, f_n(s))$ is irrotational if $\Gamma(C) = \int_C f(s)ds = 0$ for all closed curves homologuos to zero in S. The linear space $G = BL^p(s)$ of smooth Bepo Levi functions on S. is the collection of all functions u on S where grad $u = f$, f is in $L^p(s)$ and is irrotational. Define ρ on G by $\rho(u) = \|f\|_p$. Then Theorem 15 of [**8**] implies that every Cauchy sequence in (G, ρ) has a subsequence which converges pointwise except on a set E in ξ^p. A set E in on ξ^p if there exist a positive function h in $L^p(s)$ such that $u^h(s) = \infty$ for all s in E and u^h is not identically ∞. (The function $u^h{}_k$ is the Riesz potential of order 1 of Radon measure $d\lambda = h(s)ds$).

Theorem 3 of [**8**] supplies a key step in the proof of Theorem 15. A simple reconstruction of the argument shows that one need only choose the subsequence such that $\sum_{k=1}^\infty \rho(u_{n_k} - u_{n_{k+1}}) < \infty$. An agrument parallel to those in IV.1 and IV.2 then shows that sp$(\mathfrak{m})$ (with $s = p$) generates the same topology as ρ.

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