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FOCAL POINTS OF NONLINEAR EQUATIONS-A DYNAMICAL ANALYSIS

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Introduction. We consider the $2n^{th}$ order scalar nonlinear differential equation.

$$x^{(2n)}(t) = f(t, x(t)) \tag{E}$$

where f is continuous on $[0, \infty) \times R$. Let α be a real number $\geqq 0$, and k a natural number with $1 \leqq k \leqq n - 1$. The $(2k, 2(n - k))$ focal point of α for the equation (E) is the smallest $\beta > \alpha$ for which there exists a non-trivial solution of (E) satisfying the boundary conditions

$$\begin{aligned} x^{(i)}(\alpha) &= 0, \qquad 0 \leqq i \leqq 2k - 1 \text{ and} \\ x^{(i)}(\beta) &= 0, \qquad 2k \leqq i \leqq 2n - 1. \end{aligned} \tag{1}$$

For linear equations there is a large bibliography concerning focal points and conjugate points. More relevant to the present work, there are several studies on the relation between the non-existence of solutions to two point boundary value problems (disfocality, disconjugacy) and the existence of monotone solutions having prescribed asymptotic behaviour. We refer to Elias **[2]** in the linear case, and Edelson, Kreith **[4]** in the nonlinear case. Unlike the linear problem, the existence of monotone solutions in the nonlinear case has frequently been established by means of topological methods such as fixed point theorems, which may give less information but are more generally applicable.

Motivated by these considerations we will study the properties of focal point trajectories of nonlinear equations. Specifically, we will show that for the class of equations under consideration, in the disfocal case there must exist monotone solutions which we will call focal asymptotic.

DEFINITION. A trajectory $x(t)$ is said to be $(2k, 2(n - k))$ focal asymptotic on $[\alpha, \infty)$ if it satisfies the conditions

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$$(2)\qquad \begin{gathered} x^{(i)}(\alpha) = 0, \qquad 0 \leqq i \leqq 2k-1 \\ (-1)^i x^{(i)}(t) > 0,\ 2k \leqq i \leqq 2n-1 \\ \text{for all } t > \alpha. \end{gathered}$$

If in addition $\lim_{t\to\infty} x^{(2k)}(t) = 0$, then $\beta = \infty$ is said to be a $(2k, 2(n-k))$ focal point of α.

Our principal tool for the analysis of trajectories will be a theorem of combinatorial topology known as Sperner's lemma (cf. **[1]**). This lemma was used in **[3]** to prove the existence of (2, 2) system conjugate points $\hat{\eta}$ and (2, 2) system focal points $\hat{\mu}$, defined respectively by

$$x(\alpha) = x''(\alpha) = x(\hat{\eta}) = x''(\hat{\eta}) = 0$$
$$x(\alpha) = x''(\alpha) = x'(\hat{\mu}) = x'''(\hat{\mu}) = 0.$$

Main result. It will be convenient to represent (E) as a second order system of the form

$$(\Sigma)\qquad X''(t) = F(t, X)$$

$X = (Y, Z)$ where $Y \in R^k$ and $Z \in R^{n-k}$ are defined by $Y = (y_1, \ldots, y_k) = (x_1, \ldots, x_k)$ and $Z = (\zeta_1, \ldots, \zeta_{n-k}) = (x_{k+1}, \ldots, x_n)$. Then clearly the boundary conditions (1) take the form

$$(1')\qquad Y(\alpha) = Y'(\alpha) = 0,\ Z(\beta) = Z'(\beta) = 0.$$

Solutions of (1) are defined by trajectories of the initial value problem (Σ)-(2) where

$$(2)\qquad \begin{gathered} Y(\alpha) = Y'(\alpha) = 0 \\ Z(\alpha) = (1, \ldots, 1) \\ Z'(\alpha) = -(\lambda_1, \ldots, \lambda_{n-k}) = v_0 \end{gathered}$$

and the initial velocity vector v_0 satisfies $0 \leqq \lambda_i$, $1 \leqq i \leqq n-k$. A solution of the initial value problem (Σ)-(2) will be denoted by $X(t; v_0) = (Y(t; v_0), Z(t; v_0))$. Assume that K denotes the open positive cone of R^n and let ∂K be its boundary, which consists of the hyperplanes $H_i = \{x \in R^n : x_i = 0, x_j \geqq 0, 1 \leqq j \leqq n, j \neq i\}$.

We will say that the trajectory $X(t; v_0)$ egresses from K whenever there exists a $t_1 > \alpha$ such that

$$X(t; v_0) \in K,\ \alpha < t < t_1 \text{ and } X(t_1; v_0) \in \partial K.$$

If moreover there exists $\varepsilon > 0$ such that $X(t; v_0) \notin \bar{K}$ for $t \in (t_1, t_1 + \varepsilon]$, then $X(t; v_0)$ is said to egress strictly from K.

The next lemma shows that, trajectories satisfying the initial conditions (2), can be made to egress (strictly) from K on any of the hyperplanes H_i, $k+1 \leqq i \leqq n$. Let $e^i \in R^n$ be the vector defined by

$$e^i_j = -\delta_{ij} = \begin{cases} -1, & i = j \\ 0, & i \neq j \end{cases}$$

i.e., δ_{ij} is the Kronecker delta function.

LEMMA 1. *Assume that f is a continuous function and $xf(t, x) > 0$ for every $x \neq 0$. Then for each i, $k + 1 \leqq i \leqq n$, there exists a $\lambda_i > 0$ such that for any $\lambda > \lambda_i$ the trajectory $X(t; \lambda e^i)$ egresses from K on the hyperplane H_i.*

PROOF. The trajectory $X(t) \equiv X(t; \lambda e^i)$ satisfies the initial conditions

$$(3) \qquad \begin{aligned} &y_j(\alpha; \lambda e^i) = y_j'(\alpha; \lambda e^i) = 0, \ 1 \leqq j \leqq k \\ &\zeta_j(\alpha; \lambda e^i) = 1, \ 1 \leqq j \leqq n - k \\ &\zeta_j'(\alpha; \lambda e^i) = 0, \ 1 \leqq j \leqq n - k, j \neq i \\ &\zeta_i'(\alpha; \lambda e^i) = -\lambda. \end{aligned}$$

Consequently there exists a $\hat{t} > \alpha$ such that for $\alpha < t < \hat{t}$, the coordinates of X satisfy $y_j(t) > 0$ for $1 \leqq j \leqq k$, and $\zeta_j(t) > 0$ for $1 \leqq j \leqq i$. Since $x_1 f(t, x_1) > 0$, it follows that $\zeta_j(t) > 0$ for $i + 1 \leqq j \leqq n$, and for $\alpha < t < \hat{t}$. Furthermore, each component $y_j(t)$, $\zeta_j(t)$ is an increasing function as long as $\zeta_i(t) > 0$. It follows that if any component has a zero on (α, ∞), the first such zero must be in $\zeta_i(t)$. We must now show that for λ sufficiently large the trajectory $X(t; \lambda e^i)$ egresses (strictly) from the positive cone K.

By (E) and Taylor's formula, we get a point $\hat{t} \in (\alpha, t)$, $t \in \text{Dom } x$ such that

$$x_1(t) = \sum_{j=0}^{2n-1} \frac{(t - \alpha)^j}{j!} x^{(j)}(\alpha) + \frac{(t - \alpha)^{2n}}{(2n)!} x^{(2n)}(\hat{t}).$$

Thus, in view of (2) and the choice $v_0 = \lambda e^i$, we get

$$(4) \qquad x_1(t) = \sum_{j=n-k+1}^{n-1} \frac{(t-\alpha)^{2j}}{(2j)!} - \frac{(t-\alpha)^{2i+1}}{(2i+1)!} \lambda + \frac{(t-\alpha)^{2n}}{(2n)!} f(\hat{t}, x_1(\hat{t})).$$

Now for $\lambda > \lambda_0$, by the continuity of $x_i = x_i(\cdot\,; \lambda e^i) = x_i(\cdot\,; \lambda)$ and since

$$x_i(\alpha; \lambda) = x_i(\alpha; \lambda_0) = 1 \text{ and } x_i'(\alpha; \lambda) = -\lambda, \ x_i'(\alpha; \lambda_0) = -\lambda_0$$

it follows that there exists a number $\tau > \alpha$ such that

$$x_i(t; \lambda_0) > x_i(t; \lambda), \ \alpha \leqq t \leqq \tau.$$

Consequently in view of (2) and (3), we may assume

$$x_1(t; \lambda_0) > x_1(t; \lambda), \ \alpha \leqq t \leqq \tau.$$

We choose $\lambda_0 \in \mathbf{R}$ (for example $\lambda_0 = 0$) such that

$$x_1(t;\lambda_0) > 0,\ t \in \operatorname{Dom} x_1(\cdot\,;\lambda_0)$$

and assume that for every $\lambda \geqq \lambda_0$

$$x_1(t;\lambda) > 0,\ t \in \operatorname{Dom} x_1(\cdot\,;\lambda).$$

We shall prove that for every $t^* \in \operatorname{Dom} x_1(\cdot\,;\lambda_0)$

$$\psi(t;\lambda) = x_1(t;\lambda) - x_1(t;\lambda_0) < 0,\ \alpha < t \leqq t^*. \tag{5}$$

Assume that there exists a root $\hat{\tau} \in (\alpha, t^*]$ of the function $\psi(t;\lambda)$, i.e.,

$$\psi(t;\lambda) < 0,\ \alpha < t < \hat{\tau} \text{ and } \psi(\hat{\tau};\lambda) = 0. \tag{6}$$

Clearly, since $\psi^{(2n)}(t) = f(t, x_1(t;\lambda)) - f(t, x_1(t;\lambda_0))$, integrations leads (as in (4)) to

$$\psi(t;\lambda) = \frac{(t-\alpha)^{2i+1}}{(2i+1)!}(\lambda_0 - \lambda) + \frac{(t-\alpha)^{2n}}{(2n)!}[f(\hat{t}, x_1(\hat{t};\lambda)) - f(\hat{t}, x_1(\hat{t};\lambda_0))]$$

and thus we get

$$(\lambda - \lambda_0)\frac{(\hat{\tau} - \alpha)^{2i+1}}{(2i+1)!} = \frac{(\hat{\tau} - \alpha)^{2n}}{(2n)!}[f(\hat{t}, x_1(\hat{t};\lambda)) - f(\hat{t}, x_1(\hat{t};\lambda_0))].$$

Now since $0 < x_1(t;\lambda) \leqq x_1(t;\lambda_0)$, for every $\lambda > \lambda_0$, the second member of the last equality is bounded when $\lambda \to \infty$ but not the first one. Thus (5) holds.

Consider now the rectangle $R = [\alpha, t^*] \times [0, q]$, $q = x_1(t^*;\lambda_0)$. Then by (5) we get

$$G(x_1(\cdot\,;\lambda)\,|\,[\alpha, t^*]) = \{(t, x_1(t;\lambda)) : \alpha \leqq t \leqq t^*\} \subseteq R.$$

By the continuity of f, there exists $M > 0$ such that

$$|f(t, x)| \leqq M,\ (t, x) \in R.$$

Then by (4) we get

$$x_1(t^*) \leqq \sum_{j=n-k+1}^{n-1} \frac{(t^* - \alpha)^{2j}}{(2n)!} + \frac{(t^* - \alpha)^{2n}}{(2n)!} M - \frac{(t^* - \alpha)^{2i+1}}{(2i+1)!}\lambda.$$

Thus, by choosing λ large enough, say $\lambda > \lambda_i$, we get that $x_1(t^*) < 0$, a contradiction.

We need another lemma, due to Sperner (see [1]).

LEMMA [2]. *Let T^n be an n-simplex with vertices $\{e^0, e^1, \ldots, e^n\}$ and let $\{E_0, \ldots, E_n\}$ be a covering of the closure $\bar{T}^n$ by closed sets such that each closed face $[e^{i_0}, \ldots, e^{i_r}]$ of T^n is contained in the union $E_{i_0} \cup \cdots \cup E_{i_r}$. Then the intersection $\bigcap_{i=0}^n E_i$ is nonempty.*

Now, we are ready to formulate and prove our main result.

THEOREM. *For each k, $1 \leqq k \leqq n - 1$, equation (E) has a focal point trajectory of the type $(2k, 2(n-k))$ or a $(2k, 2(n-k))$ focal asymptotic trajectory, provided that assumptions of Lemma 1 hold.*

PROOF. The result will be proved by an application of Sperner's lemma, as in [2]. Let $S \subseteq \bar{K}$ be the closed $n - k$ simplex spanned by the vertices $e^0 = 0$ and λe^i, $k + 1 \leqq i \leqq n$. Using the usual notation, $[e^{i_0}, e^{i_1}, \ldots, e^{i_r}]$ denote the closed face of S spanned by the vertices $\{e^{i_0}, e^{i_1}, \ldots, e^{i_r}\}$. Here λ is chosen large enough, so that for any initial vector v_0, which starts from $e^0 = 0$ and ends on $[e^{k+1}, e^{k+2}, \ldots, e^n]$, at least one of its projections is greater than $\max\{\lambda_i\}$, where λ_i are established by the previous lemma.

Define the sets E_i, $k + 1 \leqq i \leqq n$, by

$$E_0 = \mathrm{cl}\{v_0 \in S\colon X(t; v_0) \text{ remains in } K \text{ for all } t > \alpha\}$$

and $E_i = \mathrm{cl}\{v_0 \in S\colon X(t, v_0)$ egresses strictly from K on the hyperplane $x_i = 0\}$, for $i = k + 1, k + 2, \ldots, n$.
To apply Sperner's lemma it is necessary to show

i) the sets $\{E_i\}$ form a closed covering of S and

ii) $$[e^{i_0}, e^{i_1}, \ldots, e^{i_r}] \subseteq \bigcup_{j=0}^{r} E_{i_j}.$$

It is obvious from the definitions that E_i are closed sets, and their union covers S, because every trajectory remains in K on (α, ∞) or else egresses from K on some hyperplane $x_i = 0$, $k + 1 \leqq i \leqq n$.

In order to prove (ii), let $v_0 \in [e^{i_0}, e^{i_1}, \ldots, e^{i_r}]$ and assume that $X(t; v_0) \in K$ for $\alpha < t \leqq t_1 \leqq \infty$. We will examine two cases:

(α) $0 \notin \{i_0, i_1, \ldots, i_r\}$. Then by the choice of λ and the previous lemma, it is obvious that the solution $X(t, v_0)$ egresses strictly from K, i.e., $t_1 < \infty$. Now if $j \notin \{i_0, i_1, \ldots, i_r\}$, then $x_j'(\alpha; v_0) = 0$. From the nature of the force field in K, it follows that $x_j(t; v_0)$ is a positive, increasing function on $[\alpha, t_1)$, so $x_j(t_1; v_0) > 0$ and $X(t; v_0)$ egresses from K on some hyperplane $x_i = 0$, with $i \neq j$. So $v_0 \in \bigcup_{k=0}^{r} E_{i_k}$.

(β) $0 \in \{i_0, i_1, \ldots, i_r\}$. In this case we may have $t_1 = \infty$, so $X(t; v_0)$ either egresses from K on some hyperplane H_{i_j}, $0 \leqq j \leqq r$, or remains in $\mathrm{cl}(K)$ on $[\alpha, \infty)$. In the former case $v_0 \in E_{i_j}$, and in the later case $v_0 \in E_0$.

By Sperner's lemma there exists a point $v^* \in E_0 \cap [\bigcap_{i=k+1}^{n} \mathrm{E}_i]$, and such a trajectory must satisfy the conditions

$$(8) \quad z_i(t; v^*) > 0, z_i'(r; v^*) < 0 \text{ for } 1 \leqq i \leqq n - k, \alpha < t < t_1 \leqq \infty.$$

If $t_1 < \infty$ then $z_i(t_1; v^*) = 0$ for $1 \leqq i \leqq n - k$. Since $X(t; v^*) \in \mathrm{cl}(K)$ for all $t \geqq \alpha$, we must also have $z_i'(t_1; v^*) = 0$, so $X(t; v^*)$ is a focal point trajectory, and $\beta = t_1$ is a focal point of α. If instead $t_1 = \infty$, then (8) is satisfied on (α, ∞), and $X(t; v^*)$ is focal asymptotic on $[\alpha, \infty)$.

It is interesting to determine conditions which are sufficient to guarantee that focal asymptotic trajectories necessarily define focal points at ∞, i.e., the guarantee that $\lim_{t\to\infty} x^{(2k)}(t) = 0$. We first observe that the higher derivatives must go to zero as $t \to \infty$.

LEMMA 2. *If $X(t; v^*)$ is a $(2k, 2(n-k))$ focal asymptotic trajectory on $[\alpha, \infty)$, then*

$$\lim_{t\to\infty} x^{(i)}(t; v^*) = 0, \quad 2k+1 \leqq i < 2n \tag{9}$$

PROOF. The function $u_i(t) = (-1)^i x^{(i)}(t; v^*)$, $2k \leqq i < 2n$ is positive, decreasing on $[\alpha, \infty)$, hence $\lim_{t\to\infty} u_i(t) = c_i \geqq 0$. If $c_i > 0$ then

$$u_{i-1}(t) = u_{i-1}(\alpha) + \int_\alpha^t u_{i-1}'(s)\, ds = u_{i-1}(\alpha) - \int_\alpha^t u_i(s)\, ds \leqq u_{i-1}(\alpha) - c_i(t-\alpha),$$

so $u_{i-1}(t) \to -\infty$ as $t \to \infty$. Therefore $c_i = 0$ for $2k < i < 2n$.

Finally, we derive sufficient conditions for focal points at ∞. Recall that equation (E) is said to be sublinear (resp. superlinear) if $f(t, x)$ is nonincreasing (resp. nondecreasing) in x.

THEOREM 2. *Let equation (E) be sublinear or superlinear, and assume that*

$$\int^\infty f(s, \varepsilon s^{2k})\, ds = \infty \tag{10}$$

for all $\varepsilon > 0$. If $x(t)$ is a $(2k, 2(n-k))$ focal asymptotic trajectory on $[\alpha, \infty)$, then $\lim_{t\to\infty} x^{(2k)}(t) = 0$, and $\beta = \infty$ is a focal point of α.

PROOF. Let (E) be superlinear and assume $\lim_{t\to\infty} x^{(2k)}(t) = c > 0$. Then

$$x^{(2k-1)}(t) = x^{(2k-1)}(\alpha) + \int_\alpha^t x^{(2k)}(s)ds \geqq c(t-\alpha),$$

and by successive integrations $x(t) \geqq (c/(2k)!)(t-\alpha)^{2k}$. Then

$$\begin{aligned} x^{(2n-1)}(t) &= x^{(2n-1)}(\alpha) + \int_\alpha^t x^{(2n)}(s)ds \\ &= x^{(2n-1)}(\alpha) + \int_\alpha^t f(s, x(s))\, ds \geqq x^{(2n-1)}(\alpha) \\ &\quad + \int_\alpha^t f\left(s, \frac{c}{(2k)!} s^{2k}\right) ds, \end{aligned}$$

which tends to ∞ as $t \to \infty$. This contradicts lemma 2. The sublinear case is similar, since for t sufficiently large, $x^{(2k)}(t) < 2c$.

A counterexample. The condition (10) in Theorem 2, is necessary in order for a focal asymptotic trajectory to define a focal point at $\beta = \infty$. The referees provided the next example.

Consider the fourth order linear equation

$$y^{(4)}(t) = (c/t^5)y(t),\ t \geqq 1, \tag{11}$$

where c is a constant in (0, 9/16). *Then, for any sufficiently large* α, *this equation has neither a* (2, 2) *focal point* $\beta < \infty$ *nor* (2, 2) *focal asymptotic trajectory with* $\beta = \infty$ *as a focal point.*

PROOF. We must prove that no solution of (11) can satisfy either

$$y(\alpha) = y'(\alpha) = 0 = y''(\beta) = y'''(\beta) \text{ or} \tag{12}$$

$$y(\alpha) = y'(\alpha) = 0, \lim_{t\to\infty} y''(t) = \lim_{t\to\infty} y'''(t) = 0. \tag{13}$$

Let assume that (12) holds. Then $\int_\alpha^\beta y^{(4)}(t)dt = \int_\alpha^\beta ct^{-5}y^2\, ds$ and after an integration by parts we get

$$\int_\alpha^\beta (y''(t))^2\, dt = \int_\alpha^\beta ct^{-5}y^2\, dt. \tag{14}$$

Now

$$\begin{aligned}\int_\alpha^\beta t^{-4}y^2\, dt &= \int_\alpha^\beta (-1/3)(t^{-3})'y^2\, dt \\ &= -1/3\ \beta^{-3}y(\beta^2) + \int_\alpha^\beta (2/3)t^{-3}(yy')dt \\ &\leqq \int_\alpha^\beta (2/3)(t^{-2}y)(t^{-1}y')dt.\end{aligned}$$

Consequently

$$\int_\alpha^\beta t^{-4}y^2\, dt \leqq (2/3)\left(\int_\alpha^\beta t^{-4}y^2 dt\right)^{1/2}\left(\int_\alpha^\beta t^{-2}(y')^2 dt\right)^{1/2}$$

which implies that

$$\int_\alpha^\beta t^{-4}y^2\, dt \leqq (4/9)\int_\alpha^\beta t^{-2}(y')^2\, dt. \tag{15}$$

Similarly we can obtain that

$$\int_\alpha^\beta t^{-2}(y')^2\, dt \leqq 4\int_\alpha^\beta (y'')^2\, dt. \tag{16}$$

From (14), (15) and (16) we have

$$\int_\alpha^\beta (y'')^2\, dt \leqq c\int_\alpha^\beta t^{-4}(y'')^2\, dt \leqq (16c/9)\int_\alpha^\beta (y'')^2\, dt < \int_\alpha^\beta (y'')^2\, dt$$

which is a contradiction.

Proof of (*ii*). By asymptotic theory (e.g., Stability and Asymptotic Behavior of Differential Equations by W.A. Coppel, p. 92) there are independent solutions $y_0(t)$, $y_1(t)$, $y_2(t)$, $y_3(t)$ of (11) which have the asymptotic behavior as $t \to \infty$:

$$\lim y_k^{(i)}(t)\,t^{k-1} = \begin{cases} 1/(k-i), & 0 \leqq i \leqq k \\ 0, \; k < i < n. \end{cases}$$

Suppose $y(t) = \sum_{i=0}^{3} c_i y_i(t)$ satisfies $y'''(t) \to 0$ as $t \to \infty$. The above asymptotic conditions imply $c_3 = 0$. Similarly $y''(t) \to 0$ as $t \to \infty$ implies $c_2 = 0$. Thus if y is a nontrivial solution of (11) satisying (13), $y(t) = c_0 y_0(t) + c_1 y_1(t)$. Thus $y(\alpha) = y'(\alpha) = 0$ and c_0, c_1 not both zero imply that

$$\begin{vmatrix} y_0(\alpha) & y_1(\alpha) \\ y_0'(\alpha) & y_1'(\alpha) \end{vmatrix} = 0. \tag{17}$$

Since by the above relations

$$y_0(t)y_1'(t) - y_0'(t)y_1(t) \to 1 \text{ as } t \to \infty,$$

(17) will fail to hold for all sufficiently large α.

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