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ON VALUATION RINGS

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ABSTRACT. In this note several definitions and results concerning valuation domains are extended to valuation rings. First we define immediate extension, maximally complete, pseudo-convergent sequence and maximal. It is demonstrated that maximal is equivalent to linearly compact and that maximal implies maximally complete. Note the three are equivalent in the case of valuation domains. We conclude by establishing the existence of a maximal completion of an arbitrary valuation ring.

Notation and Terminology. All rings will be commutative with identity. By a valuation ring we mean a ring whose ideals are linearly ordered by set inclusion. The principal ideals of a valuation ring form a totally ordered monoid where the ordering is reverse inclusion. This monoid will be called the value monoid. (See [**6**].)

We assume throughout that $V(V')$ is a valuation ring with value monoid M (M'), maximal ideal $\mathcal{M}$ ($\mathcal{M}'$) and residue field $k\,(k')$. We use $\subset$ for proper containment. $U(R)$ will denote the units of a ring R.

1. Maximal, Linearly Compact, Maximally Complete. V' is called an extension of V provided $V \subseteq V'$ and $\mathcal{M}' \cap V = \mathcal{M}$. V' is an immediate extension of V if V' is an extension of V and $k = k'$, $M = M'$. (Equality means the natural embeddings are bijections.) V is called maximally complete if V has no proper immediate extensions. V' is called a maximal completion of V if V' is an immediate extension of V and V' is maximally complete. These definitions stated for valuation domains can be found in [**1**, p. 91], [**5**, p. 36] or [**2**, p. 305].

A set of elements $\{r_\rho\}_{\rho\in A}$ from V is called a pseudo-convergent sequence provided A is a well-ordered set without a last element and $(r_\tau - r_\sigma)V \subset (r_\sigma - r_\rho)V$ for $\rho < \sigma < \tau$. If $\{r_\rho\}$ is such a sequence, then $(r_\sigma - r_\rho)V = (r_{\rho+1} - r_\rho)V$ for all $\rho < \sigma$. Also, either $r_\sigma V \subset r_\rho V$ for all $\rho < \sigma$ or there exists $\lambda \in A$ such that $r_\rho V = r_\sigma V$ for all $\rho, \sigma > \lambda$.

Given a pseudo-convergent sequence $\{r_\rho\}_{\rho\in A}$, for each $\rho \in A$ set $J_\rho = (r_{\rho+1} - r_\rho)V$. An element $r \in V$ is called a pseudo-limit of $\{r_\rho\}$ provided

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$(r - r_\rho)V = J_\rho$ for all $\rho \in A$. V is called maximal if every pseudo-convergent sequence in V has a pseudo-limit in V.

The above treatment of pseudo-convergence generalizes the one used by Kaplansky [**2**, p. 303] and Schilling [**5**, p. 39]. It should be noted that Ostrowski's treatment differs slightly [**4**, p. 368].

An alternate approach to pseudo-convergence is linear compactness. An arbitrary ring R is called linearly compact if every system of congruences $\{X \equiv r_\rho \bmod I_\rho\}_{\rho \in A}$ which is finitely solvable in R has a solution in R where $r_\rho \in R$, I_ρ are ideals of R and A is an index set.

The case of valuation domains is summarized by:

THEOREM 1.1. *Assume D is a valuation domain. The following are equivalent.*

i. *D is maximal.*
ii. *D is linearly compact.*
iii. *D is maximally complete.*

PROOF. i. $\Leftrightarrow$ iii. [**2**, Thm. 4]; ii. $\Leftrightarrow$ iii. [**1**, Thm. 12.6].

More generally we have.

THEOREM 1.2.

a. *V is maximal if and only if V is linearly compact.*
b. *If V is maximal, then V is maximally complete.*

PROOF.

a. Assuming V is linearly compact, we must find a pseudo-limit in V for any pseudo-convergent sequence $\{r_\rho\}_{\rho \in A}$ in V. For each $\rho \in A$, let $I_\rho = \{s \in V\colon sV \subseteq J_\rho\}$. It follows that $\{X \equiv r_\rho \bmod I_\rho\}_{\rho \in A}$ is pairwise solvable which implies finitely solvable. Let r be a solution of the entire system. Then $(r - r_\rho)V = J_\rho$ since $(r_{\rho+1} - r_\rho)V = J_\rho \supset J_{\rho+1} \supseteq (r - r_{\rho+1})V$, i.e., r is a pseudo-limit of the given sequence.

For the converse, let $\{X \equiv r_\rho \bmod I_\rho\}_{\rho \in A}$ be a finitely solvable system of congruences. We must show $\bigcap_{\rho \in A}(r_\rho + I_\rho) \neq \phi$. By eliminating repeated cosets we may assume A is totally ordered by $\rho < \sigma$ provided $r_\sigma + I_\sigma \subset r_\rho + I_\rho$. Furthermore, A contains a well-ordered cofinal subset. Since it suffices to show the intersection taken over this cofinal subset is nonempty, we may assume A is well-ordered.

If A has a last element, then $\bigcap_{\rho \in A}(r_\rho + I_\rho) \neq \phi$. If A does not have a last element, then for each $\rho \in A$ choose $s_\rho \in (r_\rho + I_\rho) - (r_{\rho+1} + I_{\rho+1})$. Note that $s_\rho + I_\rho = r_\rho + I_\rho$. For $\rho < \sigma < \tau$, $s_\rho \notin s_\sigma + I_\sigma$ and $s_\tau \in s_\sigma + I_\sigma$. Thus, $(s_\tau - s_\sigma)V \subseteq I_\sigma \subset (s_\sigma - s_\rho)V$, i.e., $\{s_\rho\}_{\rho \in A}$ is a pseudo-convergent sequence. Any limit s of this sequence satisfies $(s - s_\rho)V = (s_{\rho+1} - s_\rho)V \subseteq I_\rho$. Thus, $s \in r_\rho + I_\rho$ for all ρ.

b. Assume V' is a proper immediate extension of V. Let $s' \in V' - V$.

Since $M = M'$, $s'V' = sV'$ for some $s \in V$. Then $s' = sb'$ for some $b' \in U(V')$ where $b' \notin V$. Thus, by replacing s' with b' we may assume $s' \in U(V')$. Let $\mathscr{T} = \{(s' - r)V' \colon r \in V, r \neq 0\}$. We first show $\mathscr{T}$ does not have a smallest element. In particular, letting $r \in V$, $r \neq 0$, we find a nonzero $s \in V$ such that $(s' - s)V \subset (s' - r)V$. Since $M = M'$, there is a nonzero $t \in V$ such that $(s' - r)V' = tV'$. There exists $a' \in U(V')$ such that $s' - r = ta'$. Since $k = k'$, $a' = a + m'$ where $a \in U(V)$ and $m' \in \mathscr{M}'$. Thus, $(s' - r - ta)V' = (tm')V' \subset t'V' = (s' - r)V'$. Set $s = r + ta$. Note that $s \neq 0$ for otherwise $s' = tm'$, i.e., s' would be a nonunit of V' which contradicts the choice of s'.

For each nonzero $r \in V$, let $I_r = \{t \in V \colon tV' \subseteq (s' - r)V'\}$. The system of congruences $\{X \equiv r \bmod I_r\}$ is pairwise solvable in V but not solvable. For assume $r_0 \in V$ is a solution to the entire system, i.e., $(r_0 - r)V' \subseteq (s' - r)V'$ for all nonzero $r \in V$. This implies $(s' - r_0)V'$ is the smallest element of $\mathscr{T}$ which contradicts the above paragraph.

Note that we are unable to establish the converse of b.

QUESTION 1.3. If V is maximally complete, is V maximal?

2. Existence of a Maximal Completion. In obtaining a maximal completion of a valuation domain D, one first obtains a bound on the cardinality of D by establishing a one-to-one correspondence between D and a subset of the long power series domain $K[[G^+]]$. Here K is the residue field and G is the value group of D. For a discussion of $K[[G^+]]$, see Section 11 of [1]. We proceed in a similar manner.

For a field K and value monoid N, define $K[[N]]$ to be the set of all series of the form $\sum_{\alpha \in S} a_\alpha X^\alpha$ where $S \subseteq N - \{\infty\}$ is a well-ordered set (the ordering is induced by N) and $a_\alpha \in K$. (The use of an arbitrary value monoid N is allowed by Shores' characterization of value monoids as being precisely 0-segmental monoids [6].) Just as in the domain case, addition and multiplication can be defined so that $K[[N]]$ is a ring. In fact, $K[[N]]$ is a maximal valuation ring with residue field K and value monoid N.

THEOREM 2.1. *The cardinality of V is bounded by the cardinality of $k[[M]]$.*

PROOF. The proof is only a slight variation of the one required in the case of domains. [5, p. 37].

Since we have obtained a bound on the cardinality of V, the proof of the existence of a maximal completion of V is a set-theoretic argument similar to the one used to establish an algebraic closure of a field. For example, see [3, Thm. 66].

THEOREM 2.2. *V has a maximal completion.*

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