ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 17, Number 1, Winter 1987

ON THE AZIMI-HAGLER BANACH SPACES

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ABSTRACT. We study the X_α spaces constructed by Azimi and Hagler as examples of hereditarily l_1 spaces failing the Schur property. We show that each complemented non weakly sequentially complete subspace of X_α contains a complemented isomorph of X_α, and that X_α and X_β are isomorphic if and only if they are equal as sets.

Azimi and Hagler [1] have introduced a class of Banach spaces, the X_α spaces. Each of the spaces is hereditarily ℓ_1 and yet fails the Schur property. In this paper we discuss the isomorphic classification of the X_α spaces and show that each non weakly sequentially complete complemented subspace of an X_α space X contains a complemented isomorph of X. This lends credence to the conjecture that the X_α spaces are primary, that is, that if $X_\alpha = Y \oplus Z$, then either Y or Z is itself isomorphic to X_α. Indeed, a technique for showing that a space W is primary is to show first that if $W = Y \oplus Z$, then either Y or Z contains a complemented isomorph of W and then to use a decomposition method, based either on W being isomorphic to some infinite direct sum $\Sigma \oplus W$ [5] or on knowledge that either Y or Z is isomorphic to its Cartesian square [3]. In the case of the X_α spaces, Azimi and Hagler [1] showed that X_α is of codimension one in its first Baire class, so that if $X_\alpha = Y \oplus Z$, then precisely one summand is weakly sequentially complete. Thus our result accomplishes the first step in this program. Unfortunately, by the same dimension argument, the summand containing X_α is not isomorphic to its square, and X_α is not isomorphic to any infinite direct sum $\Sigma \oplus X_\alpha$. In the case of James' quasi-reflexive space J, Casazza [2] was able to overcome difficulties of this type, and showed J to be primary. Some of our techniques are similar to those used by Casazza in [2]. Our terminology is generally the same as that of [1] or [4], and at several points in the analysis we use perturbation arguments such as Proposition 1.a.9 of [4].

The X_α spaces are defined as follows. Let $\alpha = \{\alpha_i\}_{i=1}^\infty$ be a sequence of real numbers satisfying

AMS (MOS) subject classifications (1980). 46B99, 46B15
Received by the editors on April 25, 1985, and in revised form on June 28, 1985.

(1) $$\alpha_1 = 1 \text{ and } \alpha_i \geqq \alpha_{i+1} \text{ for } i = 1, 2, \cdots,$$

(2) $$\lim_{i\to\infty} \alpha_i = 0,$$

and

(3) $$\Sigma\alpha_i = \infty.$$

The usual unit vectors in the space W of finitely nonzero sequences (or in X_α) are denoted by $\{e_i\}$, and the biorthogonal functionals by $\{e_i^*\}$. A block is an interval of integers, and a sequence $\{F_i\}$ of blocks is admissible if $\max F_i < \min F_{i+1}$ for each i. For each block F, define a functional, also denoted by F, by $\langle F, x\rangle = \sum_{i\in F}\langle e_i^*, x\rangle$. Then X_α is the completion of W with respect to the norm

(4) $$\|x\| = \max \sum_{i=1}^{n} \alpha_i|\langle F_i, x\rangle|,$$

where the max is taken over all n and all admissible sequences $\{F_i\}_{i=1}^n$ of blocks. The functionals associated with blocks are of course bounded on X_α. We denote the natural projections associated with the unit vector basis by P_n.

From the definition of the norm it is easy to see that the unit vector basis is spreading (equivalent to each of its subsequences) and bi-monotone. That is, for each $x \in X_\alpha$ and each $n < m$, $\|(P_m - P_n)x\| \leqq \|x\|$. Further, if $\{e_{i_k}\}$ is a subsequence of $\{e_n\}$, then $[\{e_{i_k}\}]$ is complemented. Indeed, if $\{F_i\}$ is a sequence of blocks without gaps ($\max F_i + 1 = \min F_{i+1}$) such that $i_k \in F_k$, then $[\{e_{i_k}\}]$ is complemented by the projection

$$Px = \sum_{k=1}^{\infty} \langle F_k, x\rangle e_{i_k}.$$

Since $\{F_i\}$ has no gaps, any estimate of $\|Px\|$ (by (4)) is also an estimate of $\|x\|$, so $\|P\| = 1$.

In our analysis we will use the following two propositions. Proposition 1 is extracted from the proof of Theorem 1 of [1].

PROPOSITION 1.1. *If $\{u_i\} \subset X_\alpha$ converges weak* to $x^{**} \in X_\alpha^{**}$, then $x^{**} = x + \theta$ where $x \in X_\alpha$ and $\langle e_i^*, \theta\rangle = 0$ for all i.*

2. *If $\{u_i\} \subset X_\alpha$ is weakly Cauchy, then $\{u_i\}$ converges weak* to $x + \eta\theta_0$ where $x \in X_\alpha$, $\eta = \lim\langle \mathbf{N}, u_i - x\rangle$, and θ_0 is the weak* limit of $\{e_i\}$.*

PROPOSITION 2. *Let $\{v_i\}$ be a block basic sequence of $\{e_i\}$, let $F = \{M+1, M+2, \cdots\} \subset \mathbf{N}$, and suppose $\langle F, v_i\rangle = \gamma > 0$ for all i. Then for any scalar sequence $\{a_i\}$,*

$$\gamma\|\Sigma a_i e_i\| \leqq \|\Sigma a_i v_i\|.$$

PROOF. Let $(a_i)_{i=1}^N$ be a scalar sequence, and let $x = \Sigma_1^N a_i e_i$, $y = \Sigma_1^N a_i v_i$. Since $\langle F, v_i \rangle = \gamma$ for each i, there exists an admissible sequence of blocks $\{F_i\}$ such that $\langle F_i, v_i \rangle = \gamma$ and $\operatorname{supp} v_i \subset F_i$ for all i. Let $F_i = [f_i, g_i]$. Let $\{G_k\}_{k=1}^{\ell}$ be an admissible sequence with

$$\|x\| = \sum_{k=1}^{\ell} \alpha_k |\langle G_k, x \rangle|,$$

and for each k, let $G'_k = [n_k, m_k]$, where $n_k = \min\{f_i: i \in G_k\}$, $m_k = \max\{g_i: i \in G_k\}$. Then $\{G'_k\}$ is admissible and

$$\begin{aligned} \|y\| &\geqq \sum \alpha_k |\langle G'_k, y \rangle| \\ &= \sum \alpha_k \gamma |\langle G_k, x \rangle| \\ &= \gamma \|x\|. \end{aligned}$$

THEOREM 3. *Let Y be a complemented subspace of X_α. If Y is not weakly sequentially complete, then Y contains a complemented subspace isomorphic to X_α.*

PROOF. Let P be a projection onto Y, and let $Z = (I - P)X_\alpha$. The sequences $\{Pe_i\}$ and $\{(I - P)e_i\}$ are weakly Cauchy. Since Z is weakly sequentially complete [1], Proposition 1 implies that

$$(I - P)e_i \xrightarrow{w^*} y \in X_\alpha \tag{5}$$

and

$$Pe_i \xrightarrow{w^*} x + \eta\theta_0. \tag{6}$$

Now $e_i \xrightarrow{w^*} \theta_0 \in X_\alpha^{**} - X_\alpha$, and $e_i = (I - P)e_i + Pe_i$, so $\{Pe_i\}$ and $\{(I - P)e_i\}$ cannot both have weak* limits in X_α. Hence $\eta = \lim\langle \mathbf{N}, Pe_i - x \rangle \neq 0$. In fact, by standard perturbation arguments we may assume there exists $M \in \mathbf{N}$ such that $P_M y = y$ and $P_M x = x$, where x and y are as in (5) and (6). Then with $F = \{M + 1, M + 2, \cdots\}$,

$$1 = \langle F, e_i \rangle = \langle F, (I - P)e_i \rangle + \langle F, Pe_i \rangle,$$

so $\lim_i \langle F, Pe_i \rangle = 1$. Applying Proposition 1, part 1, passing to a subsequence $\{e_{i_k}\}$, and perturbing, we may assume that $Pe_{i_k} = v_k = x + w_k$ with (M and F as above)

$$P_M x = x \tag{7}$$

$$\langle F, w_k \rangle = 1 \qquad \text{for all } k, \tag{8}$$

and

(9) $\operatorname{supp} w_k \subset G_k$ where $\{G_k\}$ is an admissible sequence without gaps.

Then for any scalar sequence $\{a_k\}$,

$$\begin{aligned}\|\sum a_k v_k\| &= \|(\sum a_k)x + (\sum a_k w_k)\| \\ &\geqq \|\sum a_k w_k\| \quad (\{e_i\} \text{ is bi-monotone}) \\ &\geqq \|\sum a_k e_k\|\end{aligned}$$

by Proposition 2. Since $\|\sum a_k v_k\| \leqq \|P\| \, \|\sum a_k e_k\|$, the sequence $\{v_k\}$ is equivalent to $\{e_k\}$, and hence Y contains an isomorph of X_α. A projection onto $[\{v_k\}]$ is defined by

$$Qz = \sum_{k=1}^{\infty} \langle G_k, z\rangle v_k.$$

Q is bounded since

$$\begin{aligned}\|Qz\| &= \|\sum \langle G_k, z\rangle v_k\| \\ &\leqq \|P\| \, \|\sum \langle G_k, z\rangle e_k\| \leqq \|P\| \, \|z\|,\end{aligned}$$

since $\{G_k\}$ has no gaps.

REMARKS. 1. It is possible that no subsequence of $\{Pe_n\}$ is a block basic sequence. A typical example is the norm 2 projection P defined by $Pe_1 = 0$ and $Pe_i = e_1 + e_i$, $i \geqq 2$.

2. The arguments used in the proof of Theorem 3 may be modified to show that if $T: X_\alpha \to X_\alpha$ is a bounded linear operator, then either TX_α or $(I - T)X_\alpha$contains a complemented isomorph of X_α.

The next theorem concerns the isomorphism type of the X_α spaces.

THEOREM 4. *X_α is isomorphic to X_β if and only if the unit vector bases in X_α and X_β are equivalent.*

PROOF. Let $T: X_\alpha \to X_\beta$ be an isomorphism. Then $\{Te_i\} \subset X_\beta$ is weakly Cauchy but not weakly convergent. Thus by Proposition 1, $Te_i \xrightarrow{w^*} x + \eta\theta_0$ where $x \in X_\beta$ and $\eta = \lim_i \langle \mathrm{N}, Te_i - x\rangle \neq 0$. We assume $\eta > 0$. Passing to a subsequence and perturbing, we may assume that $v_k = Te_{i_k} = x + w_k$ where $\{w_k\}$ is a block basic sequence of $\{e_i\}$ in X_β satisfying the hypotheses of Proposition 2 with $\gamma = \eta$ and M any integer such that $P_M x = x$. Then for any scalar sequence $\{a_i\}$,

$$\begin{aligned}\|T\| \, \|\sum a_i e_i\|_{X_\alpha} &\geqq \|\sum a_k v_k\|_{X_\beta} \\ &\geqq \|\sum a_k w_k\|_{X_\beta} \quad \text{(bi-monotonicity)} \\ &\geqq \eta \|\sum a_k e_k\|_{X_\beta}.\end{aligned}$$

Applying the same argument to T^{-1} yields that the unit vectors in X_α and X_β are equivalent.

REMARKS. 1. Theorem 4 may be interpreted as saying that X_α and X_β are isomorphic if and only if they are equal as sets.

2. If α and β satisfy (1), (2), (3) and if there exists a constant A such that

$$A^{-1}\alpha_i \leqq \beta_i \leqq A\alpha_i \qquad \text{for all } i, \tag{10}$$

it is clear that X_α and X_β are isomorphic. On the other hand, since $\|\sum_1^N(-1)^i e_i\|_{X_\alpha} = \sum_1^N \alpha_i$, if X_α and X_β are isomorphic, there exists a con stant B such that for all N,

$$B^{-1}\sum_1^N \alpha_i \leqq \sum_1^N \beta_i \leqq B\sum_1^N \alpha_i. \tag{11}$$

However, there are pairs of sequences α, β, satisfying (1), (2), (3), and (11), yet satisfying no estimate of type (10).

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