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GROUPS SATISFYING THE WEAK CHAIN CONDITIONS FOR NORMAL SUBGROUPS

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1. Introduction. A group G is said to satisfy the weak maximal condition for normal subgroups if it does not contain an infinite ascending chain $G_1 < G_2 < \cdots < G_i < G_{i+1} < \cdots$ of normal subgroups such that each of the indices $|G_{i+1}: G_i|$ is infinite. The weak minimal condition for normal subgroups is defined similarly. We denote these properties by Max $-\ \infty$ and Min $-\ \infty$ for normal subgroups.

Groups satisfying the weak chain conditions for normal subgroups were first considered by L. A. Kurdačenko. In [6] he began the study of locally nilpotent groups with these conditions and obtained full information in the periodic case and in the torsion-free case, respectively.

THEOREM (Kurdačenko [6]). (i) *A periodic locally nilpotent group satisfies the condition* Max $-\ \infty$ (*resp.* Min $-\ \infty$) *for normal subgroups if and only if it is a Černikov group.*

(ii) *A torsion-free locally nilpotent group satisfies the condition* Max $-\ \infty$ (*resp.* Min $-\ \infty$) *for normal subgroups if and only if it is a nilpotent minimax group.*

Recently Kurdačenko [8] was able to give necessary and sufficient conditions for an arbitrary locally nilpotent group to satisfy the condition Min $-\ \infty$ for normal subgroups. For a short survey of results in this direction, in particular with respect to Kurdačenko's paper [7] on groups satisfying the weak chain conditions for subnormal subgroups, see Curzio [2].

In this note we are concerned with the effect of the weak chain conditions for normal subgroups within the class of (locally) solvable groups. Our first result provides the tool for short proofs of the remaining theorems.

THEOREM A. *If the group G satisfies the condition* Max $-\ \infty$ (*resp.* Min $-\ \infty$) *for normal subgroups and H is a subgroup of G with finite index, then H satisfies the condition* Max $-\ \infty$ (*resp.* Min $-\ \infty$) *for normal subgroups.*

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The following main result of this work will extend part (i) of Kurdačenko's theorem to a certain class of locally solvable groups.

THEOREM B. *Let G be a periodic locally solvable group such that there is a uniform bound n on the ranks of the chief factors of G. If G satisfies the condition* Max $-\infty$ (*resp.* Min $-\infty$) *for normal subgroups, then G is a Černikov group.*

Since chief factors of locally supersolvable groups are cyclic of prime order (cf. **[5]**, p. 11, 1.B.7), this implies

COROLLARY B1. *A periodic locally supersolvable group with the condition* Max $-\infty$ (*resp.* Min $-\infty$) *for normal subgroups is a Černikov group.*

In view of Theorem A and the very strong result of Šunkov **[12]** that a locally finite group with finite Prüfer rank is a finite extension of a locally solvable group, one obtains a further corollary.

COROLLARY B2. *A locally finite group with finite Prüfer rank is a Černikov group if it satisfies the condition* Max $-\infty$ (*resp.* Min $-\infty$) *for normal subgroups.*

Observe that Theorem B cannot be extended to arbitrary periodic locally solvable groups: McLain (cf. **[10]**, pp. 167–168) has constructed an infinite, locally finite and locally solvable group G which is not solvable and whose only non-trivial normal subgroups are the terms of its derived series. Furthermore, each non-trivial normal subgroup of G has finite index in G. Hence G satisfies both the weak maximal condition and the weak minimal condition for normal subgroups.

In the course of the proof of Theorem B, we shall make use of the following result which can also be interpreted as a new characterization of finite solvable groups.

THEOREM C. *The residually finite, periodic solvable group G satisfies the condition* Max $-\infty$ (*resp.* Min $-\infty$) *for normal subgroups if and only if G is finite.*

We mention a useful consequence of Theorems C and A which can be verified easily by the reader.

THEOREM D. *Every periodic solvable group satisfying the condition* Max $-\infty$ (*resp.* Min $-\infty$) *for normal subgroups is a finite extension of an $\mathcal{F}$-perfect group satisfying the condition* Max $-\infty$ (*resp.* Min $-\infty$) *for normal subgroups.*

Thus, in studying periodic solvable groups with the weak chain conditions for normal subgroups, one can naturally concentrate his attention on $\mathscr{F}$-perfect groups, that is, groups having no proper subgroups of finite index. Using this fact, we prove

THEOREM E. *For periodic solvable groups, the condition* Min $-\infty$ *for normal subgroups implies* Min $-$ *n, the minimal condition for normal subgroups.*

REMARK. This has also been announced (without proof) by D. I. Zaĭcev and L. A. Kurdačenko at the Eighth All-Union Symposium on Group Theory in Sumy, 1982; see **[14]**, p. 37. Now compare **[15]**

In **[11]**, Silcock has shown that metanilpotent groups with Min $-$ *n* are countable. Therefore we may state

COROLLARY E. *A periodic metanilpotent group satisfying the condition* Min $-\infty$ *for normal subgroups is countable.*

In the context of this corollary, it should be mentioned that an example of Hartley **[4]** demonstrates the existence of uncountable solvable groups (of length 3) satisfying Min $-$ *n*. On the other hand, it is not difficult to see that solvable groups with the condition Max $-\infty$ for normal subgroups are always countable. Another example of Hartley **[3]**, however, shows that this is definitely false for periodic locally solvable groups which even satisfy Max $-$ *n*, the maximal condition on normal subgroups.

The notation and terminology in this paper is standard and will follow Robinson **[10]**.

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2. Proofs. We begin by proving Theorem A. The proof is based on J. S. Wilson's proof of the corresponding statement about groups satisfying Max $-$ *n* or Min $-$ *n*; see **[13]**.

PROOF OF THEOREM A. We present a proof of the theorem only for the condition Max $-\infty$ for normal subgroups; a similar argument can be applied to the condition Min $-\infty$ for normal subgroups.

Suppose that H does not have the property Max $-\infty$ for normal subgroups. Since $G/\mathrm{Core}_G(H)$ is finite, we may assume that H is normal in G. Denote by $\mathscr{N}$ the set of all normal subgroups N of G such that N is contained in H and H/N does not satisfy the condition Max $-\infty$ for normal subgroups. Since the group G satisfies the condition Max $-\infty$ for normal subgroups, there exists an element $K \in \mathscr{N}$ such that the index $|M: K|$ is finite for all $M \in \mathscr{N}$ with $K < M$.

Let $\mathscr{S}$ be the set of all non-empty finite subsets X of G with the following property: if

$$K = K_1 < K_2 < \cdots < K_i < K_{i+1} < \cdots$$

is an infinite ascending sequence of normal subgroups of H, where each of the indices $|K_{i+1}: K_i|$ is infinite, then

$$|\mathrm{Core}_X(K_i): K| < \infty$$

for all i. Obviously, any transversal to H in G belongs to the set $\mathscr{S}$ so that $\mathscr{S}$ is not empty.

We now select a minimal element of $\mathscr{S}$, say X, and we may assume that $1 \in X$. The subset X contains at least two elements; for otherwise H/K would satisfy the condition Max $- \infty$ for normal subgroups. Consequently, the set

$$X^* = X \backslash \{1\}$$

is not empty. By the minimality of X it follows that $X^* \notin \mathscr{S}$.

Let $K = K_1 < K_2 < \cdots < K_i < K_{i+1} < \cdots$ be an infinite ascending sequence of normal subgroups of H such that all indices $|K_{i+1}: K_i|$ are infinite. Put

$$L_i = K_i \, \mathrm{Core}_{X^*}(K_i).$$

Then $K_i \leqq L_i \lhd H$ and $L_i \leqq L_{i+1}$. Suppose that $|L_{i+1}: L_i|$ is finite; then

$$|K_{i+1}\mathrm{Core}_{X^*}(K_i): K_i \, \mathrm{Core}_{X^*}(K_i)| < \infty. \tag{1}$$

Since

$$|K_{i+1} \cap \mathrm{Core}_{X^*}(K_i) : K_i \cap \mathrm{Core}_{X^*}(K_i)| \, |K_i \cap \mathrm{Core}_{X^*}(K_i): K| \leqq |K_{i+1} \cap \mathrm{Core}_{X^*}(K_{i+1}) : K|$$

and

$$|K_{i+1} \cap \mathrm{Core}_{X^*}(K_{i+1}) : K| = |\mathrm{Core}_X(K_{i+1}): K| < \infty,$$

one also has

$$|K_{i+1} \cap \mathrm{Core}_{X^*}(K_i) : K_i \cap \mathrm{Core}_{X^*}(K_i)| < \infty. \tag{2}$$

But from (1) and (2) it follows that

$$|K_{i+1} : K_i| < \infty,$$

a contradiction.

This shows that the subgroups L_i form an infinite ascending chain of normal subgroups of H such that each of the indices $|L_{i+1} : L_i|$ is infinite. By definition of $\mathscr{S}$, this implies that

$$|\mathrm{Core}_X(L_i) : K| < \infty$$

for all i.

Next L_i is contained in $K_i\mathrm{Core}_X(L_i)$, so $L_i = K_i\mathrm{Core}_X(L_i)$. Therefore L_i/K_i is a homomorphic image of the finite group $\mathrm{Core}_X(L_i)/K$ and thus is itself finite. But then $|\mathrm{Core}_{X^*}(K_i)\colon \mathrm{Core}_X(K_i)|$ is also finite, which together with the finiteness of $|\mathrm{Core}_X(K_i)\colon K|$ yields that

$$|\mathrm{Core}_{X^*}(K_i) : K| = |\mathrm{Core}_{X^*}(K_i) : \mathrm{Core}_X(K_i)|\ |\mathrm{Core}_X(K_i) : K| < \infty.$$

Consequently $X^* \in \mathscr{S}$, which is impossible. This final contradiction completes the proof of the theorem.

The proof of Theorem B depends on Theorem C, which we establish first.

Proof of Theorem C. We have only to show the necessity of the condition. If G is abelian, then G is a minimax group (cf. Baer **[1]**, Lemma 1.2) and therefore, being periodic, G satisfies the minimal condition for subgroups. Now the assertion follows at once from the residual finiteness. So let G be non-abelian and denote by A a maximal abelian normal subgroup of G containing the last non-trivial term of the derived series of G. Then, by a lemma of Learner (cf. **[9]**, p. 166, Corollary), G/A is residually finite so that we may assume by induction on the derived length of G that G/A is finite. Theorem A now yields that A satisfies the condition Max $- \infty$ (resp. Min $- \infty$) for normal subgroups, and the beginning of the proof shows that A is finite. Hence G is finite, as required.

Proof of Theorem B. Let X/Y be a chief factor of G and $C = C_G(X/Y)$. Then $C \lhd G$, and G/C is a finite solvable group of automorphisms of X/Y, since X/Y is elementary abelian of rank $\leqq n$. Thus, $G/C \hookrightarrow GL(n,p)$ for some prime p. By a well-known theorem of Zassenhaus (cf. **[10]**, p. 78, Theorem 3.23), there is a function $f(n)$ which bounds the derived length of G/C; thus $G^{f(n)} \subset C$.

Now let S be the intersection of the centralizers of all the chief factors of G. Then

i) $G^{f(n)} \subset S$,

ii) G/S is residually finite,

iii) S is locally nilpotent (cf. **[5]**, p. 16, 1.B.10).

By Theorem C, G/S is finite, and Theorem A shows that S satisfies the condition Max $- \infty$ (resp. Min $- \infty$) for normal subgroups. It follows from part (i) of Kurdačenko's theorem that S is a Černikov group, and since G/S is finite, G is also a Černikov group.

It remains to prove Theorem E. Here, in view of Theorem D, it will suffice to establish the following

PROPOSITION. *If G is an $\mathscr{F}$-perfect, periodic group with the property* Min $-\ \infty$ *for normal subgroups, then G satisfies* Min $-\ n$.

PROOF. If this is false, then G contains an infinite descending sequence of normal subgroups, say

$$N_1 > N_2 > \cdots > N_i > N_{i+1} > \cdots .$$

Put $\bar{G} = G/N$ and $\bar{N}_i = N_i/N$ where $N = \bigcap_{i=1}^{\infty} N_i$. Since G, and hence also $\bar{G}$, satisfies the condition Min $-\ \infty$ for normal subgroups, there exists a positive integer k such that

$$|\bar{N}_k/\bar{N}_{k+j}| < \infty$$

for all $j \in \mathbf{N}$. Thus each centralizer $C_{\bar{G}}(\bar{N}_k/\bar{N}_{k+j})$ has finite index in $\bar{G}$. Since G is $\mathscr{F}$-perfect, it follows that

$$\bar{G} = C_{\bar{G}}(\bar{N}_k/\bar{N}_{k+j})$$

for all $j \in \mathbf{N}$. But then $\zeta(\bar{G})$ must contain $\bar{N}_k$, contrary to the fact that $\zeta(\bar{G})$ satisfies the minimal condition for subgroups. For the latter, observe that $\zeta(\bar{G})$ must satisfy the weak minimum condition for subgroups and hence is a periodic abelian minimax group (cf. Baer [**1**], Lemma 1.2). Thus the proposition is proved.

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