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A SIMPLE CHARACTERIZATION OF THE CONTACT SYSTEM ON $J^k(E)$*

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ABSTRACT. In this note we give an invariant characterization of the contact system of $J^k(E)$ where (E, π, M) is a fibred manifold. This characterization generalizes one given in reference [1] for the case where $k = 1$. It affords a simple coordinate free proof that a section σ of $(J^k(E), \pi^k_M, M)$ is the k-jet extension of a section of (E, π, M) if σ annihilates the contact system [2].

1. The First order Case. Let (E, π, M) denote a fibred manifold with total space E, projection π and base space M. The *k-jet* bundle of local sections of (E, π, M), denoted by $J^k(E)$, has a natural fibred manifold structure over $J^\ell(E)$ for $\ell < k$ and over E and M. The canonical projections π^k_ℓ: $J^k(E) \to J^\ell(E)$, π^k_E: $J^k(E) \to E$ and π^k_M: $J^k(E) \to M$ are given by

$$\text{(1)} \quad \text{(a)} \quad \pi^k_\ell : J^k_x s \to J^\ell_x s$$

$$\text{(b)} \quad \pi^k_E : j^k_x s \to s(x)$$

and

$$\text{(c)} \quad \pi^k_M = \pi \circ \pi^k_E : j^k_x s \to x$$

respectively.

We begin by defining the contact system Ω^1 on $J^1(E)$ as the exterior differential system given pointwise by

$$\Omega^1|_{j^1_x s} = (\pi^{1*}_E - \pi^{1*}_M s^*) T^*_{s(x)} E. \tag{2}$$

It is easy to verify, from (2), that a section σ of $(J^1(E), \pi^1_M, M)$ defined on $U \subset M$, satisfies $\sigma^*\Omega^1 = 0$ iff $\sigma = j^1 s$ where $s = \pi^1_E \circ \sigma$. To see this, suppose $\sigma = j^1 s$. Then

$$\begin{aligned} \sigma^*\Omega^1|_{j^1_x s} &= j^1 s^*(\pi^{1*}_E - \pi^{1*}_M s^*) T^*_{s(x)} E \\ &= [(\pi^1_E \circ j^1 s)^* - (s \circ \pi^1_M \circ j^1 s)^*] T^*_{s(x)} E \\ &= [s^* - (s \circ id_U)^*] T^*_{s(x)} E = 0. \end{aligned}$$

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Next suppose that σ satisfies $\sigma^*\Omega^1 = 0$ and define a section s of (E, π, M) by $s := \pi^1_E \circ \sigma$. Now for each x there is a section s_x defined on a neighborhood of x such that $\sigma(x) = j^1_x s_x$. It follows that $s_x(x) = (\pi^1_E \circ \sigma)(x) = s(x)$ and, in order to show that $\sigma = j^1 s$, we need only show that all the first order partial derivatives of s_x and s agree. But this is the same as showing that, for each x, s_x and s have the same Jacobian, i.e., that

$$(s^* - s^*_x)T^*_{s(x)}E = 0.$$

This is precisely the condition given by $\sigma^*\Omega^1|_{j^1 s_x} = 0$, for

$$\begin{aligned}\sigma^*\Omega^1|_{j^1_x s_x} &= \sigma^*(\pi^{1*}_E - \pi^{1*}_M s^*_x)T^*_{s(x)}E \\ &= [(\pi^1_E \circ \sigma)^* - (s_x \circ \pi^1_M \circ \sigma)^*]T^*_{s(x)}E \\ &= [s^* - s^*_x]T^*_{s(x)}E\end{aligned}$$

because $s = \pi^1_E \circ \sigma$ and $\pi^1_M \circ \sigma = id_U$.

We note that the definition (2) leads immediately to the standard local coordinate presentation of the contact system. If (x^a, z^A) are fibred coordinates at $s(x) \in E$ and (x^a, z^A, z^A_a) are the induced coordinates at $j^1_x s \in J^1(E)$ then $T^*_{s(x)}E$ has the coordinate basis $(dx^a|_{s(x)})$, $dz^A|_{s(x)}$, and $(\pi^{1*}_E - \pi^{1*}_M s^*)dx^a|_{s(x)} = 0$, while

$$(\pi^{1*}_E - \pi^{1*}_M s^*)dz^A|_{s(x)} = (dz^A - z^A_a dx^a)|_{j^1_x s}.$$

2. The k-th order Case. The contact system on $J^k(E)$ for $k > 1$ may be defined pointwise by

$$\Omega^k|_{j^k_x s} = (\pi^{k*}_{k-1} - \pi^{k*}_M j^{k-1}s^*)T^*_{j^{k-1}_x s}J^{k-1}(E). \tag{3}$$

It is immediate from (3) that for $k = 2, 3, \ldots.,$

$$\pi^{k*}_{k-1}\Omega^{k-1} \subset \Omega^k, \tag{4}$$

for

$$\begin{aligned}\pi^{k*}_{k-1}\Omega^{k-1}|_{j^{k-1}_x s} &= \pi^{k*}_{k-1}(\pi^{k-1*}_{k-2} - \pi^{k-1*}_M j^{k-2}s^*)T^*_{j^{k-2}_x s}j^{k-2}(E) \\ &= (\pi^{k*}_{k-1} - \pi^k_M j^{k-1}s^*)\pi^{k-1*}_{k-2} T^*_{j^{k-2}_x s}J^{k-2}(E),\end{aligned}$$

and

$$\pi^{k-1*}_{k-2}(T^*_{j^{k-2}_x s}J^{k-2}(E)) \subset T^*_{j^{k-1}_x s}J^{k-1}(E).$$

We now show by induction that if σ is a section of $(J^k(E), \pi^k_M, M)$ which annihilates Ω^k then $\sigma = j^k(\pi^k_E \circ \sigma)$. The converse is left to the reader.

Assume for $\ell = 1, 2, \ldots., k - 1$, that if ϕ is a section of $(J^\ell(E), \pi^\ell_M, M)$ which satisfies $\phi^*\Omega^\ell = 0$ then $\phi = j^\ell(\pi^\ell_E \circ \phi)$. Let σ be a section of

(J^k, E, π_M^k, M) defined on $U \subset M$ and let s be the section of E defined by $s = \pi_E^k \circ \sigma$. As above, we have $\sigma(x) = j_x^k s_x$. We wish to show that s and s_x agree to *k-th* order on U, i.e., that $j^k s_x = j^k s$.

Now if $\sigma^* \Omega^k = 0$, (4) shows that $0 = \sigma^* \pi_{k-1}^{k*} \Omega^{k-1} = (\pi_{k-1}^k \circ \sigma)^* \Omega^{k-1}$ and thus, by the induction hypothesis,

$$\pi_{k-1}^k \circ \sigma = j^{k-1}(\pi_E^{k-1} \circ \pi_{k-1}^k \circ \sigma).$$

But $\pi_E^{k-1} \circ \pi_{k-1}^k \circ \sigma = \pi_E^k \circ \sigma = s$ so $\pi_{k-1}^k \circ \sigma = j^{k-1} s$.

Thus $j^{k-1} s_x = j^{k-1} s$, so s_x and s agree to $(k - 1)$st order. Now $\pi_{k-1}^k \circ \sigma = j^{k-1} s$, and the fact that $\sigma^* \Omega^k = 0$ shows that $\pi_{k-1}^k \circ \sigma$ and $j^{k-1} s_x$ have the same Jacobian at x. Thus all of the first derivatives of $j^{k-1} s$ and $j^{k-1} s_x$ agree for all x in U and hence $j^k s_x = j^k s$, i.e., $\sigma = j^k s$.

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