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DERIVATIONS FROM SUBALGEBRAS OF C^*-ALGEBRAS WITH CONTINUOUS TRACE

ALDO J. LAZAR, SZE-KAI TSUI, STEVE WRIGHT

1. Introduction. Let A be a C^*-algebra, $M(A)$ its multiplier algebra, B a C^*-subalgebra of A. Suppose δ: $B \to A$ is a derivation of B into A, i.e., a linear map for which $\delta(ab) = a\delta(b) + \delta(a)b$, for all $a, b \in B$. Derivations of this type are most easily obtained by choosing an element m of $M(A)$ and setting $\delta(b) = mb - bm$, $b \in B$. Such derivations are said to be inner in $M(A)$, with generator m. In this paper, we begin the investigation of C^*-algebras A with the following property: for each C^*-subalgebra B of A and each derivation δ: $B \to A$, there is an element $m \in M(A)$ for which $\delta(b) = mb - bm$, for all $b \in B$. We will say that a C^*-algebra with this property is hereditarily cohomologically trivial (HCT for short).

The HCT C^*-algebras are of interest for a number of reasons. The problem of studying them was first raised (without the terminology just introduced) by Kaplansky on p. 7 of [**11**], who was motivated by Sakai's famous theorem [**16**] that all derivations of simple, unital C^*-algebras are inner, and the fact that any derivation of a semisimple subalgebra into a central simple algebra can be extended to an inner derivations of the larger algebra. E. Christensen in [**4**, **5**] has attacked the difficult and as yet still unanswered question of whether $B(H)$, the algebra of all bounded operators on a Hilbert space H, is HCT, and has shown [**4**, Section 5] that all finite von Neumann algebras have this property. Akemann and Johnson have pointed out in the introduction to [**2**] the importance of investigating those pairs (B, A) of C^*-algebras A and C^*-subalgebras B of A for which every derivation of B into A is inner in $M(A)$; since the HCT C^*-algebras have this property for all C^*-subalgebras, a knowledge of them will be very useful in coming to grips with this more general problem of Akemann and Johnson (indeed, in the remarks which end [**13**], the present authors indicate at least one instance in which such a payoff actually occurs). Finally, J. Cuntz (private communication) has pointed out that knowledge of the structure of derivations of a C^*-subalgebra B into a C^*-algebra A would be useful in the study of the non-commutative K-theory of Kasparov [**12**]. In particular, the very nice

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behavior of such derivations for HCT algebras would allow in that context the extension in certain situations of arguments from ordinary K-theory to Kasparov's KK-theory.

We thus begin, as always, with the search for examples. It is easy to see that all commutative and elementary C^*-algebras are HCT. To advance beyond these, we note that the HCT algebras are evidently contained in the class of C^*-algebras A for which every derivation $\delta: A \to A$ is inner in $M(A)$, and the most important examples of C^*-algebras with this latter property are von Neumann algebras **[10, 15]**, simple C^*-algebras **[16]**, and C^*-algebras with continuous trace whose spectrum is paracompact **[1]**. As we have noted earlier, Christensen has shown that all finite von Neumann algebras are HCT, and it is not known if $B(H)$ is also. It therefore seems most promising to look for other HCT examples among either the simple C^*-algebras or the C^*-algebras with continuous trace. The purpose of the paper before the reader is to do this for algebras with continuous trace; in a subsequent paper **[13]**, the authors determine the structure of the separable HCT C^*-algebras, and in particular show that a separable, simple C^*-algebra is HCT if and only if it is elementary.

If X is a locally compact space, we will denote by $C_0(X)$ the algebra of all continuous, complex-valued functions on X which vanish at infinity, and if n is a positive integer, M_n will denote the algebra of complex $n \times n$ matrices. Our goal is to prove the following theorem.

THEOREM 1.1. *Let A be a C^*-algebra with continuous trace, and suppose every irreducible representation of A acts on a separable Hilbert space. Then A is HCT if and only if it has a* (*unique*) *direct sum decomposition $A_1 \oplus A_2 \oplus A_3$ such that*

(i) *A_1 is commutative*;

(ii) *A_2 is the restricted direct sum of a* (*possibly finite*) *sequence $\{A_n\}$ of C^*-algebras with A_n isomorphic to $C_0(X_n) \otimes M_{k_n}$ for a Stonean, locally compact Hausdorff space X_n and an integer $k_n \geqq 2$*;

(iii) *A_3 is the restricted direct sum of a family of infinite-dimensional elementary C^*-algebras.*

2. Preliminaries. In this section we include all the basic preliminary lemmas which we will use in the proof of the main theorem. The first three lemmas are easily verified and we will hence omit their proofs.

LEMMA 2.1. *If A is an HCT C^*-algebra and I is a closed, two-sided ideal of A, then I is HCT.*

LEMMA 2.2. *If A and B are HCT C^*-algebras, then $A \oplus B$ is HCT.*

LEMMA 2.3. *All commutative C^*-algebras are HCT.*

The following notation will be useful. If X is a locally compact space,

$C_0(X)$ will denote the C^*-algebra of all complex-valued, continuous functions on X vanishing at infinity, and $C(X)$ will denote the algebra of all bounded, complex-valued continuous functions on X. M_n will denote the algebra of all $n \times n$ complex matrices. If A is a C^*-algebra and $b \in A$, we will denote the derivation $a \to ba - ab$, $a \in A$, by adb.

LEMMA 2.4. *Let X be a compact, Hausdorff, Stonean space. Then $C(X) \otimes M_n$ is HCT.*

PROOF. We may suppose that $A = C(X) \otimes M_n$ acts on a Hilbert space H. Since X is Stonean, it follows that A is injective, and hence there is a projection P of norm 1 of $B(H)$ onto A.

Let B be a C^*-subalgebra of A, $\delta\colon B \to A$ a derivation. Now δ extends to a $B(H)$-valued derivation $\bar{\delta}$ of the closure B^- of B in the weak operator topology on $B(H)$ [**14**], Theorem 4], and since B is liminal, B^- is a type I von Neumann algebra. Thus, by [**4**], there exists $x \in B(H)$ with $\bar{\delta} = adx|_{B^-}$. Let $a = P(x) \in A$. Since P is a conditional expectation of $B(H)$ onto A, we have

$$\delta(b) = P(\delta(b)) = P(x)b - bP(x) = (ada)(b),\ b \in B.$$

LEMMA 2.5. *Let X be a locally compact Hausdorff space, n a fixed positive integer $\geqq 2$. Then $C_0(X) \otimes M_n$ is HCT if and only if X is Stonean.*

PROOF. ($\Leftarrow$). Let $A = C_0(X) \otimes M_n$. Then $M(A)$ is the C^*-algebra of all M_n-valued, bounded, continuous functions on X, and hence is isomorphic to $C(\beta X) \otimes M_n$, where βX denotes the Stone-Čech compactification of X. Since X is Stonean so is βX [**18**, Theorem 14.1.4], and thus, by Lemma 2.4, $M(A)$, and hence A, is HCT.

($\Rightarrow$). If X is not Stonean, then there exist disjoint open sets U and V whose closures have nonvoid intersection. Set $Z = X\backslash(U \cup V)$. Let $C = \{a \in A : a \equiv 0 \text{ on } Z\}$. Let χ denote the characteristic function of V. Define $b\colon X \to M_n$ by

$$b(x) = \begin{cases} \begin{pmatrix} \chi(x) & 0 \\ 0 & 0 \end{pmatrix}, & x \in U \cup V, \\ 0 & , x \in Z \end{cases}$$

(for simplicity, we take $n = 2$). Define δ on C by $\delta(c)\,(x) = b(x)c(x) - c(x)b(x)$, $x \in X$. Since χ is continuous on $U \cup V$, δ is a derivation of C into A.

Suppose $\delta = adm|_C$ for some $m \in M(A) \approx C(\beta X) \otimes M_2$. If $x \in U \cup V$, it follows from Urysohn's lemma that $M_2 = \{c(x) : c \in C\}$, so, for each $x \in U \cup V$, there is a complex number $\lambda(x)$ for which $m(x) - b(x) = \lambda(x)I$, i.e., for each $x \in U \cup V$,

$$\begin{pmatrix} \chi(x) + \lambda(x) & 0 \\ 0 & \lambda(x) \end{pmatrix} = \begin{pmatrix} m_{11}(x) & m_{12}(x) \\ m_{21}(x) & m_{22}(x) \end{pmatrix},$$

where $m_{ij} \in C(X)$; $i, j = 1, 2$. Hence

$$\begin{aligned} m_{11}(x) &= \lambda(x) \quad &&, x \in U, \\ m_{11}(x) &= 1 + \lambda(x), &&x \in V, \\ m_{22}(x) &= \lambda(x) \quad &&, x \in U \cup V, \end{aligned}$$

whence

$$(m_{11} - m_{22})(U) = \{0\} = (m_{11} - m_{22} - 1)(V).$$

Thus, for any $x_0 \in U^- \cap V^- \neq \emptyset$, we have $m_{11}(x_0) - m_{22}(x_0) = m_{11}(x_0) - m_{22}(x_0) - 1 = 0$, which is absurd. Thus no such $m \in M(A)$ exists and A is not HCT.

3. The proof of Theorem 1.1. The main portion of the proof of Theorem 1.1 will proceed in a series of lemmas which analyze the topology on the spectrum of a C^*-algebra with continuous trace which is HCT. We begin by recalling for the convenience of the reader the structural features of a C^*-algebra with continuous trace of which we will make use.

Let A be a C^*-algebra with continuous trace. By Proposition 4.5.3 and Theorem 10.5.4 of **[6]**, the spectrum $\hat{A}$ of A is Hausdorff and A is isomorphic to the C^*-algebra defined by a continuous field of elementary C^*-algebras on $\hat{A}$. We will always identify A with the C^*-algebra defined by this continuous field.

If U is an open subset of $\hat{A}$, we set $A_U = \{a \in A : a \equiv 0 \text{ on } \hat{A} \backslash U\}$. Then by Proposition 10.7.7 of **[6]**, for each point x of $\hat{A}$, there is a neighborhood U of x for which A_U is isomorphic to the C^*-algebra defined by the field of elementary C^*-algebras relative to a continuous field of Hilbert spaces on U (see §10.7.1 of **[6]** for the construction of the field of elementary C^*-algebras relative to a continuous field of Hilbert spaces).

We set $X = \hat{A}$. In what follows, we will denote the dimension of the representation space of a point $x \in X$ by dim x. For each positive integer n, we set $X_n = \{x \in X : \dim x = n\}$, and we set $X_\infty = \{x \in X : \dim x \text{ is infinite}\}$.

Suppose now that A is a C^*-algebra with continuous trace which is HCT. The next lemma gives the consequences of n-homogeneity for such an algebra, and the four lemmas which follow it analyze the topology on the sets X_n, $n = 1, 2, \ldots, \infty$.

LEMMA 3.1. *If A is n-homogeneous with $n \geqq 2$, then there is a Stonean, locally compact Hausdorff space X such that A is isomorphic to $C_0(X) \otimes M_n$.*

PROOF. Let $X = \hat{A}$. Then X is locally compact and Hausdorff, and by **[8]** or **[17]**, the field $\mathscr{F} = \{A(x)\colon x \in X\}$ of elementary C^*-algebras associated with A is locally trivial. Hence, for each $x \in X$, there is an open neighborhood U of x such that A_U is isomorphic to $C_0(U) \otimes M_n$. Since A_U is a closed, two-sided ideal in A, we conclude by Lemma 2.1 that A_U, and hence $C_0(U) \otimes M_n$, is HCT. By Lemma 2.5, U is Stonean. X is hence covered by open, Stonean subsets, and is thus itself Stonean.

We now let $\mathscr{P}$ denote the family of all pairs (U, φ_U), where U is an open subset of X and φ_U is an isomorphism of $\mathscr{F}|_U$ onto the trivial field with base U and fibre M_n (see §10.1.3 of **[6]** for the description of such an isomorphism). We partially order $\mathscr{P}$ as follows: $(U, \varphi_U) \leqq (V, \varphi_V)$ if $U \subseteq V$ and $\varphi_V|_U = \varphi_U$. It is straightforward to check that $(\mathscr{P}, \leqq)$ is inductive and hence has a maximal element (W, φ_W). By local compactness of X, local triviality of $\mathscr{F}$, and maximality, W is dense in X, and φ_W induces an algebra isomorphism of A_W onto $C_0(W) \otimes M_n$. Thus, by Corollary 3.5 of **[3]**, there exist elements. $\{u_{ij}\colon i, j = 1, \ldots, n\} \subseteq M(A_W)$ such that $\{u_{ij}(x)\}$ is an $n \times n$ system of matrix units in $A(x) \approx M_n$ for each $x \in W$. Now each element u_{ij} induces a derivation of A_W into itself, and since A is HCT, we may find $m'_{ij} \in M(A)$ and $\lambda_{ij} \in C(W)$ such that

$$u_{ij}(x) = m'_{ij}(x) + \lambda_{ij}(x)1_x, \quad x \in W;\ i, j = 1, \ldots, n.$$

(Here 1_x denotes the identity in $A(x)$.) Since X is Stonean, we may extend each λ_{ij} to an element $\bar{\lambda}_{ij}$ of $C(X)$ **[9**, Problem 6*M*], and setting

$$m_{ij}(x) = m'_{ij}(x) + \bar{\lambda}_{ij}(x)1_x, \quad x \in X,$$

while noting that the field $x \to 1_x$ is in $M(A)$ by local triviality of $\mathscr{F}$, we obtain $m_{ij} \in M(A)$ such that $m_{ij}(x) = u_{ij}(x)$, $x \in W$.

Claim 1. $\{m_{ij}(x)\}$ is an $n \times n$ system of matrix units in $A(x)$ for each $x \in X$.

To see this, simply notice that if δ_{jk} denotes the Kronecker delta, each of the functions

$$x \to \|m_{ij}(x)m_{k\ell}(x) - \delta_{jk}m_{i\ell}(x)\|,$$
$$x \to \|m_{ij}(x)^* - m_{ji}(x)\|,$$

is continuous on X and vanishes on W, while recalling that W is dense in X.

Let $a \in A$. Since dim $x = n$ for each $x \in X$, it follows from Claim 1 that there exist bounded, complex-valued functions f_{ij} on X for which

$$\begin{aligned} a(x) &= \sum_{i,j} m_{ii}(x)a(x)m_{jj}(x) \\ &= \sum_{i,j} f_{ij}(x)m_{ij}(x), \quad x \in X. \end{aligned}$$

Claim 2. $f_{ij} \in C_0(X)$; $i, j, = 1, \ldots, n$.

To verify this, note first that $m_{ii}am_{jj} \in A$ and $\|m_{ij}(x)\| = 1$ for $x \in X$, and so $|f_{ij}| = \|m_{ii}am_{jj}\| \in C_0(X)$. We must therefore prove that each f_{ij} is continuous.

Let $x_0 \in W$. Let V be a precompact neighborhood of x_0 with $V^- \subseteq W$ and for which there is an $f \in C_0(X)$ such that $f \equiv 1$ on V and $f \equiv 0$ on $X \backslash W$. Then $f \cdot a \in A_W$, and so there exist $g_{ij} \in C_0(W)$ such that

$$(f \cdot a)(x) = \sum_{i,j} g_{ij}(x)u_{ij}(x) = \sum_{i,j} g_{ij}(x)m_{ij}(x), \quad x \in W.$$

But $f \cdot a = a$ on V, and since $\{m_{ij}(x)\}$ is a basis for $A(x)$, $x \in X$, we conclude that $f_{ij} = g_{ij}$ on V. Since $x_0 \in W$ is arbitrary, $f_{ij}|_W$ is continuous on W for each i and j.

Since X is Stonean, we may extend $f_{ij}|_W$ to an element $\bar{f}_{ij}$ of $C(X)$. By Corollary 3.5 of [3], the functions

$$x \to \|m_{ii}(x)a(x)m_{jj}(x) - \bar{f}_{ij}(x)m_{ij}(x)\|$$

are continuous on X and vanish on W, whence $m_{ii}(x)a(x)m_{jj}(x) = \bar{f}_{ij}(x)m_{ij}(x)$ for $x \in X$. So, by uniqueness we conclude that $f_{ij} = \bar{f}_{ij}$ on X, i.e., f_{ij} is continuous for each i and j.

We have hence decomposed each $a \in A$ as a sum $\sum_{i,j} f_{ij}m_{ij}$, where each $f_{ij} \in C_0(X)$. Conversely, each such sum defines an element of A by Corollary 3.5 of [3]. It follows that the map $(f_{ij}) \to \sum_{i,j} f_{ij}m_{ij}$ defines an isomorphism of $C_0(X) \otimes M_n$ onto A.

LEMMA 3.2. *For each positive integer n, X_n is open and closed in X, and if $n \geqq 2$, X_n is also Stonean.*

PROOF. Let n be a fixed integer $\geqq 1$. Let $Y = \{x \in X: \dim x \geqq n + 1\}$. By Proposition 3.6.3 of [6], Y is open in X, and so in order to show that X_n is open and closed, it suffices to show that Y is closed.

Suppose Y is not closed. Then there is a net $\{x_\alpha\} \subseteq Y$ and $x_0 \in X$ with $\dim x_0 = k \leqq n$ and $x_\alpha \to x_0$. Since A_U is an ideal for each open set U of X, we may hence suppose by Lemma 2.1 and the remarks which opened this section that A is defined by a field of elementary C^*-algebras relative to a continuous field $(H(x), \Gamma)$ of Hilbert spaces on X.

Since $\dim x_0 = k < \infty$, we can find a neighborhood Q of x_0 and vector fields $\xi_1, \ldots, \xi_k \in \Gamma$ such that $\{\xi_1(x), \ldots, \xi_k(x)\}$ is orthonormal in $H(x)$ for each $x \in Q$.

Claim 1. For each $\gamma \in \Gamma$, there is a neighborhood $U = U_\gamma$ of x_0 such that $\gamma(x) = \sum_1^k(\gamma(x), \xi_i(x))\xi_i(x)$, $x \in U$.

For suppose not. Then we can find $\gamma \in \Gamma$ and a net $y_\alpha \to x_0$ with $\gamma(y_\alpha) \perp \xi_i(y_\alpha)$, $i = 1, \ldots, k$, and $\gamma(y_\alpha) \neq 0$, for all α. Set $V = \{x: \gamma(x) \neq 0\}$, and for $x \in V$, set $\gamma_1(x) = \gamma(x)/\|\gamma(x)\|$, $q(x) =$ rank-1 partial isometry on $H(x)$ which maps $\xi_1(x)$ to $\gamma_1(x)$. Set $B = A_V$. If $a \in B$, and $\delta(a)$ is defined by

$$\delta(a)\,(x) = \begin{cases} q(x)a(x) - a(x)q(x), & x \in B \\ 0 & , \ x \in X\backslash V, \end{cases}$$

then δ defines a derivation of B into A.

Since A is HCT, there exists $m \in M(A)$ such that $\delta = adm|_B$. Then for each $x \in V$, there is a complex number $\lambda(x)$ such that

$$m(x) = q(x) + \lambda(x)I_{H(x)}.$$

Now it follows from Corollary 3.5 of [3] that the vector field $x \to m(x)\xi_1(x) = \eta(x)$ is in Γ, and

$$\eta(x) = \gamma_1(x) + \lambda(x)\xi_1(x), \quad x \in V.$$

We have, for each α,

$$\begin{aligned} \lambda(y_\alpha) &= (\lambda(y_\alpha)\xi_1(y_\alpha), \xi_1(y_\alpha)) \\ &= (\lambda(y_\alpha)\xi_1(y_\alpha) + \gamma_1(y_\alpha), \xi_1(y_\alpha)) \\ &= (\eta(y_\alpha), \xi_1(y_\alpha)), \end{aligned}$$

and so $\lambda(y_\alpha) \to (\eta(x_0), \xi_1(x_0)) = \lambda_0$ since $\eta \in \Gamma$. Hence

$$\begin{aligned} \|\eta(x_0) - \lambda_0\xi_1(x_0)\| &= \lim_\alpha \| \eta(y_\alpha) - \lambda(y_\alpha)\xi_1(y_\alpha)\| \\ &= \lim_\alpha \| \gamma_1(y_\alpha)\| = 1. \end{aligned} \tag{1}$$

But, for $1 \leqq i \leqq k$,

$$\begin{aligned} (\eta(x_0) - \lambda_0\xi_1(x_0), \xi_i(x_0)) &= \lim_\alpha(\eta(y_\alpha) - \lambda(y_\alpha)\xi_1(y_\alpha), \xi_i(y_\alpha)) \\ &= \lim_\alpha(\gamma_1(y_\alpha), \xi_i(y_\alpha)) = 0, \end{aligned}$$

which contradicts (1) since $\{\xi_i(x_0)\}_{i=1}^k$ is a basis for $H(x_0)$. This verifes Claim 1.

For $x \in X$, set $p(x) =$ projection of $H(x)$ onto the linear span of $\{\xi_1(x), \ldots \xi_k(x)\}$.

Claim 2. If $m \in M(A)$, then there exists a neighborhood $U = U_m$ of x_0 such that

$$m(x) = p(x)m(x)p(x) + (I_{H(x)} - p(x))m(x)\,(I_{H(x)} - p(x)), \quad x \in U. \tag{2}$$

It follows from Corollary 3.5 of [3] that for $\gamma \in \Gamma$, the vector field $x \to m(x)\gamma(x)$ is in Γ. Thus by applying Claim 1 to the fields $x \to m(x)\xi_i(x)$, $x \to m(x)^*\,\xi_i(x)$, $i = 1, \ldots, k$, we find a neighborhood U of x_0 for which

$$\{m(x)\xi_i(x), m(x)^*\xi_i(x)\} \subseteq \text{span}\,\{\xi_1(x), \ldots, \xi_k(x)\}, \quad i=1, \ldots, k, \ x \in U.$$

But this says that (2) holds in U.

We can now derive a contradiction from the assumed existence of the net $\{x_\alpha\} \subseteq Y$ with $x_\alpha \to x_0$. For each $x \in Q \cap Y$, there is an open neighborhood V_x of x contained in $Q \cap Y$ and a $\gamma_x \in \Gamma$ which does not vanish on V_x and for which

$$\gamma_x \perp \{\xi_1, \ldots, \xi_k\} \text{ on } V_x. \tag{3}$$

Let $\mathscr{F}$ be a maximal family of disjoint, open subsets of $Q \cap Y$ such that, for each $O \in \mathscr{F}$, there is an $x \in Q \cap Y$ with $O \subseteq V_x$. Let U denote the union of the members of $\mathscr{F}$. Then U is open in X, and dense in $Q \cap Y$ by maximality.

We define a nonvanishing vector field ξ on U as follows: for $x \in U$, choose $O \in \mathscr{F}$, $y_0 \in Q \cap Y$, $\gamma_{y_0} \in \Gamma$, and V_{y_0} such that $x \in O$, $O \subseteq V_{y_0}$, and γ_{y_0} satisfies (3) on V_{y_0}. Then set $\xi(x) = \gamma_{y_0}(x)$. Since U is the disjoint union of the members of $\mathscr{F}$, ξ is well-defined, nonvanishing, and by (3),

$$\xi(x) \perp \{\xi_1(x), \ldots, \xi_k(x)\}, \quad x \in U. \tag{4}$$

Now, set $B = A_U$. For each $x \in U$, set $b(x) =$ the rank-1 operator on $H(x)$ with kernel $\{\xi(x)\}^\perp$ and which maps $\xi(x)$ to $\|\xi(x)\|^2\xi_1(x)$. By construction, ξ agrees on a neighborhood of each point of U with an element of Γ. So, by the definition of $b(x)$, if we set, for $a \in B$,

$$\delta(a)\,(x) = \begin{cases} b(x)a(x) - a(x)b(x), & x \in U, \\ 0 & , x \in X\backslash U, \end{cases}$$

then it is straightforward to check that δ is a derivation of B into A.

There exists $m \in M(A)$ with $\delta = adm|_B$. Then for each $x \in U$, there is a complex number $\lambda(x)$ such that $m(x) = b(x) + \lambda(x)I_{H(x)}$. By Claim 2, we hence find a neighborhood W of x_0 such that

$$b(x) = p(x)b(x)p(x) + (I_{H(x)} - p(x))b(x)(I_{H(x)} - p(x)), \quad x \in U \cap W. \tag{5}$$

But $x_0 \in Q \cap Y^-$, and since U is dense in $Q \cap Y$, $x_0 \in U^-$. Thus for points in $U \cap W$, we must have both (4) and (5) holding, but by the definition of b and the nonvanishing of ξ on U, this is impossible. We conclude that Y is closed, and so X_n is open and closed.

Now A_{X_n} is a direct summand of A, and so is HCT. But A_{X_n} is also n-homogeneous with spectrum X_n, and so, by Lemma 3.1, X_n is Stonean for $n \geqq 2$.

Set $X_t = \{x \in X\colon \dim x \geqq 2\}$.

LEMMA 3.3. *X_t is an open, Stonean subset of X.*

PROOF. Let $x_0 \in X_t$. By restricting to a suitable neighborhood of x_0, we may suppose A arises from a continuous field $(H(x), \Gamma)$ of Hilbert spaces on X. Choose an open precompact neighborhood U of x_0 and γ_1, $\gamma_2 \in \Gamma$ such that $\{\gamma_1(x), \gamma_2(x)\}$ is orthonormal for $x \in U$. For $i, j = 1, 2$, set

$$u_{ij}(x) = \begin{cases} \text{rank-1 partial isometry on } H(x) & \\ \text{sending } \gamma_j(x) \text{ to } \gamma_i(x) & , x \in U, \\ \qquad 0 & , x \in X \backslash U, \end{cases}$$

and set $e(x) =$ projection of $H(x)$ onto span $\{\gamma_1(x), \gamma_2(x)\}$, $x \in X$. Let $B = A_U$. If $C_0(U)$ denotes the algebra of all elements of $C(X)$ which vanish on $X \backslash U$, let $C =$ the C^*-algebra generated by $\{f \cdot u_{ij}: f \in C_0(U),\ i, j = 1, 2\}$. Then $C \subseteq B$, and it follows from Corollary 3.5 of [3] that $M(C) \subseteq M(B)$. For $m \in M(B)$, set $E(m)(x) = e(x)m(x)e(x)$. Since $\gamma_i \in \Gamma$ and $e(x) = u_{11}(x) + u_{22}(x)$, $x \in U$, it is straightforward to check that E defines a projection of norm 1 of $M(B)$ onto $M(C)$. Since B is an ideal in A, it is HCT, and so by the argument of Lemma 2.4, so is C. But C is isomorphic to $C_0(U) \otimes M_2$, and so by Lemma 2.5, U is Stonean. X_t is hence the union of open, Stonean subsets, and the lemma follows.

LEMMA 3.4. *X_∞ is open and closed in X.*

PROOF. $X \backslash X_\infty = \bigcup_1^\infty X_n$, and thus X_∞ is closed by Lemma 3.2.

Suppose X_∞ is not open. Then there is $x_0 \in X_\infty$ with $x_0 \in (\bigcup_n X_n)^-$. By Lemma 3.3, x_0 has a compact, open, Stonean neighborhood U. Let $U_n = U \cap X_{n+1}$, $n \geqq 1$. Then each U_n is compact and open, and $x_0 \in (\bigcup_n U_n)^-$. Since dim x_0 is infinite, we can find a sequence $\{y_k\} \subseteq U$ with $y_k \in U_{n_k}$ for distinct n_k. Since U is compact, $\{y_k\}$ has a cluster point $y_0 \in U$. Since the U_n's are disjoint, $y_0 \notin \bigcup_n U_n$.

As before, we may suppose A arises from a continuous field $(\Gamma, H(x))$ of Hilbert spaces on X. We claim that $H(y_0)$ is separable. Let $\{e_\alpha : \alpha \in \mathscr{A}\}$ be an orthonormal set in $H(y_0)$. Let $\xi_\alpha \in \Gamma$ have $\xi_\alpha(y_0) = e_\alpha$. For each fixed k, there is at most a countable set $\mathscr{A}_k$ of pairs (α, β) with $\|\xi_\alpha(y_k) - \xi_\beta(y_k)\| \geqq 1$. If $H(y_0)$ is nonseparable, we may take $\mathscr{A}$ uncountable and hence find a pair $(\alpha_0, \beta_0) \notin \bigcup_k \mathscr{A}_k$, whence $\|\xi_{\alpha_0}(y_k) - \xi_{\beta_0}(y_k)\| < 1$, for all k. But since $y_0 \in \{y_k\}^-$, we conclude from the continuity of $x \to \|\xi_{\alpha_0}(x) - \xi_{\beta_0}(x)\|$ that $\|e_{\alpha_0} - e_{\beta_0}\| = \|\xi_{\alpha_0}(y_0) - \xi_{\beta_0}(y_0)\| \leqq 1$, which is not possible. Hence $H(y_0)$ is separable.

Let $\{e_n\}$ be an orthonormal basis for $H(y_0)$. We will inductively construct a sequence $\{\gamma_n\} \subseteq \Gamma$ for which

(i) $\gamma_n(y_0) = e_n$, for all n;

(ii) $\{\gamma_1(x), \ldots, \gamma_n(x)\}$ is orthonormal for

$$x \in U \backslash \left(\bigcup_{k=1}^{n-1} U_k\right),\ n \geqq 2.$$

We may find a $\gamma_1' \in \Gamma$ and an open and closed neighborhood W of y_0

in U with $\gamma_1'(y_0) = e_1$ and $\|\gamma_1'(x)\| = 1$ for $x \in W$. Since U is compact and Stonean, $V = U \backslash W$ is compact and Stonean. If $V = \emptyset$, set $\gamma_1 = \gamma_1'$. Otherwise, each point y of V is contained in a neighborhood V_y of V on which there is a $\gamma_y \in \Gamma$ with $\|\gamma_y\| = 1$. Since V is compact and Stonean, we may disjointify a finite clopen cover of V by suitable V_y's to obtain a finite, disjoint, clopen cover $\{V_1, \ldots, V_k\}$ of V for which we may find $\xi_j \in \Gamma$ with $\|\xi_j(x)\| = 1$, $x \in V_j$, $j = 1, \ldots, k$. We may hence patch the γ_1' and ξ_j's together to obtain $\gamma_1 \in \Gamma$ with $\gamma_1(y_0) = e_1$ and $\|\gamma_1(x)\| = 1$, for all $x \in U$.

Suppose now that $\{\gamma_1, \ldots, \gamma_n\}$ have been chosen to satisfy (i) and (ii). $V_n = U \backslash (\bigcup_{k=1}^n U_k)$ is a compact, open, Stonean subset of X each point of which has dimension $\geqq n + 1$. We may hence find a clopen neighborhood W of y_0 in V_n and $\xi' \in \Gamma$ for which $\{\gamma_1, \ldots, \gamma_n, \xi'\}$ is orthonormal on W and $\xi'(y_0) = e_{n+1}$. If $V_n \backslash W \neq \emptyset$, each of its points has a neighborhood on which there is a $\xi \in \Gamma$ with $\{\gamma_1, \ldots, \gamma_n, \xi\}$ orthonormal, and we can now proceed as in the previous paragraph to construct $\gamma_{n+1} \in \Gamma$ with $\gamma_{n+1}(y_0) = e_{n+1}$ and $\{\gamma_1, \ldots, \gamma_{n+1}\}$ orthonormal on V_n. This finishes the construction of the γ_n's; we note that, by (ii),

(iii) $\{\gamma_1(x), \ldots, \gamma_n(x)\}$ is orthornomal for $x \in U_n$, $n \geqq 2$.

Now, set $B = A_{\bigcup_n U_n}$ and let

$$b(x) = \begin{cases} \text{rank-1 partial isometry on } H(x) \text{ which} & \\ \text{sends } \gamma_1(x) \text{ to } \gamma_n(x) & , x \in U_n, \\ \qquad 0 & , x \in X \backslash (\bigcup_n U_n). \end{cases}$$

Then $\delta = adb|_B$ is a derivation from B into A, and so there exists $m \in M(A)$ with $\delta = adm|_B$. There is a number $\lambda(x)$ with $m(x) = b(x) + \lambda(x) I_{H(x)}$, $x \in \bigcup_n U_n$.

Let $\{y_\alpha\}$ be a net in $\bigcup_n U_n$ converging to y_0. For each α, there is a unique integer $n(\alpha)$ with $y_\alpha \in U_{n(\alpha)}$. We may suppose that $n(\alpha) \geqq 2$, for each α, and so by (iii), for each α,

$$\begin{aligned} (m(y_\alpha)\gamma_1(y_\alpha), \gamma_1(y_\alpha)) &= (m(y_\alpha)\gamma_1(y_\alpha) - \gamma_{n(\alpha)}(y_\alpha), \gamma_1(y_\alpha)) \\ &= (m(y_\alpha)\gamma_1(y_\alpha) - b(y_\alpha)\gamma_1(y_\alpha), \gamma_1(y_\alpha)) \\ &= \lambda(y_\alpha)\,(\gamma_1(y_\alpha), \gamma_1(y_\alpha)) \\ &= \lambda(y_\alpha). \end{aligned}$$

Hence $\lim_\alpha \lambda(y_\alpha)$ exists since $x \to (m(x)\gamma_1(x), \gamma_1(x))$ is continuous on X. Set $\lambda_0 = \lim_\alpha \lambda(y_\alpha)$.

We have for each α,

$$\|m(y_\alpha)\gamma_1(y_\alpha) - \lambda(y_\alpha)\gamma_1(y_\alpha)\| = \|b(y_\alpha)\gamma_1(y_\alpha)\| = \|\gamma_{n(\alpha)}(y_\alpha)\| = 1. \tag{6}$$

Let k be a fixed positive integer. For each α,

(7) $\qquad (m(y_\alpha)\gamma_1(y_\alpha) - \lambda(y_\alpha)\gamma_1(y_\alpha), \gamma_k(y_\alpha)) = (\gamma_{n(\alpha)}(y_\alpha), \gamma_k(y_\alpha)).$

For each α, we can find $\alpha_1 > \alpha$ with $n(\alpha_1) > k$, and so by (iii), $(\gamma_{n(\alpha_1)}(y_{\alpha_1}), \gamma_k(y_{\alpha_1})) = 0$. Thus the left hand side of (7) has 0 as a cluster point. Since

$$\lim_\alpha (m(y_\alpha)\gamma_1(y_\alpha) - \lambda(y_\alpha)\gamma_1(y_\alpha), \gamma_k(y_\alpha)) = (m(y_0)\gamma_1(y_0) - \lambda_0\gamma_1(y_0), \gamma_k(y_0)),$$

we conclude that $(m(y_0)\gamma_1(y_0) - \lambda_0\gamma_1(y_0), \gamma_k(y_0)) = 0$, for all k, whence by (i), $m(y_0)\gamma_1(y_0) - \lambda_0\gamma_1(y_0) = 0$. But by (6),

$$\begin{aligned} \|m(y_0)\gamma_1(y_0) - \lambda_0\gamma_1(y_0)\| &= \lim_\alpha \|m(y_\alpha)\gamma_1(y_\alpha) - \lambda(y_\alpha)\gamma_1(y_\alpha)\| \\ &= 1. \end{aligned}$$

This contradiction shows that X_∞ is open.

LEMMA 3.5. *Suppose A is as before, and that each of its irreducible representations acts on a separable Hilbert space. Then X_∞ is discrete in X.*

PROOF. By Lemmas 3.3 and 3.4, we may suppose that $X = X_\infty$ and X is Stonean. If X is not discrete, we can find a disjoint sequence $\{U_n\}$ of nonempty, compact, open subsets of X with $U = \bigcup_n U_n$ precompact and such that $X \setminus U \neq \emptyset$. Taking an element y_k from each U_k and considering a cluster point y_0 of $\{y_k\}$ in U^-, we can, from the hypothesis that A has only separably acting irreducible representations, now apply the proof of Lemma 3.4 unchanged to deduce that A is not HCT.

PROOF OF THEOREM 1.1. ($\Rightarrow$). This is a straightforward consequence of Lemmas 2.1, 3.1, 3.2, 3.4, and 3.5.

($\Leftarrow$). If A has the indicated structure, then $M(A_2 \oplus A_3)$ is a direct sum of a family of injective C^*-algebras, and so, by Proposition 5.4 of [7], is injective. If we identify $A_2 \oplus A_3$ with its image in its universal representation, there is therefore a projection of norm 1 of $B(H)$ onto $M(A_2 \oplus A_3)$, where H is the representation space of the universal representation of $A_2 \oplus A_3$. Since $A_2 \oplus A_3$ is liminal, the argument of Lemma 2.4 now shows that $A_2 \oplus A_3$ is HCT. An application of Lemmas 2.2 and 2.3 then shows that A is HCT.

REMARKS. (1) We conjecture that Theorem 1.1 holds without the assumption of separably acting irreducible representations. The proof we give shows that the algebras with the structure as specified in Theorem 1.1 without this assumption are HCT; the problem comes in showing that an HCT C^*-algebra with continuous trace has its set X_∞ of infinite-dimensional irreducible representations discrete in its spectrum. The following proposition shows that if the associated field of elementary C^*-algebras is trivial over X_∞, Theorem 1.1 does hold.

PROPOSITION. *Suppose X is a locally compact Hausdorff space, H an* *infinite dimensional Hilbert space, $K(H)$ the algebra of compact operators on H. Then $C_0(X) \otimes K(H)$ is HCT if and only if X is discrete.*

PROOF. ($\Leftarrow$). This follows from the proof of Theorem 1.1.

($\Rightarrow$). Let $A = C_0(X) \otimes K(H)$. If X is not discrete, we may find a sequence $\{U_n\}$ of nonempty, disjoint, open subsets of X with $U = \bigcup_n U_n$ precompact and $X \backslash U \neq \emptyset$. For each n, choose $x_n \in U_n$ and let $\{f_n\}$ be a sequence in $C(X)$ with $0 \leqq f_n \leqq 1$, $f_n(x_n) = 1$, $f_n(X \setminus U_n) = \{0\}$. Let $\{e_n\}$ be an orthonormal sequence in H, and let T be any operator in $B(H)$ of norm 1 with $Te_n = e_{n+1}$. Define $b\colon X \to B(H)$ by $b(x) = \sum_n f_n(x)T^n$, $x \in X$. Set $B = A_U$, and define the derivation $\delta\colon B \to A$ by $\delta = adb|_B$. If there is an $m \in M(A)$ with $\delta = adm|_B$, then there is a sequence $\{\lambda_n\}$ of complex numbers for which $m(x_n) = T^n + \lambda_n \cdot I_H$. By Corollary 3.5 of [**3**], m is a bounded, continuous function of X into $B(H)$ equipped with the *-strong topology, and since U^- is compact, $m(U^-)$ is a *-strongly compact subset of $B(H)$. Thus $\{m(x_n)\}^-$ is *-strongly compact, which by the choice of T is not possible. Thus no such $m \in M(A)$ exists, and A is not HCT.

(2) A close examination of its proof shows that the conclusion of Theorem 1.1 is equivalent to the following assumption on A: for each closed, two-sided ideal I of A, every derivation of I into A is inner in $M(A)$. That this formally weaker assumption is equivalent to hereditary cohomological triviality for C^*-algebras with continuous trace is really not surprising, since the ideal structure of such algebras is so rich.

(3) Hereditary cohomological triviality does not pass to quotients. For a simple example, let N denote the positive integers with the discrete topology, and let X denote the Stone-Čech compactification of N. Then X is compact and Stonean, so $A = C(X) \otimes M_2$ is HCT by Lemma 2.5. Let I denote the ideal of all $a \in A$ with $\lim_n \|a(n)\| = 0$. Then A/I is isomorphic to $C(X \backslash N) \otimes M_2$, and since $X \backslash N$ is not Stonean [**9**, Problem 6R], Lemma 2.5 shows that A/I is not HCT.

(4) Hereditary cohomological triviality is not preserved by tensor products, as Lemma 2.5 or the proposition in Remark 1 shows. Suppose $\{A_\alpha\}$ is a family of C^*-algebras with the following property: there exists a constant $K > 0$ such that, for each α, whenever $\delta\colon B \to A_\alpha$ is a derivation from a C^*-subalgebra B of A_α, there is an $m \in M(A_\alpha)$ with $\delta = adm|_B$ *and* $\|m\| \leqq K \|\delta\|$. Is the restricted direct sum of the A_α's HCT?

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SCHOOL OF MATHEMATICAL SCIENCES, TEL-AVIV UNIVERSITY, TEL-AVIV, ISRAEL

DEPARTMENT OF MATHEMATICS, OAKLAND UNIVERSITY, ROCHESTER, MI 48063, U.S.A.