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THE IDEAL STRUCTURE OF THE SPACE OF κ-UNIFORM ULTRAFILTERS ON A DISCRETE SEMIGROUP

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1. Introduction. Throughout this paper $(S, +)$ will denote an infinite discrete semigroup. It is well known that the operation $+$ on S extends uniquely to βS, the Stone-Čech compactification of S, so that $(\beta S, +)$ is a left-topological semigroup with S contained in its topological center. (By left-topological we mean that, for each $p \in \beta S$, the function λ_p, defined by $\lambda_p(q) = p + q$, is continuous. The topological center is the set of points at which also ρ_p is continuous, where $\rho_p(q) = q + p$. See [**2**] for an elementary derivation of this extension.)

Since βS is the maximal left-topological compactification of S [**2**, Theorem 2.4], its algebraic structure is of inherent interest. Each compact left-topological semigroup has a smallest two-sided ideal (called, for obscure historical reasons, the minimal ideal) which is the union of all of the minimal right ideals and is also the union of all of the minimal left ideals, [**3**, Theorem II. 2.2]. It is this ideal structure with which we are primarily concerned in this paper.

In an earlier paper [**9**] we characterized the minimal right ideals and minimal ideals of $(\beta N, +)$ and $(\beta N, \cdot)$. It was observed later that the same results held for any discrete semigroup. We were led by this observation to consider the extent to which these and other earlier results extended to certain natural subsemigroups of βS.

The points of βS are the ultrafilters on S, each point $x \in S$ being identified with the principal ultrafilter $\hat{x} = \{A \subseteq S: x \in A\}$. For $A \subseteq S$, we let $\bar{A} = \{p \in \beta S: A \in p\}$. The set $\{\bar{A}: A \subseteq S\}$ forms a basis for the open sets of βS (as well as a basis for the closed sets). See [**5**] or [**7**] for a detailed construction of βS as a space of ultrafilters.

Associated with each ultrafilter p is a cardinal, $\|p\|$, called the *norm* of p. It is the minimum of $\{|A|: A \in p\}$. An ultrafilter p with $\|p\| \geqq \kappa$ is called κ-uniform. The space $U_\kappa(S)$ of κ-uniform ultrafilters on S is closed in βS and, as we shall see in §2, is very often a subsemigroup of βS. (For

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those more familiar with other constructions of βS, $U_\kappa(S)$ is the set of all points of βS which are not in the closure of any subset of S whose cardinal is smaller than κ.) See §7 and §14 of [5] for extensive information about spaces of uniform ultrafilters.

In §3 we characterize the minimal right ideals and the minimal ideal of $U_\kappa(S)$, provided $U_\kappa(S)$ is a semigroup. (The failure to characterize the minimal left ideals arises from a lack of ability rather than a lack of interest.)

In §4 we succeed in characterizing the closure of the minimal ideal of $U_\kappa(S)$ under certain conditions on κ and S. Fortunately these conditions are always satisfied when $\kappa = 1$, that is when $U_\kappa(S) = \beta S$. Consequently we obtain the fortuitous corollary: The closure of the minimal ideal of βS is an ideal of βS.

In §5 we turn our attention from minimal ideals to idempotents. Probably the major result of this section is: If S is cancellative and p is an idempotent in $\beta S \backslash S$, then each neighborhood of p contains 2^c idempotents and contains copies of the free group on 2^c generators.

We close this introduction with some remarks about notation. Given a subset A of S and given $x \in S$, $A - x = \{y \in S\colon y + x \in A\}$. (Then, given $p, q \in \beta S$ and $A \subseteq S$, $A \in p + q$ if and only if $\{x \in S\colon A - x \in p\} \in q$.) Observe that for $A, B \subseteq S$ and $x, y \in S$, $S\backslash(A - x) = (S\backslash A) - x$, $(A - x) \cap (B - x) = (A \cap B) - x$, $(A - x) \cup (B - x) = (A \cup B) - x$, and $(A - x) - y = A - (y + x)$.

A cardinal κ is an ordinal (the first ordinal of the given size) and each ordinal is the set of its predecessors. Exponentiation always denotes cardinal exponentiation. We write ω for the first infinite ordinal and c for the cardinality of the continuum (2^ω). The letters α, γ, δ, and κ will always denote cardinals. (Thus we write "let $\omega \leqq \kappa \leqq |S|$" in lieu of "let κ be a cardinal such that $\omega \leqq \kappa \leqq |S|$".)

Given a set A and a cardinal κ, $[A]^\kappa = \{B \subseteq A\colon |B| = \kappa\}$ and $[A]^{<\kappa} = \{B \subseteq A\colon |B| < \kappa\}$.

2. $U_\kappa(S)$. In this section we determine when $U_\kappa(S)$ is a right ideal, a left ideal, or a subsemigroup of βS. We begin by displaying some definitions and elementary facts.

DEFINITION 2.1. (a) For $p \in \beta S$, $\|p\| = \min\{|A|\colon A \in p\}$.

(b) $U_\kappa(S) = \{p \in \beta S\colon \|p\| \geqq \kappa\}$.

(c) A set $\mathscr{A}$ has the κ-uniform finite intersection property if and only if $|\bigcap \mathscr{F}| \geqq \kappa$ whenever $\mathscr{F}$ is a finite non-empty subset of $\mathscr{A}$.

Observe that $U_1(S) = \beta S$ and $U_\omega(S) = \beta S\backslash S$.

LEMMA 2.2. *If $\mathscr{A} \subseteq \mathscr{P}(S)$ and $\mathscr{A}$ has the κ-uniform finite intersection property then there exists $p \in U_\kappa(S)$ such that $\mathscr{A} \subseteq p$.*

PROOF. This is a routine Zorn's Lemma argument. See for example [5, Lemma 7.2].

The following induced filter is useful throughout our study of $U_\kappa(S)$.

DEFINITION 2.3. Let $p \in \beta S$. Let $\kappa = 1$ or $\kappa \geqq \omega$. $C_\kappa(p) = \{A \subseteq S: |\{x \in S: A - x \notin p\}| < \kappa\}$.

Thus $C_\kappa(p)$ is the set of subsets of S which κ-almost always translate to a member of p. (And $C_1(p)$ is the set of subsets which always translate to a member of p.)

LEMMA 2.4. *Let $p \in \beta S$ and let $\kappa \leqq |S|$ (with $\kappa = 1$ or $\kappa \geqq \omega$).*

(a) *$C_\kappa(p)$ is a filter on S.*

(b) $p + U_\kappa(S) = \{q \in \beta S: C_\kappa(p) \subseteq q\}$.

PROOF (a). For all $x \in S$, $S - x = S$ so $C_\kappa(p) \neq \varnothing$. Trivially $C_\kappa(p)$ is closed under supersets and $\varnothing \notin C_\kappa(p)$. If $A, B \in C_\kappa(p)$, then $\{x \in S: (A \cap B) - x \notin p\} = \{x \in S: A - x \notin p\} \cup \{x \in S: B - x \notin p\}$ so $A \cap B \in C_\kappa(p)$.

(b). First let $q \in p + U_\kappa(S)$ and pick $r \in U_\kappa(S)$ such that $q = p + r$. Let $A \in C_\kappa(p)$. Then $|\{x \in S: A - x \notin p\}| < \kappa$ so $\{x \in S: A - x \notin p\} \notin r$. Thus $S\backslash\{x \in S: A - x \notin p\} \in r$. That is $\{x \in S: A - x \in p\} \in r$ so $A \in p + r = q$.

Now let $q \in \beta S$ such that $C_\kappa(p) \subseteq q$. For each $A \in q$, let $D(A) = \{x \in S: A - x \in p\}$. Observe that, if $A, B \in q$, then $D(A \cap B) = D(A) \cap D(B)$. Further, if $A \in q$, then $S\backslash A \notin C_\kappa(p)$ (since $C_\kappa(p) \subseteq q$) so $|D(A)| \geqq \kappa$. Thus $\{D(A): A \in q\}$ has the κ-uniform finite intersection property. Pick $r \in U_\kappa(S)$ such that $\{D(A): A \in q\} \subseteq r$. Then $q \subseteq p + r$ and therefore, since q and $p + r$ are ultrafilters, $q = p + r$.

We are of course not interested in studying the semigroup $U_\kappa(S)$ unless it is a semigroup (And, since $U_1(S) = \beta S$ is always a semigroup we exclude $\kappa = 1$ from the following result, even though it is technically valid then.)

THEOREM 2.5. *Let $\omega \leqq \kappa \leqq |S|$. The following statements are equivalent.*

(a) *$U_\kappa(S)$ is a subsemigroup of βS.*

(b) *For all $p \in U_\kappa(S)$ and all $A \in [S]^{<\kappa}$, $S\backslash A \in C_\kappa(p)$.*

(c) *For all $A \in [S]^{<\kappa}$ and all $B \in [S]^\kappa$ there $F \in [B]^{<\omega}$ such that $|\bigcap_{x \in F} A - x| < \kappa$.*

PROOF. To see that (a) implies (b), let $p \in U_\kappa(S)$ and let $A \in [S]^{<\kappa}$. Suppose that $S\backslash A \notin C_\kappa(p)$. Then $C_\kappa(p) \cup \{A\}$ has the finite intersection property. (If $B \in C_\kappa(p)$ and $B \cap A = \varnothing$, then $B \subseteq S\backslash A$ so $S\backslash A \in C_\kappa(p)$.) Pick $q \in \beta S$ such that $C_\kappa(p) \cup \{A\} \subseteq q$. Pick, by Lemma 2.4(b), $r \in U_\kappa(S)$ such that $p + r = q$. Since $A \in q$, $q \notin U_\kappa(S)$ so $p + r \notin U_\kappa(S)$, a contradiction.

To see that (b) implies (c), let $A \in [S]^{<\kappa}$ and let $B \in [S]^\kappa$. Suppose that for each $F \in [B]^{<\omega}$, $|\bigcap_{x \in F} A - x| \geqq \kappa$. Then $\{A - x: x \in B\}$ has the κ-uniform finite intersection property so pick $p \in U_\kappa(S)$ such that $\{A - x: x \in B\} \subseteq p$. Then $B \subseteq \{x \in S: A - x \in p\}$ so $S \backslash A \notin C_\kappa(p)$, a contradiction.

To see that (c) implies (a), let $p, r \in U_\kappa(S)$. Let $q = p + r$. Then by Lemma 2.4(b), $C_\kappa(p) \subseteq q$. Suppose that $q \notin U_\kappa(S)$ and pick $A \in q$ such that $|A| < \kappa$. Let $D = \{x \in S: A - x \in p\}$. Then $D \in r$ so $|D| \geqq \kappa$. Pick $B \in [D]^\kappa$. Pick $F \in [B]^{<\omega}$ such that $|\bigcap_{x \in F} A - x| < \kappa$. Then $\bigcap_{x \in F} A - x \in p$ so $p \notin U_\kappa(S)$, a contradiction.

We observe that the statement of Theorem 2.4(c) is an algebraic statement about S. It is this sort of characterization in which we are primarily interested.

THEOREM 2.6. *Let $\omega \leqq \kappa \leqq |S|$. Statements* (a) *and* (b) *are equivalent and imply statement* (c). *If κ is regular all three statements are equivalent.*

(a) *$U_\kappa(S)$ is a right ideal of βS.*

(b) *For all $A \in [S]^{<\kappa}$ and all $x \in S$, $|A - x| < \kappa$.*

(c) *For all $x, y \in S$, $|\rho_x^{-1}[\{y\}]| < \kappa$.*

PROOF. To see that (a) implies (b), let $A \in [S]^{<\kappa}$ and let $x \in S$. Suppose that $|A - x| \geqq \kappa$ and pick $p \in U_\kappa(S)$ such that $A - x \in p$. Then $A \in p + x$ so $p + x \notin U_\kappa(S)$, a contradiction. (Recall that we are identifying x with $\hat{x}$. Since $x \in \{y \in S: A - y \in p\}$, $\{y \in S: A - y \in p\} \in \hat{x}$ and thus $A \in p + \hat{x}$.)

To see that (b) implies (a), let $p \in U_\kappa(S)$, let $q \in \beta S$, and suppose that $p + q \notin U_\kappa(S)$. Pick $A \in p + q$ such that $|A| < \kappa$. Since $A \in p + q$, $\{x \in S: A - x \in p\} \neq \emptyset$. Pick $x \in S$ such that $A - x \in p$. Then $|A - x| \geqq \kappa$.

To see that (b) implies (c), observe that $\rho_x^{-1}[\{y\}] = \{y\} - x$.

Finally assume that κ is regular. To see that (c) implies (b), let $A \in [S]^{<\kappa}$ and let $x \in S$. Then $A - x = \rho_x^{-1}[A] = \bigcup_{y \in A} \rho_x^{-1}[\{y\}]$. Since κ is regular, $|A| < \kappa$, and for each $y \in A$, $|\rho_x^{-1}[\{y\}]| < \kappa$, we have $|A - x| < \kappa$.

COROLLARY 2.7. *Let $\omega \leqq \kappa \leqq |S|$. If right cancellation holds in S, then $U_\kappa(S)$ is a right ideal of βS.*

PROOF. Since ρ_x is one-to-one, for each $A \subseteq S$, $|A - x| \leqq |A|$.

THEOREM 2.8. *If κ is any non-regular* (*infinite*) *cardinal there is a semigroup S satisfying statement* (c) *of Theorem* 2.6 *but not statement* (b).

PROOF. Let $\delta = \mathrm{cf}(\kappa)$ and let $\langle \eta_\sigma \rangle_{\sigma < \delta}$ be cofinal and increasing in κ. Define $f: \kappa \to \delta$ by $f(\tau) = \min \{\sigma < \delta: \tau < \eta_\sigma\}$. Let $S = \kappa$ and define an operation $*$ on S by

$$\xi * \tau = \begin{cases} \xi \text{ if } \xi < \delta \\ f(\xi) \text{ if } \xi \geqq \delta. \end{cases}$$

Then * is clearly associative. Now given $\tau, \sigma \in S$, if $\sigma \geqq \delta$, then $\rho_\tau^{-1}[\{\sigma\}] = \emptyset$, and if $\sigma < \delta$, then $\rho_\tau^{-1}[\{\sigma\}] \subseteq \{\sigma\} \cup \eta_\sigma$. Thus statement (c) holds. Let $A = \delta$ and let $\tau \in S$. Then $\rho_\tau^{-1}[A] = S$. Thus statement (b) fails.

Notice the similarity between the conditions of Theorems 2.5 and 2.9. The proofs are nearly the same also so we omit the proof of Theorem 2.9.

THEOREM 2.9. *Let $\omega \leqq \kappa \leqq |S|$. The following statements are equivalent.*
(a) *$U_\kappa(S)$ is a left ideal of βS.*
(b) *For all $p \in \beta S$ and all $A \in [S]^{<\kappa}$, $S \backslash A \in C_\kappa(p)$.*
(c) *For all $A \in [S]^{<\kappa}$ and all $B \in [S]^\kappa$ there exists $F \in [B]^{<\omega}$ such that $\bigcap_{x \in F} A - x = \emptyset$.*

COROLLARY 2.10. *Let $\omega \leqq \kappa \leqq |S|$. If left and right cancellation hold in S, then $U_\kappa(S)$ is a left ideal (in fact an ideal) of βS.*

PROOF. By Corollary 2.7, $U_\kappa(S)$ is a right ideal of βS. Suppose $U_\kappa(S)$ is not a left ideal and pick $A \in [S]^{<\kappa}$ and $B \in [S]^\kappa$ such that $\bigcap_{x \in F} A - x \neq \emptyset$ whenever $F \in [B]^{<\omega}$. Pick $x \in B$ and for all $y \in B\backslash\{x\}$, pick $z_y \in (A - x) \cap (A - y)$. For each $u \in A$, let $D_u = \{y \in B\backslash\{x\}: z_y + x = u\}$. Then $B\backslash\{x\} = \bigcup_{u \in A} D_u$. Pick $u \in A$ such that $|D_u| > |A|$. Given $y, v \in D_u$, $z_y + x = z_v + x$ so, by right cancellation $z_y = z_v$. Let z be that member of S such that $z_y = z$ for all $y \in D_u$. For each $v \in A$, let $E_v = \{y \in D_u: z + y = v\}$. Then $D_u = \bigcup_{v \in A} E_v$ (since for $y \in D_u$, $z + y = z_y + y \in A$). Since $|D_u| > |A|$, pick $v \in A$ such that $|E_v| \geqq 2$. Pick distinct $t, y \in E_v$. Then $z + t = z + y$, contradicting left cancellation.

Corollaries 2.7 and 2.10 raise the natural question of whether left cancellation in S is sufficient to guarantee that $U_\kappa(S)$ is a left ideal of βS. It turns out that the answer is "yes" (Theorem 2.11) and "no" (Theorem 2.12). Recall that $U_\omega(S) = \beta S\backslash S$.

THEOREM 2.11. *$U_\omega(S)$ is a left ideal of βS if and only if whenever $A \in [S]^{<\omega}$ and $\langle x_n \rangle_{n<\omega}$ and $\langle z_n \rangle_{n<\omega}$ are sequences in S with $x_n \neq x_m$ for $n \neq m$, there exist $n < m < \omega$ such that $z_m + x_n \notin A$. In particular, if left cancellation holds in S, then $U_\omega(S)$ is a left ideal of βS.*

PROOF. Necessity. Let A, $\langle x_n \rangle_{n<\omega}$ and $\langle z_n \rangle_{n<\omega}$ be given and let $B = \{x_n: n < \omega\}$. Pick, by Theorem 2.9(c), $F \in [B]^{<\omega}$ such that $\bigcap_{x \in F} A - x = \emptyset$. Pick $m < \omega$ such that $F \subseteq \{x_n: n < m\}$. Then $z_m \notin \bigcap_{n<m} A - x_n$ so there exists $n < m$ such that $z_m \notin A - x_n$.

Sufficiency. We show that condition (c) of Theorem 2.9 holds. Let $A \in [S]^{<\omega}$ and let $B \in [S]^\omega$. Enumerate B faithfully as $\{x_n: n < \omega\}$. Suppose that for all $F \in [B]^{<\omega}$, $\bigcap_{x \in F} A - x \neq \emptyset$. For each $m < \omega$ (with $m > 0$) pick $z_m \in \bigcap_{n<m} A - x_n$. Then for $n < m < \omega$, $z_m + x_n \in A$, a contradiction.

Finally assume left cancellation holds in S, and let $A \in [S]^{<\omega}$, $\langle x_n \rangle_{n<\omega}$ and $\langle z_n \rangle_{n<\omega}$ be given with $x_n \neq x_m$ when $n \neq m$. Let $m = |A| + 1$. Then $|\{z_m + x_n: n < m\}| = m > |A|$ so for some $n < m$, $z_m + x_n \notin A$.

THEOREM 2.12. *Let $\kappa > \omega$. There exists a semigroup S such that $|S| = \kappa$ and left cancellation holds in S but $U_\kappa(S)$ is not a subsemigroup of βS (and in particular, not a left ideal).*

PROOF. Let $L = \{x_\sigma: \sigma < \kappa\} \cup \{y_n: n < \omega\} \cup \{z_F: F \in [\kappa]^{<\omega}\}$ be an alphabet of distinct letters. Let $S = \{a_1 a_2 \ldots a_t$: each $a_i \in L$ and, if $1 \leqq i \leqq t - 1$, $a_i = z_F$, and $a_{i+1} = x_\sigma$, then $\sigma \notin F\}$. (The empty word is not in S.)

Given words w_1 and w_2 on L, let $w_1 w_2$ denote the usual concatenation of words. Let $w_1 = a_1 a_2 \ldots a_t$ and $w_2 = b_1 b_2 \ldots b_s$ be members of S. Define

$$w_1 + w_2 = \begin{cases} w_1 w_2 \text{ unless } a_t = z_F,\ b_1 = x_\sigma, \text{ and } \sigma \in F \\ a_1 a_2 \ldots a_{t-1} y_n b_2 b_3 \ldots b_s \text{ if } a_t = z_F,\ b_1 = x_\sigma, \\ \qquad \sigma \in F, \text{ and } n = |\{\tau \in F: \tau < \sigma\}|. \end{cases}$$

Then clearly $w_1 + w_2 \in S$. A routine consideration of cases establishes that $+$ is associative on S.

To see that left cancellation holds, let $w_1, w_2, w_3 \in S$ and assume $w_1 + w_2 = w_1 + w_3$. Assume that $w_1 = a_1 a_2 \ldots a_t$, $w_2 = b_1 b_2 \ldots b_s$, and $w_3 = c_1 c_2 \ldots c_v$. If $w_1 + w_2 = w_1 w_2$ and $w_1 + w_3 = w_1 w_3$ then $w_1 w_2 = w_1 w_3$ so, by cancellation in the free semigroup on L, we have $w_2 = w_3$. We may thus assume that $a_t = z_F$ for some $F \in [\kappa]^{<\omega}$ and, without loss of generality, that $b_1 = x_\sigma$ for some $\sigma \in F$. Let $n = |\{\tau \in F: \tau < \sigma\}|$. Then $w_1 + w_2 = a_1 a_2 \ldots a_{t-1} y_n b_2 b_3 \ldots b_s$. If $c_1 \neq x_\eta$ for any $\eta \in F$ then $w_1 + w_3 = a_1 a_2 \ldots a_{t-1} a_t c_1 c_2 \ldots c_v$ and hence $a_t = y_n$, a contradiction. We thus assume $c_1 = x_\eta$ for some $\eta \in F$ and let $m = |\{\tau \in F: \tau < \eta\}|$. Then $w_1 + w_3 = a_1 a_2 \ldots a_{t-1} y_m c_2 c_3 \ldots c_v$. Then $y_m = y_n$ so $m = n$ and hence $\sigma = \eta$. Since then $a_1 a_2 \ldots a_{t-1} y_n b_2 b_3 \ldots b_s = a_1 a_2 \ldots a_{t-1} y_n c_2 c_3 \ldots c_v$, we have $s = v$, $b_2 = c_2, \ldots, b_s = c_s$. That is $w_2 = w_3$ as required.

Finally, to see that $U_\kappa(S)$ is not a subsemigroup of βS, we show that condition (c) of Theorem 2.5 fails. Let $A = \{y_n: n < \omega\}$ and let $B = \{x_\sigma: \sigma < \kappa\}$. Then $A \in [S]^{<\kappa}$ and $B \in [S]^\kappa$. Suppose we have $F \in [B]^{<\omega}$ such that $|\bigcap_{x \in F} A - x| < \kappa$. Let $G = \{\sigma < \kappa: x_\sigma \in F\}$. Given any $H \in [\kappa]^{<\omega}$ such that $G \subseteq H$, we have $z_H \in \bigcap_{x \in F} A - x$. (For if $x \in F$ then $x = x_\sigma$ for some $\sigma \in H$ and $z_H + x_\sigma = y_n$ for some $n < \omega$.) Thus $\{z_H: H \in [\kappa]^{<\omega}$ and $G \subseteq H\} \subseteq \bigcap_{x \in F} A - x$, a contradiction.

In the next two sections we shall describe the minimal right ideals and the minimal ideal of $U_\kappa(S)$ (provided $U_\kappa(S)$ is a semigroup). As is well known, if $U_\kappa(S)$ is an ideal of βS then the minimal ideal of $U_\kappa(S)$ is the minimal ideal of βS. (See, for example, [**9**, Lemma 7.2].) We shall see in the next theorem that there are many examples where the minimal ideals are different.

DEFINITION 2.13. If $U_\kappa(S)$ is a subsemigroup of βS, then $M_\kappa(S)$ is the minimal ideal of $U_\kappa(S)$.

THEOREM 2.14. *Let L be a set of cardinals (each infinite or 1) with $|L| \geqq 2$. For each $\kappa \in L$, let $(S_\kappa, +_\kappa)$ be any semigroup such that $|S_\kappa| = \kappa$ and for all $\gamma \leqq \kappa$, $U_\gamma(S_\kappa)$ is a subsemigroup of βS_κ. Assume that $S_\kappa \cap S_\gamma = \varnothing$ for distinct $\kappa, \gamma \in L$. Let $S = \bigcup_{\kappa \in L} S_\kappa$ For $x \in S_\kappa$ and $y \in S_\gamma$, define*

$$x + y = \begin{cases} x +_\kappa y & \text{if } \kappa = \gamma \\ x & \text{if } \kappa < \gamma \\ y & \text{if } \gamma < \kappa. \end{cases}$$

Then (a) *For each κ, if $\omega \leqq \kappa \leqq |S|$, then $U_\kappa(S)$ is a subsemigroup of βS.*

(b) *For each $\kappa \geqq \omega$, if there exists $\delta \in L$ such that $\delta \geqq \kappa$, then $U_\kappa(S)$ is neither a right nor a left ideal of βS.*

(c) *If $\gamma \leqq |\bigcup \{S_\delta: \delta \in L$ and $\delta \leqq \gamma\}| < \kappa \leqq |S|$ and either $\gamma = 1$ or $\gamma \geqq \omega$, then $M_\kappa(S) \neq M_\gamma(S)$.*

PROOF. (a). Let $\omega \leqq \kappa \leqq |S|$. To see that $U_\kappa(S)$ is a subsemigroup of βS we apply Theorem 2.5. Let $A \in [S]^{<\kappa}$ and let $B \in [S]^\kappa$. Since $|A| < |B|$, pick $\delta \in L$ such that $|A \cap S_\delta| < |B \cap S_\delta|$. Let $\gamma = |B \cap S_\delta|$. so that $\gamma \leqq \kappa$ and $\gamma \leqq \delta$. By assumption $U_\gamma(S_\delta)$ is a subsemigroup of βS_δ so pick $F \in [B \cap S_\delta]^{<\omega}$ such that $|\bigcap_{x \in F}(A \cap S_\delta) - x| < \gamma$. (Here the minus is taken in S_δ.) Pick $y \in (B \cap S_\delta) \backslash A$ and let $G = F \cup \{y\}$. We show that $\bigcap_{x \in G} A - x \subseteq A \cup (\bigcap_{x \in F}(A \cap S_\delta) - x)$. (The minus on the left is taken in S.) To this end let $z \in \bigcap_{x \in G} A - x$. If $z \in S_\delta$, then $z \in \bigcap_{x \in F}(A \cap S_\delta) - x$. If $z \in S_\alpha$ for some $\alpha > \delta$, then $z + y = y \notin A$ so $z \notin \bigcap_{x \in G} A - x$. Finally, if $z \in S_\alpha$ for some $\alpha < \delta$, then $z + y = z$ so $z \in A$. Thus we have $|\bigcap_{x \in G} A - x| \leqq |A| + |\bigcap_{x \in F}(A \cap S_\delta) - x| < \kappa + \gamma = \kappa$.

(b). Pick $\alpha, \delta \in L$ such that $\alpha < \delta$ and $\kappa \leqq \delta$. Pick $y \in S_\alpha$ and pick $B \in [S_\delta]^\kappa$. Let $A = \{y\}$. Then $A \in [S]^{<\kappa}$. Then $S_\delta \subseteq A - y$ so by Theorem 2.6, $U_\kappa(S)$ is not a right ideal. Also $y \in \bigcap_{x \in B} A - x$ so by Theorem 2.9, $U_\kappa(S)$ is not a right ideal.

(c) Assume $\gamma \leqq |\bigcup \{S_\delta: \delta \in L$ and $\delta \leqq \gamma\}| < \kappa \leqq |S|$ and either $\gamma = 1$ or $\gamma \geqq \omega$. Let $B = \bigcup\{S_\delta: \delta \in L$ and $\delta \leqq \gamma\}$ and pick $p \in U_\gamma(S)$ such that $B \in p$. It suffices to show that $(p + U_\gamma(S)) \cap M_\kappa(S) = \varnothing$. (For $p + U_\gamma(S)$ is a right ideal of $U_\gamma(S)$ and hence $(p + U_\gamma(S)) \cap M_\gamma(S) \neq \varnothing$.) To this end, let $q \in U_\gamma(S)$. For any $x \in S$ and any $y \in B$, $y + x \in B$. Thus for each $x \in S$, $B \subseteq B - x$ and hence $S = \{x \in S\colon B - x \in p\}$. Thus $B \in p + q$ so $p + q \notin U_\kappa(S)$.

Theorem 2.14 provides plentiful examples of distinct minimal ideals but it does not guarantee that these ideals are non-isomorphic.

THEOREM 2.15. *If $\omega \leqq \delta$ and $2^{2^\delta} < \kappa$, then there is a semigroup S such that $M_\delta(S)$ and $M_\kappa(S)$ are not isomorphic.*

PROOF. Let $L = \{\delta, \kappa\}$ and let S_δ and S_κ be the free semigroups on δ and κ generators respectively. Let $S = S_\delta \cup S_\kappa$ be as in Theorem 2.14. As in the proof of Theorem 2.14 we have a right ideal, and hence a minimal right ideal R of $U_\delta(S)$, disjoint from $M_\kappa(S)$. Suppose we have an isomorphism $\phi: M_\kappa(S) \to M_\delta(S)$. Then $\phi^{-1}[R]$ is a minimal right ideal of $M_\kappa(S)$ and hence of $U_\kappa(S)$. Clearly $U_\kappa(S)$ is isomorphic to $U_\kappa(S_\kappa)$. (Given $p \in \beta S$, if $p \in U_\kappa(S)$, then $S_\delta \notin p$ so $S_\kappa \in p$. Identify p with $\{A \in p: A \subseteq S_\kappa\}$, an ultrafilter on S_κ.) By [3, Theorem II. 2.2] $\phi^{-1}[R]$ contains a maximal group. By [**12**, Lemma 1.4], this maximal group contains a copy of G_κ, the free group on κ generators. Thus $|\phi^{-1}[R]| \geqq \kappa$. But for each $q \in R$, $S_\delta \in q$ so $|R| \leqq 2^{2^\delta}$, a contradiction.

3. The minimal right ideals and the minimal ideal of $U_\kappa(S)$. Given a minimal right ideal R of $U_\kappa(S)$ and $p \in R$, $R = p + U_\kappa(S)$. We are concerned here with determining those ultrafilters p such that $p + U_\kappa(S)$ is a minimal right ideal.

LEMMA 3.1 *Let $\kappa \leqq |S|$ such that $\kappa \geqq \omega$ or $\kappa = 1$. If $U_\kappa(S)$ is a subsemigroup of βS, $A \subseteq S$, $p \in U_\kappa(S)$, and $S \backslash A \notin C_\kappa(p)$, then $C_\kappa(p) \cup \{A\}$ has the κ-uniform finite intersection property.*

PROOF. Since, by Lemma 2.4, $C_\kappa(p)$ is a filter, it suffices to let $D \in C_\kappa(p)$ and show that $|D \cap A| \geqq \kappa$. Suppose instead that $|D \cap A| < \kappa$.

Now $|\{x \in S: A - x \in p\}| \geqq \kappa$ since $S \backslash A \notin C_\kappa(p)$. Also $|\{x \in S: D - x \notin p\}| < \kappa$ since $D \in C_\kappa(p)$. Therefore, $|\{x \in S: (A \cap D) - x \in p\}| \geqq \kappa$. Pick $B \subseteq \{x \in S: (A \cap D) - x \in p\}$ such that $|B| = \kappa$. Since $A \cap D \in [S]^{<\kappa}$ and $B \in [S]^\kappa$, pick by Theorem 2.5, $F \in [B]^{<\omega}$ such that $|\bigcap_{x \in F} (A \cap D) - x| < \kappa$. But $\bigcap_{x \in F}(A \cap D) - x \in p$ and $p \in U_\kappa(S)$, a contradiction.

THEOREM 3.2. *Let $\kappa \leqq |S|$ with $\kappa = 1$ or $\kappa \geqq \omega$, assume that $U_\kappa(S)$ is a subsemigroup of βS, and let $p \in U_\kappa(S)$. Then $p + U_\kappa(S)$ is a minimal right ideal of $U_\kappa(S)$ if and only if whenever $B \in [S]^{<\kappa}$ and $A \subseteq S$ with $S \backslash A \notin C_\kappa(p)$, there exists $F \in [S \backslash B]^{<\omega}$ such that $\bigcup_{x \in F} A - x \in C_\kappa(p)$.*

PROOF. Necessity. Let $B \in [S]^{<\kappa}$ and let $A \subseteq S$ with $S \backslash A \notin C_\kappa(p)$. Then by Lemma 3.1, $C_\kappa(p) \cup \{A\}$ has the κ-uniform finite intersection property so pick $r \in U_\kappa(S)$ such that $C_\kappa(p) \cup \{A\} \subseteq r$.

Suppose that for each $F \in [S \backslash B]^{<\omega}$, $\bigcup_{x \in F} A - x \notin C_\kappa(p)$. We show now that $C_\kappa(p) \cup \{S \backslash (\bigcup_{x \in F} A - x): F \in [S \backslash B]^{<\omega}\}$ has the κ-uniform finite intersection property. Since $\{S \backslash (\bigcup_{x \in F} A - x): F \in [S \backslash B]^{<\omega}\}$ is closed under finite intersections, it suffices to let $F \in [S\backslash B]^{<\omega}$ and show that $C_\kappa(p) \cup \{S\backslash(\bigcup_{x\in F} A - x)\}$ has the κ-uniform finite intersection property. But this follows from Lemma 3.1 since, by assumption $\bigcup_{x\in F} A - x \notin C_\kappa(p)$. Pick $q \in U_\kappa(S)$ such that $C_\kappa(p) \cup \{S\backslash(\bigcup_{x\in F} A - x): F \in [S\backslash B]^{<\omega}\} \subseteq q$.

By Lemma 2.4(b), $q, r \in p + U_\kappa(S)$. Since $p + U_\kappa(S)$ is a minimal right ideal, $q + U_\kappa(S)$ is a right ideal, and $q + U_\kappa(S) \subseteq p + U_\kappa(S)$ we have $q + U_\kappa(S) = p + U_\kappa(S)$. Thus $r \in q + U_\kappa(S)$. Pick $s \in U_\kappa(S)$ such that $q + s = r$. Now $A \in r$ so $\{x \in S: A - x \in q\} \in s$ so $|\{x \in S: A - x \in q\}| \geqq \kappa$. Since $|B| < \kappa$ pick $x \in S\backslash B$ such that $A - x \in q$. But also, since $\{x\} \in [S\backslash B]^{<\omega}$, $S\backslash(A - x) \in q$, a contradiction.

Sufficiency. It suffices to let $q, r \in p + U_\kappa(S)$ and show that there is some $s \in U_\kappa(S)$ with $q + s = r$. (For suppose we have a right ideal R of $U_\kappa(S)$ with $R \subsetneqq p + U_\kappa(S)$. If $q \in R$, $r \in (p + U_\kappa(S))\backslash R$, and $s \in U_\kappa(S)$ then $q + s \in R$ so $q + s \neq r$.)

Let $q, r \in p + U_\kappa(S)$. Then $C_\kappa(p) \subseteq q$ and $C_\kappa(p) \subseteq r$. For each $A \in r$, let $D(A) = \{x \in S: A - x \in q\}$. As in the proof of Lemma 2.4(b), it suffices to show that, for each $A \in r$, $|D(A)| \geqq \kappa$.

Let $A \in r$. Since $C_\kappa(p) \subseteq r$ we have $S\backslash A \notin C_\kappa(p)$. Pick $F_0 \in [S]^{<\omega}$ such that $\bigcup_{x\in F_0} A - x \in C_\kappa(p)$. Let $\tau < \kappa$ (τ an ordinal) and assume that for all $\sigma < \tau$ we have chosen $F_\sigma \in [S\backslash \bigcup_{\eta<\sigma} F_\eta]^{<\omega}$ such that $\bigcup_{x\in F_\sigma} A - x \in C_\kappa(p)$. Then $|\bigcup_{\sigma<\tau} F_\sigma| < \kappa$ so by assumption we may pick $F_\tau \in [S\backslash \bigcup_{\sigma<\tau} F_\sigma]^{<\omega}$ such that $\bigcup_{x\in F_\tau} A - x \in C_\kappa(p)$. (Observe that if $\kappa = 1$ the basis step was the only step in the preceding induction.) Now for each $\tau < \kappa$, $\bigcup_{x\in F_\tau} A - x \in q$ since $C_\kappa(p) \subseteq q$, so pick $x_\tau \in F_\tau$ such that $A - x_\tau \in q$. Then $\{x_\tau: \tau < \kappa\} \subseteq D(A)$. Since $\{F_\tau: \tau < \kappa\}$ is a pairwise disjoint collection we have $|D(A)| \geqq \kappa$ as required.

COROLLARY 3.3. *Let $p \in \beta S$. Then $p + \beta S$ is a minimal right ideal of βS if and only if whenever $A \subseteq S$ with $S\backslash A \notin C_1(p)$, there exist $F \in [S]^{<\omega}$ such that $\bigcup_{x\in F} A - x \in C_1(p)$.*

It would of course be interesting to know when $U_\kappa(S) + p$ is a minimal left ideal. This task is made more difficult by the fact that $U_\kappa(S) + p$ is not generally closed [9, Corollary 9.16], and therefore does not consist of all ultrafilters containing some fixed filter. The best characterization we have been able to come up with amounts to little more than a translation of the definition of minimal left ideals.

THEOREM 3.4. *Let $\kappa \leqq |S|$ with $\kappa = 1$ or $\kappa \geqq \omega$, assume that $U_\kappa(S)$ is a subsemigroup of βS, and let $p \in U_\kappa(S)$. Statements* (b) (i) *and* (c) (i) *are equivalent. Statements* (b) (ii) *and* (c) (ii) *are equivalent. Statements* (a), (b), *and* (c) *are equivalent and imply statement* (d). *Finally statement* (d) *implies statement* (c) (i).

(a) $p \in M_\kappa(S)$.

(b) (i) $C_\kappa(p) \subseteq p$, *and*

(ii) $p + U_\kappa(S)$ *is a minimal right ideal of* $U_\kappa(S)$.

(c) (i) *For all* $A \in p$, $|\{x \in S: A - x \in p\}| \geqq \kappa$,

(ii) *For all* $A \subseteq S$, *if* $|\{x \in S: A - x \in p\}| \geqq \kappa$, *then for all* $B \in [S]^{<\kappa}$ *there exists* $F \in [S\backslash B]^{<\omega}$ *such that* $\bigcup_{x \in F} A - x \in C_\kappa(p)$.

(d) *For all* $A \in p$ *and all* $B \in [S]^{<\kappa}$, *there exists* $F \in [S\backslash B]^{<\omega}$ *such that* $\bigcup_{x \in F} A - x \in C_\kappa(p)$.

PROOF. The equivalence of (b) (i) and (c) (i) is trivial. The equivalence of (b) (ii) and (c) (ii) is Theorem 3.2. That (c) implies (d) is trivial.

To see that (a) implies (b), pick a minimal right ideal R of $U_\kappa(S)$ such that $p \in R$ (since $M_\kappa(S)$ is the union of all of the minimal right ideals of $U_\kappa(S)$). Then $p + U_\kappa(S) \subseteq R$ so $p + U_\kappa(S) = R$. Thus (b) (ii) holds. Since $p \in R$, $p \in p + U_\kappa(S)$ so by Lemma 2.4, $C_\kappa(p) \subseteq p$.

To see that (b) implies (a), observe that by (b) (ii), $p + U_\kappa(S) \subseteq M_\kappa(S)$ and by (b) (i), $p \in p + U_\kappa(S)$.

Finally we show that (d) implies (c) (i). Let $A \in p$, let $B = \{x \in S: A - x \in p\}$, and suppose that $|B| < \kappa$. Pick $F_0 \in [S\backslash B]^{<\omega}$ such that $\bigcup_{x \in F_0} (A - x) \in C_\kappa(p)$. (Again we observe that if $\kappa = 1$, the induction stops here.) Let $\tau < \kappa$ and assume we have chosen for $\sigma < \tau$, $F_\sigma \in [S\backslash(B \cup \bigcup_{\eta<\sigma} F_\eta)]^{<\omega}$ such that $\bigcup_{x \in F_\sigma} A - x \in C_\kappa(p)$. Then $|B \cup \bigcup_{\sigma<\tau} F_\sigma| < \kappa$ so we may choose $F_\tau \in [S\backslash(B \cup \bigcup_{\sigma<\tau} F_\sigma)]^{<\omega}$ such that $\bigcup_{x \in F_\tau} A - x \in C_\kappa(p)$.

We claim that for each $\tau < \kappa$, $|S\backslash \bigcup_{x \in F_\tau} (B - x)| < \kappa$. Indeed, let $y \in S\backslash \bigcup_{x \in F_\tau}(B - x)$. Then for each $x \in F_\tau$, $y + x \notin B$ so $A - (y + x) \notin p$. Thus $(\bigcup_{x \in F_\tau} A - x) - y \notin p$. Therefore $S\backslash \bigcup_{x \in F_\tau}(B - x) \subseteq \{y \in S: (\bigcup_{x \in F_\tau} A - x) - y \notin p\}$. Since $\bigcup_{x \in F_\tau} A - x \in C_\kappa(p)$ we have $|\{y \in S: (\bigcup_{x \in F} A - x) - y \notin p\}| < \kappa$ as required.

Now let q be any element of $U_\kappa(S)$. Since, for each τ, $|S\backslash \bigcup_{x \in F_\tau}(B - x)| < \kappa$ we have $\bigcup_{x \in F_\tau}(B - x) \in q$. Pick for each $\tau < \kappa$ some $x_\tau \in F_\tau$ such that $B - x_\tau \in q$. $E = \{x_\tau: \tau < \kappa\}$. Since $\{F_\tau: \tau < \kappa\}$ is a pairwise disjoint family, $|E| = \kappa$.

Now $B \in [S]^{<\kappa}$, $E \in [S]^\kappa$, and $U_\kappa(S)$ is a subsemigroup of βS so by Theorem 2.5 we may pick $G \in [E]^{<\omega}$ such that $|\bigcap_{x \in G} B - x| < \kappa$. But for each $x \in G$, $B - x \in q$ so $|\bigcap_{x \in G} B - x| \geqq \kappa$, a contradiction.

The statement of Theorem 3.4(d) is clearly nicer than that of Theorem 3.4(c). That is the former only requires that members of p satisfy a certain condition. Unfortunately, we have been unable to determine if statement (d) is equivalent to the other statements. We do however have the following theorem. Observe that the hypothesis implies that $U_\kappa(S)$ is a subsemigroup of βS. Observe also that the hypothesis holds if $U_\kappa(S)$ is a right ideal of βS.

COROLLARY 3.5. *Let* $\kappa \leqq |S|$ *with* $\kappa = 1$ *or* $\kappa \geqq \omega$, *assume that for each*

$B \in [S]^{<\kappa}$ *one has* $|\{x \in S\colon |B - x| \geqq \kappa\}| < \kappa$, *and let* $p \in U_\kappa(S)$. *Then* $p \in M_\kappa(S)$ *if and only if for all* $A \in p$ *and all* $B \in [S]^{<\kappa}$, *there exists* $F \in [S \backslash B]^{<\omega}$ *such that* $\bigcup_{x \in F} A - x \in C_\kappa(p)$.

PROOF. Let $A \subseteq S$ such that $|\{x \in S\colon A - x \in p\}| \geqq \kappa$ and let $B \in [S]^{<\kappa}$. By Theorem 3.4, we need only show that there exists $F \in [S \backslash B]^{<\kappa}$ with $\bigcup_{x \in F} A - x \in C_\kappa(p)$. Now $|\{x \in S\colon A - x \in p\}| \geqq \kappa$ and $|\{x \in S\colon |B - x| \geqq \kappa\}| < \kappa$ so pick $x \in S$ such that $A - x \in p$ and $|B - x| < \kappa$. Pick $G \in [S \backslash (B - x)]^{<\omega}$ such that $\bigcup_{y \in G} (A - x) - y \in C_\kappa(p)$. Let $F = G + x$. Then $F \in [S \backslash B]^{<\omega}$. (If $z \in F \cap B$, then $z = y + x$ for some $y \in G$ so $y \in G \cap (B - x)$.) Also $\bigcup_{z \in F} A - z = \bigcup_{y \in G} A - (y + x) = \bigcup_{y \in G} (A - x) - y \in C_\kappa(p)$.

COROLLARY 3.6. *Let* $p \in \beta S$. *Then* $p \in M_1(S)$ *if and only for all* $A \in p$ *there exists* $F \in [S]^{<\omega}$ *such that* $\bigcup_{x \in F} A - x \in C_1(p)$.

4. The closure of the minimal ideal of $U_\kappa(S)$. We have in this section an example of the phenomenon that generalization sometimes leads to simplification. The reader may wish to compare this section with Section 3 of [8], which consisted essentially of a proof of a special case of Theorem 4.5.

DEFINITION 4.1. (a) Let $A \subseteq S$. We say that A is κ-large if and only if there exists $B \in [S]^{<\kappa}$ such that whenever $F \in [S \backslash B]^{<\omega}$, $|\{x \in A\colon x + F \subseteq A\}| \geqq \kappa$.

(b) $\Delta_\kappa(S) = \{p \in U_\kappa(S)\colon$ For all $A \in p$ and all $B \in [S]^{<\kappa}$, there exists $G \in [S \backslash B]^{<\omega}$ such that $\bigcup_{t \in G} A - t$ is κ-large.$\}$

LEMMA 4.2. *Let* $\kappa \leqq |S|$ *with* $\kappa = 1$ *or* $\kappa \geqq \omega$ *and let* $p \in U_\kappa(S)$ *such that* $C_\kappa(p) \subseteq p$. *Then each* $A \in C_\kappa(p)$ *is* κ*-large.*

PROOF. Let $A \in C_\kappa(p)$ and let $B = \{x \in S\colon A - x \notin p\}$. Then $|B| < \kappa$. Let $F \in [S \backslash B]^{<\omega}$. Since $C_\kappa(p) \subseteq p$, $A \in p$. Thus $A \cap \bigcap_{y \in F} A - y \in p$. Since $A \cap \bigcap_{y \in F} A - y \subseteq \{x \in A\colon x + F \subseteq A\}$ and $p \in U_\kappa(S)$ we are done.

Lemma 4.3. *Let* $\kappa \leqq |S|$ *with* $\kappa = 1$ *or* $\kappa \geqq \omega$ *and assume* $U_\kappa(S)$ *is a subsemigroup of* βS. *Then* $M_\kappa(S) \subseteq \Delta_\kappa(S)$.

PROOF. Let $p \in M_\kappa(S)$. Let $A \in p$ and let $B \in [S]^{<\kappa}$. Pick, by Theorem 3.4(d), $G \in [S \backslash B]^{<\omega}$ such that $\bigcup_{t \in G} A - t \in C_\kappa(p)$. By Lemma 4.2, $\bigcup_{t \in G} A - t$ is κ-large.

LEMMA 4.4. *Let* $\kappa \leqq |S|$ *with* $\kappa = 1$ *or* $\kappa \geqq \omega$, *assume* $U_\kappa(S)$ *is a subsemigroup of* βS, *and let* $A \subseteq S$. *If* A *is* κ*-large then there exists* $p \in U_\kappa(S) \cap \bar{A}$ *such that* $p + U_\kappa(S) \subseteq \bar{A}$.

PROOF. Assume A is κ-large. By Lemma 2.4(b) it suffices to show that there exists $p \in U_\kappa(S) \cap \bar{A}$ such that $A \in C_\kappa(p)$. Suppose instead that for each $p \in U_\kappa(S) \cap \bar{A}$, $A \notin C_\kappa(p)$. Pick $B \in [S]^{<\kappa}$ such that, whenever $F \in [S \backslash B]^{<\omega}$, $|\{x \in A: x + F \subseteq A\}| \geqq \kappa$. Now, given $p \in U_\kappa(S) \cap \bar{A}$, $A \notin C_\kappa(p)$ so $|\{x \in S: A - x \notin p\}| \geqq \kappa$ so we may pick $x \in S \backslash B$ such that $(S \backslash A) - x \in p$. Thus $U_\kappa(S) \cap \bar{A} \subseteq \bigcup_{x \in S \backslash B} \overline{(S \backslash A) - x}$. Since $U_\kappa(S) \cap \bar{A}$ is compact and each $\overline{(S \backslash A) - x}$ is open we may pick $F \in [S \backslash B]^{<\omega}$ such that $U_\kappa(S) \cap \bar{A} \subseteq \bigcup_{x \in F} \overline{(S \backslash A) - x}$. Now $|\{x \in A: x + F \subseteq A\}| \geqq \kappa$ so pick $p \in U_\kappa(S)$ such that $\{x \in A: x + F \subseteq A\} \in p$. Then $p \in U_\kappa(S) \cap \bar{A}$ so pick $z \in F$ such that $(S \backslash A) - z \in p$. Pick $y \in (S \backslash A - z) \cap \{x \in A: x + F \subseteq A\}$. They $y + z \in S \backslash A$ but $z \in F$ so $y + z \in A$, a contradiction.

We could, of course, not like to have any assumptions in the following theorem beyond the necessary one that $U_\kappa(S)$ is a semigroup. We do not know if any version of the assumption which we add is necessary. We observe at any rate that our assumption does hold, by Theorem 2.6, if $\kappa \geqq \omega$ and $U_\kappa(S)$ is a right ideal of βS. It also clearly holds if $\kappa = 1$. Observe that since $U_\kappa(S)$ is closed in βS, it does not matter whether we consider $\operatorname{cl} M_\kappa(S)$ to be taken in βS or in $U_\kappa(S)$.

THEOREM 4.5. *Let $\kappa \leqq |S|$ with $\kappa = 1$ or $\kappa \geqq \omega$, assume $U_\kappa(S)$ is a subsemigroup of βS, and assume that $|\{x \in S: |B + x| < \kappa$ for some $B \in [S]^\kappa\}| < \kappa$. Then $\operatorname{cl} M_\kappa(S) = \Delta_\kappa(S)$.*

PROOF. To see that $\Delta_\kappa(S)$ is closed, let $p \in U_\kappa(S) \backslash \Delta_\kappa(S)$ and pick $A \in p$ and $B \in [S]^{<\kappa}$ such that for all $G \in [S \backslash B]^{<\omega}$, $\bigcup_{t \in G} A - t$ is not κ-large. Then $\bar{A}$ is a neighborhood of p missing $\Delta_\kappa(S)$. By Lemma 4.3, $M_\kappa(S) \subseteq \Delta_\kappa(S)$. Consequently we have $\operatorname{cl} M_\kappa(S) \subseteq \Delta_\kappa(S)$.

To see that $\Delta_\kappa(S) \subseteq \operatorname{cl} M_\kappa(S)$, let $p \in \Delta_\kappa(S)$ and let $D \in p$. We show that $\bar{D} \cap M_\kappa(S) \neq \emptyset$. Let $E = \{x \in S: |B + x| < \kappa$ for some $B \in [S]^\kappa\}$. By assumption $|E| < \kappa$.

Since $E \in [S]^{<\kappa}$, $D \in p$, and $p \in \Delta_\kappa(S)$, pick $G \in [S \backslash E]^{<\omega}$ such that $\bigcup_{t \in G} D - t$ is κ-large. Let $A = \bigcup_{t \in G} D - t$ and pick, by Lemma 4.4, $q \in U_\kappa(S) \cap \bar{A}$ such that $q + U_\kappa(S) \subseteq \bar{A}$. Now $q + U_\kappa(S)$ is a right ideal of $U_\kappa(S)$ so pick a minimal right ideal R of $U_\kappa(S)$ such that $R \subseteq q + U_\kappa(S)$ and pick $r \in R$.

Since $R \subseteq q + U_\kappa(S) \subseteq \bar{A}$, we have $A \in r$. Since $R \subseteq M_\kappa(S)$ (which is, you will recall, the union of all minimal right ideals of $U_\kappa(S)$) we have $r \in M_\kappa(S)$. By Theorem 3.4 we have $C_\kappa(r) \subseteq r$. Since $A \in r$, $S \backslash A \notin C_\kappa(r)$ so $|\{x \in S: A - x \in r\}| \geqq \kappa$.

Now for any $x \in S$, $A - x = \bigcup_{t \in G}(D - t) - x = \bigcup_{t \in G} D - (x + t)$. Thus, if $x \in S$ and $A - x \in r$, then for some $t \in G$, $D - (x + t) \in r$. Thus $\{x \in S: A - x \in r\} \subseteq \bigcup_{t \in G}\{x \in S: D - (x + t) \in r\}$ and hence we may pick $t \in G$ such that $|\{x \in S: D - (x + t) \in r\}| \geqq \kappa$. Pick $B \subseteq \{x \in S: D - (x + t) \in r\}$ such that $|B| = \kappa$.

Since $G \subseteq S \backslash E$, we have $|B + t| \geqq \kappa$ (and hence $|B + t| = \kappa$). Since

$B \subseteq \{x \in S: D - (x + t) \in r\}$ we have $B + t \subseteq \{y \in S: D - y \in r\}$ and hence $|\{y \in S: D - y \in r\}| \geqq \kappa$. Pick $s \in U_\kappa(S)$ such that $\{y \in S: D - y \in r\} \in s$. Then $D \in r + s$. Since $r \in M_\kappa(S)$, $r + s \in M_\kappa(S)$ and hence $\bar{D} \cap M_\kappa(S) \neq \emptyset$, as required.

With the appropriate right-left switch and with D chosen to be non-dense, Example V.1.1. of **[3]** (due originally to Ruppert in **[13]**) shows that the closure of the minimal ideal of a compact left-topological semigroup need not be a right ideal. The following corollary shows in particular that the closure of the minimal ideal of βS is always an ideal.

COROLLARY 4.6. *Let $\kappa \leqq |S|$ with $\kappa = 1$ or $\kappa \geqq \omega$. If $U_\kappa(S)$ is a right ideal of βS, then* $\mathrm{cl}M_\kappa(S)$ *is a right ideal of βS (and hence an ideal of $U_\kappa(S)$).*

PROOF. Since the closure of any left ideal is again a left ideal and since, by Theorem 4.5, $\mathrm{cl}M_\kappa(S) = \Delta_\kappa(S)$, we need only show that $\Delta_\kappa(S)$ is a right ideal of βS. To this end, let $p \in \Delta_\kappa(S)$ and let $q \in \beta S$. To see that $p + q \in \Delta_\kappa(S)$, let $A \in p + q$ and let $B \in [S]^{<\kappa}$. Then $\{x \in S: A - x \in p\} \in q$ so pick $x \in S$ such that $A - x \in p$. By Theorem 2.6, $|B - x| < \kappa$ so pick $G \in [S\backslash(B - x)]^{<\kappa}$ such that $\bigcup_{t \in G}(A - x) - t$ is κ-large. Then $G + x \in [S\backslash B]^{<\kappa}$ and $\bigcup_{y \in G+x} A - y = \bigcup_{t \in G}(A - x) - t$.

5. Idempotents in βS, for cancellative S. We restrict our attention in this section to semigroups S in which both left and right cancellation hold. In this event, by Corollaries 2.7 and 2.10, each $U_\kappa(S)$ is an ideal of βS. Consequently **[9**, Lemma 7.2**]** if $\omega \leqq \kappa \leqq |S|$, then $M_\kappa(S) = M_1(S)$. That is, we are only concerned with one minimal ideal, the minimal ideal of βS.

As is well known, the minimal ideal of βS contains idempotents. (See, for example **[3**, Theorem II.2.2**]**.) We shall see here that, given any idempotent p of $\beta S\backslash S$ (whether or not it is in the minimal ideal), each neighborhood contains an algebraic and topological copy of a certain subsemigroup of βN (where $N = \{1, 2, 3, \ldots\} = \omega\backslash\{0\}$ under addition). Since this subsemigroup is known to contain 2^c idempotents and 2^c copies of the free group on 2^c generators **[12]**, a similar result holds for each neighborhood of p.

We shall have need of the following generalization of Theorem 2.2 of **[12]**. The proof may be taken nearly verbatim from **[12]**.

THEOREM 5.1 *Let T be a compact Hausdorff left-topological semigroup, let $\phi: S \to T$, and let $\mathscr{A} \subseteq \mathscr{P}(S)$. If*

(1) *for each $x \in S$, $\rho_{\phi(x)}$ is continuous and*

(2) *there exists $B \in \mathscr{A}$ such that for each $x \in B$, there exists $A \in \mathscr{A}$ such that $\phi(y + x) = \phi(y) + \phi(x)$ whenever $y \in A$,*

then $\phi^\beta(p + q) = \phi^\beta(p) + \phi^\beta(q)$ whenever $p, q \in \bigcap_{A \in \mathscr{A}} \bar{A}$.

The function ϕ with which we shall apply this result is based on the finite sums of a sequence, written in decreasing order of indices. (See **[11]** for a discussion of the history of the relationship between ultrafilters and finite sums.)

DEFINITION 5.2 Let $H \subseteq \omega$ and let $\langle x_n \rangle_{n \in H}$ be a (possibly finite) sequence in S.

(a) If F is a finite non-empty subset of H define $\sum_{n \in F} x_n$ inductively on $|F|$ by

(i) $\sum_{n \in \{m\}} x_n = x_m$ and

(ii) if $|F| > 1$ and $m = \max F$ then $\sum_{n \in F} x_n = x_m + \sum_{n \in F \setminus \{m\}} x_n$.

(b) $FS(\langle x_n \rangle_{n \in H}) = \{\sum_{n \in F} x_n \colon F \text{ is a finite non-empty subset of } H\}$.

Thus, for example, $\sum_{n \in \{1,3,4,5\}} x_n = x_5 + x_4 + x_3 + x_1$. Our initial efforts are directed at obtaining sequences in which the expressions in $FS(\langle x_n \rangle_{n<\omega})$ are unique.

LEMMA 5.3. *Assume S is cancellative, let $p \in \beta S \backslash S$, and let $G \in [S]^{<\omega}$. Then* $|\{x \in S\colon \{y \in S\colon y + x \neq y\} \cap \bigcap_{t \in G}\{y \in S\colon y + x + t \neq y\} \notin p\}| < \omega$.

PROOF. For notational convenience let us temporarily adjoin 0 to S. Then we are concerned with $D = \{x \in S\colon \bigcap_{t \in G \cup \{0\}} \{y \in S\colon y + x + t \neq y\} \notin p\}$. Suppose $|D| > |G| + 1$. Then for each $x \in D$ there exists $t \in G \cup \{0\}$ such that $\{y \in S\colon y + x + t \neq y\} \notin p$, that is $\{y \in S\colon y + x + t = y\} \in p$. Pick $t \in G \cup \{0\}$ and distinct $x_1, x_2 \in D$ such that $\{y \in S\colon y + x_1 + t = y\} \in p$ and $\{y \in S\colon y + x_2 + t = y\} \in p$. Pick y in the intersection of these two sets. Then $y + x_1 + t = y + x_2 + t$. Then by right and left cancellation (or just left cancellation if $t = 0$), $x_1 = x_2$.

The derivation of the finite sums in the following lemma uses an old argument of F. Galvin. (See **[10]**.)

LEMMA 5.4. *Assume S is cancellative, let p be an idempotent in $\beta S \backslash S$, and let $A \in p$. There is a sequence $\langle x_n \rangle_{n<\omega}$ in S such that*

(1) *$FS(\langle x_n \rangle_{n<\omega}) \subseteq A$ and*

(2) *if F and G are finite non-empty subsets of ω and $\sum_{n \in F} x_n = \sum_{n \in G} x_n$, then $F = G$.*

PROOF. Let $A_0 = A$. Let $B_0 = \{x \in S\colon A_0 - x \in p\}$ and let $C_0 = \{x \in S\colon \{y \in S\colon y + x \neq y\} \in p\}$. Since $p + p = p$, $B_0 \in p$. By Lemma 5.3, C_0 is cofinite so $C_0 \in p$. Pick $x_0 \in A_0 \cap B_0 \cap C_0$ and let $A_1 = A_0 \cap B_0 \cap C_0 \cap (A_0 - x_0) \cap \{y \in S\colon y + x_0 \neq y\}$. Inductively let $m \geqq 1$ and assume we have $\langle x_n \rangle_{n<m}$ and $A_m \in p$. Let $G = FS(\langle x_n \rangle_{n<m})$. Let $B_m = \{x \in S\colon A_m - x \in p\}$, let $C_m = \{x \in S\colon \{y \in S\colon y + x \neq y\} \cap \bigcap_{t \in G} \{y \in S\colon y + x + t \neq y\} \in p\}$, and let $D_m = \{x \in S\colon \text{for all } t, s \in G, x \neq s \text{ and } x + t \neq s\}$. Again $B_m \in p$ since $p + p = p$ and $C_m \in p$ by

Lemma 5.3. Trivially D_m is cofinite and hence $D_m \in p$. Pick $x_m \in A_m \cap B_m \cap C_m \cap D_m$ and let $A_{m+1} = A_m \cap B_m \cap C_m \cap D_m \cap (A_m - x_m) \cap \{y \in S: y + x_m \neq y\} \cap \bigcap_{t \in G} \{y \in S: y + x_m + t \neq y\}$.

We first repeat Galvin's argument showing by induction on $|F|$ that if F is a finite non-empty subset of ω and $m = \min F$, then $\sum_{n \in F} x_n \in A_m$ (and hence $FS(\langle x_n \rangle_{n<\omega}) \subseteq A$). If $|F| = 1$, this result is trivial so assume $|F| > 1$ and let $r = \min(F \backslash \{m\})$. Then $\sum_{n \in F \backslash \{m\}} x_n \in A_r \subseteq A_{m+1} \subseteq A_m - x_m$. Thus $\sum_{n \in F \backslash \{m\}} x_n + x_m \in A_m$, that is $\sum_{n \in F} x_n \in A_m$ as required.

Now we show by induction on m that if F and G are distinct non-empty subsets of $\{0, 1, 2, \ldots, m\}$ then $\sum_{n \in F} x_n \neq \sum_{n \in G} x_n$. This is vacuously true for $m = 0$ so assume $m \geqq 1$. We may assume $\max F \geqq \max G$ and, using the induction hypothesis, that $m = \max F$.

We first consider the possibility that also $m = \max G$. We assume without loss of generality that $|G| \leqq |F|$. If $|G| \geqq 2$ and $\sum_{n \in F} x_n = \sum_{n \in G} x_n$, then we have $x_m + \sum_{n \in F \backslash \{m\}} x_n = x_m + \sum_{n \in G \backslash \{m\}} x_n$ so by left cancellation, $\sum_{n \in F \backslash \{m\}} x_n = \sum_{n \in G \backslash \{m\}} x_n$, a contradiction to our induction hypothesis. Thus we must have $G = \{m\}$ (so that, since $F \neq G$, $|F| \geqq 2$). Let $r = \max(F \backslash \{m\})$. Assume now $|F| \geqq 3$ and let $t = \sum_{n \in F \backslash \{m, r\}} x_n$. Then $t \in FS(\langle x_n \rangle_{n<r})$ and $x_m \in A_m \subseteq A_{r+1} \subseteq \{y \in S: \ y + x_r + t \neq y\}$. Thus $x_m + x_r + t \neq x_m$ but $x_m = \sum_{n \in G} x_n$ and $x_m + x_r + t = \sum_{n \in F} x_n$. Consequently we must have $|F| = 2$, that is $F = \{m, r\}$. But $x_m \in A_m \subseteq A_{r+1} \subseteq \{y \in S: y + x_r \neq y\}$ so $x_m + x_r \neq x_m$.

We must have then that $\max G < m$. Let $s = \sum_{n \in G} x_n$. Then $s \in FS(\langle x_n \rangle_{n<m})$ so, since $x_m \in D_m$, $x_m \neq s$. Thus we may assume that $|F| > 1$. Let $t = \sum_{n \in F \backslash \{m\}} x_n$. Then $t \in FS(\langle x_n \rangle_{n<m})$ so, since $x_m \in D_m$, $x_m + t \neq s$ as required.

DEFINITION 5.5. Define $I \subseteq \beta N$ by $I = \bigcap_{n<\omega} \overline{N2^n}$.

It is easy to see that I is a subsemigroup of $(\beta N, +)$. Further all idempotents of βN are in I. (Given $p \in \beta N$ such that $p + p = p$ and given $n < \omega$, there is some $j \in \{0, 1, \ldots, 2^n - 1\}$ such that $N2^n + j \in p$. But then $N2^n + j + j \in p + p = p$. Thus $j = 0$.)

THEOREM 5.6. *Let $\langle x_n \rangle_{n<\omega}$ be a sequence in S such that $\sum_{n \in F} x_n \neq \sum_{n \in G} x_n$ whenever F and G are distinct finite non-empty subsets of ω. Define ϕ: $FS(\langle x_n \rangle_{n<\omega}) \to N \subseteq \beta N$ by $\phi(\sum_{n \in F} x_n) = \sum_{n \in F} 2^n$ and extend ϕ to the rest of S arbitrarily. For each $m < \omega$ let $B_m = FS(\langle x_n \rangle_{m \leq n<\omega})$. Then the restriction of ϕ^β to $\bigcap_{m<\omega} \overline{B_m}$ is an isomorphism and a homeomorphism onto I.*

PROOF. Since $\sum_{n \in F} x_n \neq \sum_{n \in G} x_n$ for $F \neq G$, ϕ is well defined. It suffices to show that ϕ^β restricted to $\bigcap_{m<\omega} \overline{B_n}$ is a one-to-one homomorphism onto I. (Since ϕ^β is continuous, $\bigcap_{m<\omega} \overline{B_m}$ is compact, and βN is Hausdorff, the homeomorphism assertion follows.) To see that the restriction is a homomorphism, we invoke Theorem 5.1. Observe that ρ_n is continuous in βN for each $n \in N$. Let $x \in B_0$ and pick $F \subseteq \omega$ such that $x = \sum_{n\in F} x_n$. Let $m = \max F$. We claim that whenever $y \in B_{m+1}$ one has $\phi(y + x) = \phi(y) + \phi(x)$. Indeed pick G such that $y = \sum_{n\in G} x_n$ and note that $\min G > m$ so $y + x = \sum_{n\in G} x_n + \sum_{n\in F} x_n = \sum_{n\in G\cup F} x_n$. Thus $\phi(y + x) = \sum_{n\in G\cup F} 2^n = \sum_{n\in G} 2^n + \sum_{n\in F} 2^n = \phi(y) + \phi(x)$, as required.

We now show that $\phi^\beta{}_{\overline{B_0}}$ is one-to-one. Let p and q be distinct members of $\overline{B_0}$. Observe that if $C \in p$, then $\phi[C] \in \phi^\beta(p)$. (For otherwise, $N\backslash\phi[C] \in \phi^\beta(p)$ and hence there is some $D \in p$ with $\phi^\beta[\bar{D}] \subseteq \overline{N\backslash\phi[C]}$. Picking $x \in C \cap D$ one would have $\phi(x) \in N\backslash\phi[C]$, a contradiction.) Pick $C \in p\backslash q$, so that $S\backslash C \in q$. Then $\phi[C \cap B_0] \in \phi^\beta(p)$ and $\phi[B_0\backslash C] \in \phi^\beta(q)$. Since ϕ is one-to one on B_0, $\phi[C \cap B_0] \cap \phi[B_0\backslash C] = \emptyset$ and hence $\phi^\beta(p) \neq \phi^\beta(q)$.

Finally we show $\phi^\beta[\bigcap_{n<\omega} \overline{B_n}] = I$. Given $p \in \bigcap_{n<\omega} \overline{B_n}$ and $n < \omega$ we have $\phi[B_n] \in \phi^\beta(p)$ as above. Since $\phi[B_n] = N2^n$ we have $N2^n \in \phi^\beta(p)$. Thus $\phi^\beta[\bigcap_{n<\omega} \overline{B_n}] \subseteq I$. To see that $I \subseteq \phi^\beta[\bigcap_{n<\omega} \overline{B_n}]$, let $r \in I$. Then $\{\phi^{-1}[C]\colon C \in r\} \cup \{B_n\colon n < \omega\}$ has the finite intersection property. (Given $C \in r$ and $m < \omega$, pick $x \in C \cap N2^m$. Pick $F \in [\omega]^{<\omega}$ such that $x = \sum_{n\in F} 2^n$ and observe that $\min F \geqq m$. Thus $\sum_{n\in F} x_n \in \phi^{-1}[C] \cap B_m$.) Pick $p \in \beta S$ such that $\{\phi^{-1}[C]\colon C \in r\} \cup \{B_n\colon n < \omega\} \subseteq p$. Then $p \in \bigcap_{n<\omega} \overline{B_n}$. Also, given $C \in r$, since ϕ is onto N, $\phi[\phi^{-1}[C]] = C$. Since $\phi^{-1}[C] \in p$, $C \in \phi^\beta(p)$. Then $r \subseteq \phi^\beta(p)$ and, since both are ultrafilters $r = \phi^\beta(p)$.

The assertion about the number of idempotents in the following corollary generalizes an old unpublished result of van Douwen [**6**]. (He proved the same assertion in the event S is commutative.)

COROLLARY 5.7. *Assume S is cancellative and let $p \in \beta S\backslash S$ such that $p + p = p$. Then each neighborhood of p contains a topological and algebraic copy of I. In particular, each neighborhood of p contains 2^c pairwise disjoint copies of the free group on 2^c generators (and hence contains 2^c idempotents).*

PROOF. Let $A \in p$. By Lemma 5.4 we may pick a sequence $\langle x_n\rangle_{n<\omega}$ in S such that $FS(\langle x_n\rangle_{n<\omega}) \subseteq A$ and $\sum_{n\in F} x_n \neq \sum_{n\in G} x_n$ whenever F and G are distinct finite non-empty subsets of ω. By Theorem 5.6 there is a topological and algebraic copy of I contained in $\overline{FS(\langle x_n\rangle_{n<\omega})}$ (and hence contained in $\bar{A}$).

To see the "in particular" assertions let $J = \bigcap_{1\leq n<\omega} \overline{Nn}$ (so that $J \subseteq I$). It is a result of Chou [**4**] that βN has 2^c minimal right ideals. Since each of these has an idempotent (in fact each has 2^c idempotents) and idempotents are in J, one has that J (and hence I) contains 2^c idempotents in the minimal ideal of βN. By [**12**, Theorem 3.9], if p is an idempotent in the minimal ideal of βN (so that $p + \beta N + p$ is a group [**3**, Theorem

II.2.2]), then $(p + \beta N + p) \cap J$ contains a copy of the free group on 2^c generators (which necessarily has p as its identity).

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