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EXTREMA AND NOWHERE DIFFERENTIABLE FUNCTIONS

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ABSTRACT. We give simple, unified constructions of continuous, nowhere differentiable functions that (1) have no proper local maxima or proper local minima, (2) have no proper local maxima but have proper local minima at every point of a dense set, and (3) have proper local maxima at every point of a dense set, and proper local minima at every point of another dense set.

1. Introduction. A function f: $\mathbf{R} \to \mathbf{R}$ is said to have a proper (or strict) local maximum at p in $\mathbf{R}$ provided there exists $\varepsilon > 0$ such that if $0 < |x - p| < \varepsilon$, then $f(x) < f(p)$. A function f: $\mathbf{R} \to \mathbf{R}$ is said to have a local maximum at $p \in \mathbf{R}$ provided there exists $\varepsilon > 0$ such that if $|x - p| < \varepsilon$, then $f(x) \leqq f(p)$. The terms proper local minimum and local minimum are defined in the obvious way. The main purpose of this paper is to give simple, unified constructions of the following examples.

EXAMPLES 1.1. There exist continuous, nowhere differentiable, real valued functions f, g, and h of a real variable such that

A. The function f has no proper local maxima and no proper local minima, and, furthermore $f^{-1}(y)$ is a perfect subset of $\mathbf{R}$ for every $y \in \mathbf{R}$.

B. The function g has no proper local maxima, but has proper local minima at every point of a dense subset of $\mathbf{R}$.

C. The function h has proper local maxima at every point of a dense subset of $\mathbf{R}$ and has proper local minima at every point of another dense subset of $\mathbf{R}$.

We also mention the following result which concerns *all* local extrema of continuous, nowhere monotone functions. In particular, it applies to the function f of Example A.

THEOREM 1.2. *If f: $\mathbf{R} \to \mathbf{R}$ is a continous function which is not monotone over any interval, then the set of points where f has a local maximum and the set of points where f has a local minimum are both dense in* $\mathbf{R}$. *Further, both of these sets are sets of first category.*

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The dense sets mentioned in Theorem 1.2 can be uncountable. This is easy to see using the function f of Example 1.1(A), which obviously is not monotone over any interval. It is easy to check that $f^{-1}(0)$ is the set of absolute minima for f, and $f^{-1}(1)$ is the set of absolute maxima for f. Since these sets are non-empty perfect sets, they both have the cardinality of the continuum.

The examples are constructed using standard techniques. In each example, we construct a Cauchy sequence (f_n) of "sawtooth" functions in the space of continuous functions, and the limit of this sequence is the desired example. We believe that our constructions, taking into account the definition of the functions and the verfications of the properties, are especially simple because, for every n, the construction of f_{n+1} from f_n is repetitive and easy to visualize. In addition, the three examples are unified in the sense that given one of the constructions, any of the others can be obtained from it by simple and obvious modifications. Thus, the effort needed to construct and verify these three examples is reduced. The constructions are given in §2, 3 and 4, and the proof of Theorem 1.2 is given in §5. We conclude this section with a discussion of some known results which are related to the examples in 1.1.

EXAMPLE A. It is obvious that functions like f which have $f^{-1}(y)$ perfect for all y in **R** have no proper local extrema. Continuous functions which have $f^{-1}(y)$ perfect for all y in **R** have been constructed. The first was by J. Gilles [**4**] (also see [**2**, Remark 4.4] for a discussion of an error in the paper by Gilles). Another construction of such a function is attributed to J. Foran in [**1**, p. 223]. Our function f has no finite or infinite derivative at any point.

EXAMPLE B. As far as we know, our construction in §3 is the only explicit construction of a function having the properties of Example B. Further, it can be shown that the standard Baire category argument cannot be used to prove that such functions exist because the set of all functions having the property of Example B is a set of first category in the space of continuous functions. Thus, it seems that Example B is new. In [**5**], we constructed a continuous function having proper local maxima on a dense set, but this is not as strong as Example B.

EXAMPLE C. The existence of continuous functions which have proper local extrema on a dense set was proved by A.M. Bruckner and K.M. Garg [**2**] using the Baire category technique. An explicit construction of a function having the properties of Example C was given by F.S. Cater [**3**]. His construction, like that of Gilles and ours in [**5**], uses infinite series. The function h of Example C, like f and g, is the limit of a sequence of functions which are numerically explicit and easy to visualize.

2. Construction of the function f of Example 1.1. Let $C(\mathbf{R})$ denote the set of all bounded, continuous functions $F: \mathbf{R} \to \mathbf{R}$, and define the sup norm on $C(\mathbf{R})$ in the usual way:

$$\|F\| = \sup \{|F(x)|: x \in \mathbf{R}\}.$$

In all three examples we will start our sequence, with f_0 defined as follows: f_0 is the unique continuous, piecewise linear function $f_0: \mathbf{R} \to \mathbf{R}$ such that for every integer n,

$$f_0(n) = \begin{cases} 1 \text{ if } n \text{ is even} \\ 0 \text{ if } n \text{ is odd} \end{cases}$$

and which is linear over $[n, n + 1]$ for every integer n.

We define for each $n \geqq 0$ the discrete, countable sets $P_n = \{q(2^{-3n}): q \text{ is an integer}\}$. Thus P_0 is the set of integers, P_1 is the set of fractions of the form $(q/8)$, and so on.

Notation. If p, p' are consecutive in P_n, we denote the nine consecutive elements $\{p + k \cdot 2^{-3n-3}: 0 \leqq k \leqq 8\}$ in P_{n+1} by $p, p_1, p_2, \ldots, p_6, p_7, p'$. We denote the set of non-negative integers by ω.

To finish the construction of the sequence $\{f_n\}$ we need to describe how to construct f_{n+1} from f_n. We do this by looking at each $p \in P_n$ and defining f_{n+1} over the interval $[p, p']$, where p' is the immediate successor of p in P_n. There are two cases depending on whether f_n is decreasing over $[p, p']$ or increasing over $[p, p']$. The instructions, however, read the same in each case (see figures 1 and 2).

2.1. If f_n is defined on $[p, p']$, where $p \in P_n$ and $p, p_1, p_2, \ldots p_7, p'$ are consecutive in P_{n+1}, then define f_{n+1} to be the unique continuous, piecewise linear function which is linear over $[p, p_1]$, $[p_1, p_3]$, $[p_3, p_5]$, $[p_5, p_7]$, and $[p_7, p']$ and such that

a. $f_{n+1}(p) = f_n(p)$ and $f_{n+1}(p') = f_n(p')$
b. $f_{n+1}(p_1) = f_n(p_4)$
c. $f_{n+1}(p_2) = f_n(p_2)$
d. $f_{n+1}(p_3) = f_n(p)$
e. $f_{n+1}(p_4) = f_n(p_4)$
f. $f_{n+1}(p_5) = f_n(p')$
g. $f_{n+1}(p_6) = f_n(p_6)$
h. $f_{n+1}(p_7) = f_n(p_4)$

this by elementary considerations in Euclidean Geometry. To prove (5) let $j < n + 1$. By the induction hypothesis we are given that the maxi-

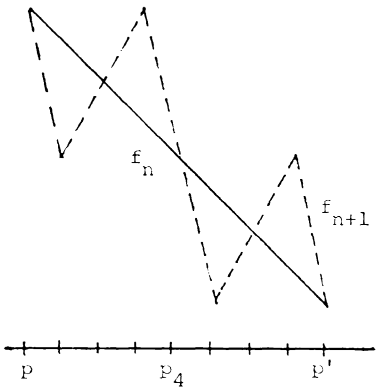

FIGURE 1.

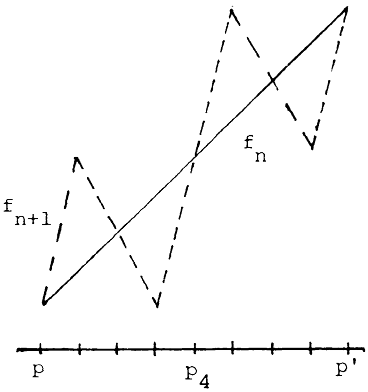

FIGURE 2.

We now state the conditions on the sequence $\{f_n\}$ that we need in order to prove part A of Example 1.1.

We proceed by induction, and start with f_0 and P_0 as noted above. Assume we have constructed f_i and P_i for $0 \leqq i \leqq n$ such that the following hold

2.2. Induction hypotheses.

(1) f_i is a continuous, piecewise linear function which is linear over $[p, p']$ for each $p \in P_n$ (where p' is the immediate successor of p in P_n).

(2) $\|f_{i-1} - f_i\| < 1/2^i$.

(3) If $p \in P_{i-1}$, then $f_i(p) = f_{i-1}(p)$.

(4) Let p, p' be consecutive in P_{i-1}. Then f_i is defined from f_{i-1} over $[p, p']$ using the instructions of 2.1.

(5) If p, p' are consecutive in P_j when $j < i$, then

$\max\{f_j(x): p \leqq x \leqq p'\} = \max\{f_i(x): p \leqq x \leqq p'\}$ and

$\min\{f_j(x): p \leqq x \leqq p'\} = \min\{f_i(x): p \leqq x \leqq p'\}$.

(6) If p, p' are consecutive in P_{i-1} and s is the slope of the line segment which is the graph of f_{i-1} over $[p, p']$, then the slopes of f_i over $[p, p_1]$, $[p_3, p_5]$, and $[p_7, p']$ all equal $\pm 4s$ and the slopes of f_i over $[p_1, p_3]$ and $[p_5, p_7]$ both equal $\pm\ 2s$.

The inductive step. Condition (4) of 2.2 tells how to construct f_{n+1} from f_n; so we need to show that the other conditions in 2.2 hold for $n + 1$.

Conditions (1) and (3) are obvious. To see that conditions (2) and (6) hold, let p, p' be consecutive in P_n. Consider the graph of f_n over $[p, p']$ to be the diagonal of a rectangle with vertices $(p, f_n(p))$, $(p, f_n(p'))$, $(p', f_n(p))$, and $(p', f_n(p'))$. In going from f_n to f_{n+1} we partition the interval $[p, p']$ into eight intervals of equal length. Thus the diagonal is also partitioned into eight segments of equal length. Conditions (2) and (6) follow from mum of f_j over $[p, p']$ equals the maximum of f_n over $[p, p']$. Let $\{s_k: 0 \leqq k \leqq m\}$ be the sequence of all consecutive points in P_n such that $p = s_0 < s_1 < \cdots < s_{m-1} < s_m = p'$. By the way f_{n+1} was constructed from f_n over each interval $[s_k, s_{k+1}]$ $(0 \leqq k \leqq m)$, we see that the maximum value of f_{n+1} over $[s_k, s_{k+1}]$ equals the maximum value of f_n over $[s_k, s_{k+1}]$ (which is either $f_n(s_k)$ or $f_n(s_{k+1})$ since f_n is linear over $[s_k, s_{k+1}]$). It follows that the maximum value of f_{n+1} over $[p, p']$ equals the maximum value of f_n (or f_j) over $[p, p']$. A similar argument works for minimums. This completes the induction and the construction of f_n for all $n \geqq 0$.

As discussed above we take $f = \lim_{n\to\infty} f_n$. Then f is continuous, and for all $x \in \mathbf{R}$ we have $f(x) = \lim_{n\to\infty} f_n(x)$.

2.3 Claim. *For every $y \in \mathbf{R}$, the set $f^{-1}(y)$ is a perfect (possibly empty) subset of* $\mathbf{R}$.

Proof. Let $y \in \mathbf{R}$, $x \in f^{-1}(y)$, and $\varepsilon > 0$. We need to show that there exists $z \in \mathbf{R}$ such that $0 < |x - z| < \varepsilon$ and $f(z) = f(x) = y$. Choose n so large that if p, p' are consecutive in P_n, then $p' - p < \varepsilon$. Choose $p, p' \in P_n$ such that $p \leqq x < p'$. If $x \in P$ then also choose n so large that $x \in P_n$ (i.e., $x = p$), and take $z = p_3$. Hence, we assume $x \notin P$. Since f_n is linear over $[p, p']$, one of $f_n(p), f_n(p')$ is the maximum value of f_n over $[p, p']$ and the other is the minimum such value. By 2.2 (5) it follow that $f(x)$ lies in the closed interval I which has end points $f(p) = f_n(p)$ and $f(p') = f_n(p')$. Let q, q' be consecutive in $\{p, p_1, \ldots, p_7, p'\}$ such that $q < x < q'$. Note that for every $y \in I$, the piecewise linear function f_{n+1} crosses level y at least two times (usually three times); so there exists $a \in [p, p']$ such that $a \neq x$ and $f_{n+1}(a) = y = f(x)$, since f_{n+1} is linear (hence one-one) over $[q, q']$, $a \notin [q, q']$. Let r, r' be consecutive in $\{p, p_1, \ldots p_7, p'\}$ such that $a \in [r, r']$. Since f_{n+1} is linear over $[r, r']$, we see that $b = f(a)$ lies in the closed interval with end points $f(r) = f_{n+1}(r)$ and $f(r') = f_{n+1}(r')$. By the intermediate value theorem, there exists $z \in [r, r']$ such that $f(z) = y$. Since $[r, r'] \cap (q, q') = \varnothing$, we see that $z \neq x$. Thus, x is not isolated in $f^{-1}(y)$.

2.4 Claim. *The function f is nowhere differentiable.*

Proof. Let $x \in \mathbf{R}$. By 2.2 (6) there exists for all $i \in \omega$ consecutive a_i, b_i in P_i such that $a_i \leqq x \leqq b_i$ and

$$|(f_i(a_i) - f_i(b_i))(a_i - b_i)^{-1}| \geqq i$$

which implies by 2.2(3) that

$$|(f(a_i) - f(b_i))(a_i - b_i)^{-1}| \geqq i.$$

Thus f does not have a derivative at x.

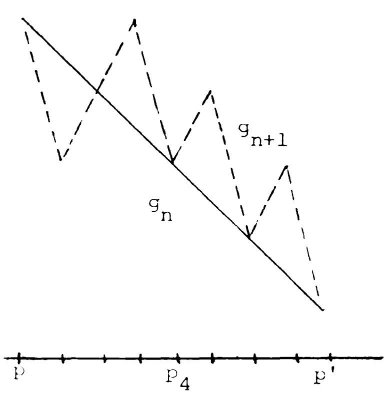

FIGURE 3.

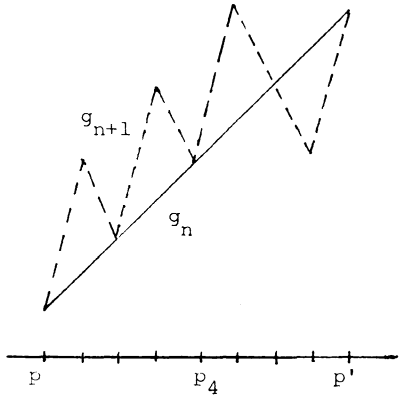

FIGURE 4.

2.5 CLAIM. *The function f has no finite or infinite derivative at any point.*

PROOF. This follows at once from 2.4 (which shows that the difference-quotients have $\pm\infty$ as a cluster point) and 2.3 (which shows that the difference-quotients have 0 as a cluster point).

It can be shown that f has infinite one-sided derivatives at certain points $x \notin P$.

3. Construction of the function g in Example 1.1(B). The function g is similar to f except that we want g to have proper local minima on a dense set. Loosely speaking, it is conditions 2.1(f) in the decreasing case, and 2.1(d) in the increasing case which kill off proper local minima for the function f. Thus, we change those conditions to read as follows (for notational convenience we denote this new version of f_n by g_n).

3.1. (2.1(f) (decreasing case) revised for g). $g_i(p_5) = g_{i-1}(p_2)$ (see Fig. 3).

3.2. (2.1(d) (increasing case) revised for g). $g_i(p_3) = g_{i-1}(p_6)$ (see Fig. 4).

We leave it to the reader to make the appropriate changes in the instructions 2.1 so that g_{n+1} is linear over the correct intervals (see figures 3 and 4).

Let $g = \lim_{n\to\infty} g_n$. Thus g is a continuous nowhere differentiable function, and g has a proper local minimum at each point of a dense set D. This last claim follows easily from the following lemma which was suggested by the referee.

LEMMA 3.3. *Let p, q be consecutive in P_n such that $g_n(p) < g_n(q)$, then for every $k \geqq 0$ the following statement holds*

$S(k)$: For every x in the closed interval having end points p, q,

$$g_{n+k}(x) - g_n(p) \geqq (1/2)\,|x - p|.$$

PROOF. (By induction on k). If s is the slope of a line segment which is part of the graph of g_n, then $|s| \geqq 1 \geqq 1/2$; so $S(0)$ holds. We assume $S(k)$ and prove $S(k+1)$. Let x be in the closed interval with end points p, q. Pick r, s consecutive in P_{n+k} such that x is in the closed interval with end points r, s, and such that $g_{n+k}(r) < g_{n+k}(s)$. By the way $g_{(n+k)+1}$ was constructed from $g_{(n+k)}$ (see figures 3 and 4) it is clear that

$$g_{n+k+1}(x) - g_{n+k}(r) \geqq (1/2)\,|x - r|.$$

Since r is in the closed interval with end points p, q, $S(k)$ implies

$$g_{n+k}(r) - g_n\,(p) \geqq (1/2)\,|r - p|.$$

By adding these two inequalities we get

$$g_{n+k+1}(x) - g_n\,(p) \geqq (1/2)\,|x - p|$$

which completes the proof.

The set D where g has a proper local minimum is given by $D = \bigcup\{D_n : n \in \omega\}$, where $D_n = \{p \in P_n : g_n$ has a proper local minimum at $p\}$. This can be seen as follows. If $p \in D_n$, then, by 3.3, $g_{n+k}(x) - g_n(p) \geqq 1/2\ |x - p|$ for $k \geqq 0$ whenever $x \in [p - 2^{-3n}, p + 2^{-3n}]$. Since $g_n(p) = g(p)$, letting $k \to \infty$ yields $g(x) - g(p) \geqq 1/2\ |x - p|$. The proof that g has no proper local maxima uses ideas similar to those in the proof of 2.3.

4. Construction of the function h in Example 1.1(C). We indicated in §3 how to avoid killing off proper local minima. For the function h, we want to avoid killing off both proper local minima and proper local maxima; so we need to change the definition of $g_n(p_3)$ (decreasing case) and $g_n(p_5)$ (increasing case). For notational convenience we denote these new functions by h_n.

4.1. (2.1(d) (decreasing case) revised for h). $h_{n+1}(p_3) = h_n(p_6)$.

4.2. (2.1(f) (increasing case) revised for h). $h_{n+1}(p_5) = h_n(p_2)$.

As was the case for f, the instructions for h read the same in both the increasing and decreasing cases.

An alternative suggested by the referee is to define $\hat{g}_n = 1 - g_n$, for $n \geqq 0$, and modify $\hat{g}_n$ to get $\hat{h}_{n+1}$ in the same manner that f_n was modified to get g_{n+1} (i.e., to avoid killing off local minima). Then $\hat{h} = \lim \hat{h}_n$ satisfies Example 1.1 C, and $h = 1 - \hat{h}$, where h is the function defined above.

5. Proof of Theorem 1.2. We break the proof into three very simple lemmas.

LEMMA 5.1. *If f: $\mathbf{R} \to \mathbf{R}$ is any function (not necessarily continuous) and $X = \{x : f$ has a local maximum at $x\}$ then $Y = f(X)$ is countable.*

PROOF. For each y in Y, pick x_y in X such that $f(x_y) = y$, and pick rational numbers a_y, b_y such that (1) $a_y < x_y < b_y$ and (2) for all x in (a_y, b_y), $f(x) \leqq f(x_y) = y$. The map that assigns y to the interval (a_y, b_y) is one-to-one; so Y is countable.

LEMMA 5.2. *If f: $\mathbf{R} \to \mathbf{R}$ is continuous and nowhere constant, then* $X = \{x: f$ has a local maximum at $x\}$ *is a set of first category.*

PROOF. By the hypothesis on f, the set $f^{-1}(y)$ is nowhere dense for all y in $\mathbf{R}$. Thus, $X = \bigcup \{f^{-1}(y) \cap X: y \in f(X) = Y\}$ is a set of first category by Lemma 5.1.

LEMMA 5.3. *If f: $\mathbf{R} \to \mathbf{R}$ is continuous and nowhere monotone, then the set* $X = \{x: f$ has a local maximum at $x\}$ *is dense in* $\mathbf{R}$.

PROOF. Let (a, b) be an arbitrary open interval. Since f is not decreasing on (a, b) there exist points s and t in (a, b) such that $s < t$ and $f(s) < f(t)$. By continuity, there exists an open interval $(r, w) \subset (s, t)$ such that, for all x in (r, w), $f(x)$ is in $(f(s), f(t))$. Since f is not increasing on (r, w), there exist points u, v in (r, w) such that $u < v$ and $f(u) > f(v)$. By continuity, there exists x in $[s, v]$ such that $f(x)$ is the maximum value of f on $[s, v]$. Now both $f(s), f(v) < f(u)$ and $u \in (s, v)$, hence x is a point where f has a local maximum.

Similar arguments work for local minima, and that completes the proof of Theorem 1.2.

Lemma 5.1 improves the result stated in [**6**, p. 117] which only shows that Y has measure zero, and is essentially given in [**7**; Prob. 5, p. 33].

We wish to thank Anatole Beck for reference [**7**], Andrew M. Bruckner for reference [**4**] and other interesting references, and the referee for several helpful comments.

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