ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 16, Number 3, Summer 1986

BEZIER-CURVES WITH CURVATURE AND TORSION CONTINUITY

HANS HAGEN

ABSTRACT. One of the main problems in computer-aided design is how to input shape information to the computer. In the analytic description and approximation of arbitrary shaped curves the Bezier-curves are of great importance (see [5]). A Bezier-curve is a segmented curve. The segments $x_\ell(u) := \sum_m^{i=0} b_{m\ell+i} \cdot B_i^m(u-u_\ell / u_{\ell+1}-u_\ell)$ of a Bezier-curve of degree m over the parameter interval $u_\ell \leqq u \leqq u_{\ell+1}$ use the Bernstein-polynomials as blending functions. The coefficients $b_{m\ell+i}$ are called Bezier points. They form the so called Bezier polygon, which implies the Bezier-curve.

A.R. Forrest analyzed the Bezier techniques in [4] and extended these techniques to generalized blending functions.

W. J. Gordon and R. F. Riesenfeld provided in [5] an alternative development in which the Bezier methods emerge as an application of the Bernstein polynomial approximation operator to vector-valued functions.

As connecting conditions between the curve-segments are always chosen the so called C^2- or C^3- continuity. (A segmented curve is said to have $C^{(k)}$-continuity if an only if $X^{(k)}(t_i^+) = X^{(k)}(t_i^-)$ at the connecting points t_i; $i = 1, \ldots, n$, where $X^{(k)} := (\partial/\partial t^k)X$; $k \in N$.)

In this paper we create, after a brief survey of the fundamentals of differential geometry, a tangent, a curvature, and a torsion continuity, using the geometric invariants of a curve.

Considering $C^2-(C^3-)$ continuity, we have only one choice for $b_{m(\ell+1)+2}(b_{m(\ell+1)+3})$, $0 = \ell \leqq k$. In the third part of this paper we show that curvature continuity offers a "straight line of alternatives" and torsion continuity offers a "plane of alternatives."

We give also constructions for the Bezier polygons of Bezier curves with curvature - and torsion - continuity, which are convenient for a graphic terminal.

1. Fundamentals of differential geometry.

DEFINITION 1.1. (a) A parametrized C^r-curve is a C^r-differentiable map $X: I \to E^n$ of an open interval I of the real line R into the euclidean space E^n.

(b) A parametrized C^r-curve $X: I \to E^n$ is said to be regular if $\dot{X}(t) \neq 0$, for all $t \in I$, where $\dot{X} = \partial/\partial t X$.

Received by the editors on November 13, 1983 and in revised form on January 11, 1985.

REMARK. Let $X: I \to E^n$ and $\tilde{X}: \tilde{I} \to E^n$ be two curves. A diffeomorphism $\phi: \tilde{I} \to I$ such that $\tilde{x} = x \circ \phi$ is called a parameter transformation. The map ϕ is called orientation preserving if $\phi' > 0$. Relationship by a parameter transformation is an equivalence relation on the set of all parametrized curves in E^n. A C^r-curve is an equivalence class of parametrized C^r-curves.

DEFINITION 1.2. (a) Let $X: I \to E^n$ be a C^r-curve. A moving frame along $X(I)$ is a collection of vector fields,

$$e_i: I \to E^n, \ 1 \leqq i \leqq n,$$

such that, for all $t \in I$, $\langle e_i, e_j \rangle = \delta_{ij}$.

(b) A moving frame is called a Frenet-frame, if, for all k, $1 \leqq k \leqq n$, the k-th derivative $X^{(k)}(t)$ of $X(t)$ lies in the span of the vectors $e_1(t), \ldots, e_k(t)$.

PROPOSITION 1.3. *Let $X: I \to E$ be a curve such that, for all $t \in I$, the vectors $X^{(1)}(t), X^{(2)}(t), \ldots, X^{(n-1)}(t)$ are linearly independent. Then there exists a unique Frenet-frame with the following properties*:

(i) *For $1 \leqq k \leqq n - 1$, $X^{(1)}(t), \ldots, X^{(k)}(t)$ and $e_1(t), \ldots, e_k(t)$ have the same orientation,*

(ii) *$e_1(t), \ldots, e_n(t)$ has the positive orientation.*

PROOF. See [1, p. 11].

PROPOSITION 1.4. (a) *Let $X(t)$, $t \in I$, be a curve in E^n together with a moving frame $\{e_i(t)\}$, $1 \leqq i \leqq n$, $t \in I$. Then the following equations for the derivatives hold*:

$$\dot{X}(t) = \sum_{i=1}^{n} \alpha_i(t) e_i(t),$$

$$\dot{e}_i(t) = \sum_{j=1}^{n} w_{ij}(t) e_j(t),$$

where $w_{ij}(t) := \langle \dot{e}_i(t), e_j(t) \rangle = -w_{ji}(t)$.

(b) *If $\{e_i(t)\}$ is the Frenet-frame*

$$\alpha_1(t) = \|X^{(1)}(t)\|,$$

then $\alpha_i(t) = 0$, for $i > 1$, and $w_{ij}(t) = 0$, for $j > i + 1$.

PROOF. See [1, p. 12].

DEFINITION 1.5. Let $X: I \to E^n$ be a curve satisfying the conditions of (1.3) and consider its Frenet-frame. The i-th curvature of X, $i = 1, \ldots, n - 1$, is the function

$$\kappa_i(t) := \frac{w_{i,i+1}(t)}{\|X^{(1)}(t)\|}.$$

For the Frenet-frame we may now write the Frenet-equations in the following form:

$$(1.6)\qquad \dot e_i(t) = \|\dot X\| \begin{bmatrix} 0 & \kappa_1 & 0 & \cdots & & & 0 \\ -\kappa_1 & 0 & \kappa_2 & \cdots & & & 0 \\ & \cdot & & & & & \\ \cdot & -\kappa_2 & \cdot & & & & \\ \cdot & & & \cdot & & & \cdot \\ \cdot & & & & & & \cdot \\ \cdot & & & & & & \cdot \\ \cdot & & & & & \kappa_{n-1} & \\ 0 & \cdots & & & & -\kappa_{n-1} & 0 \end{bmatrix} e_i(t).$$

REMARK. The i-th curvature of a curve $X(t)$, $i = 1, \ldots, n-1$, is a geometric invariant.

It is a fundamental result of (local) differential geometry that these curvature functions determine curves satisfying the nondegeneracy conditions of (1.3)!

THEOREM 1.7. (a) *Let $X: I \to E^n$ and $\tilde X: I \to E^n$ be two curves satifying the hypotheses of* (1.3), *insuring the existence of a unique distinguished Frenet-frame. Denote these Frenet-frames by* $\{e_i(t)\}$ *and* $\{\tilde e_i(t)\}$ *respectively,* $1 \leqq i \leqq n$. *Suppose, relative to these frames, that* $\kappa_i(t) = \tilde\kappa_i(t)$, $1 \leqq i \leqq n-1$, *and assume* $\|X^{(1)}(t)\| = \|\tilde X^{(1)}(t)\|$. *Then there exists a unique isometry* $B = E^n \to E^n$ *such that* $\tilde X = B \circ X$.

(b) *Let* $\kappa_1(s), \ldots, \kappa_{n-1}(s)$ *be differentiable functions defined on a neighborhood of* $0 \in R$ *with* $\kappa_i(s) > 0$, $1 \leqq i \leqq n-2$. *Then there exists an interval I containing* 0 *and a curve* $X: I \to E^n$ *parametrized by arc length which satisfies the conditions* (1.3) *and whose i-th curvature function is* $\kappa_i(s)$, $1 \leqq i \leqq n-1$.

PROOF. See [**1**, p. 14–15].

If we investigate regular plane curves and regular space curves, we will always choose the Frenet-frame as the moving frame on our curve. The Frenet equations for a plane curve are

$$(1.8)\qquad \begin{aligned} e_1(t) &:= \frac{X^{(1)}(t)}{\|X^{(1)}(t)\|}, \\ \dot e_1(t) &= w_{12} e_2(t), \\ \dot e_2(t) &= -w_{12} e_1(t). \end{aligned}$$

There is only one curvature: $\kappa(t) := (w_{12}(t))/\|\dot X(t)\|$. The curvature of a planar curve is given by the formula

$$\kappa(t) = \frac{\det(\dot{X}(t), \ddot{X}(t))}{\|\dot{X}(t)\|^3}. \tag{1.9}$$

The Frenet equations for a space curve are

$$\dot{e}_i(t) = \|\dot{X}(t)\| \begin{bmatrix} 0 & \kappa_1(t) & 0 \\ -\kappa_1(t) & 0 & \kappa_2(t) \\ 0 & -\kappa_2(t) & 0 \end{bmatrix} e_i(t), \quad i = 1, 2, 3, \tag{1.10}$$
$$e_1(t) := \frac{\dot{X}(t)}{\|\dot{X}(t)\|},$$

and the curvatures $\kappa_1(t)$ and $\kappa_2(t)$ will be denoted $\kappa(t)$ and $\tau(t)$ and called the "curvature" and "torsion" of the curve. The curvature of a space curve is given by the formula

$$\kappa(t) = \frac{\|[\dot{X}(t), \ddot{X}(t)]\|}{\|\dot{X}(t)\|^3}, \tag{1.11}$$

where $[\,,\,]: E^3 \times E^3 \to E^3$ is the cross product, and the torsion of a space curve

$$\tau(t) = \frac{\det(\dot{X}(t), \ddot{X}(t), \dddot{X}(t))}{\|[\dot{X}, \ddot{X}]\|^2}. \tag{1.12}$$

Here e_1 is called tangent vector, e_2 principal normal vector, and e_3 binormal vector.

PROPOSITION 1.13. *A space curve is planar if and only if its torsion vanishes identically.*

PROOF. See [2, p. 40].

2. Tangent, curvature, and torsion continuity for curves. We now create "geometric continuities" using the geometric invariants described in Chapter 1.

DEFINITION 2.1, Let $X: I \to E^3$ be a curve such that, for all $t \in I$, the vectors $X^{(1)}(t)$, $X^{(2)}(t)$ are linearly independent.

(i) This curve is said to have tangent continuity if and only if $(\dot{X}/\|\dot{X}\|)(t)$ is continuous.

(ii) This curve is said to have curvature continuity if and only if $(\dot{X}/\|\dot{X}\|(t)$ and $\kappa(t)$ are continuous.

(iii) This curve is said to have torsion continuity if and only if $(\dot{X}/\|\dot{X}\|)(t)$ and $\kappa(t)$ and $\tau(t)$ are continuous.

REMARKS. 1) Since a space curve is planar if and only if its torsion vanishes identically, it is sufficient to consider tangent and curvature continuity for a planar curve.

2) A segmented curve is said to have $C^{(k)}$-continuity if and only if $X^{(k)}(t_i^+) = X^{(k)}(t_i^-)$ at the connecting points t_i, $i = 1, \ldots, n$.

3) Curvature continuity includes the "natural spline condition" for cubic splines given by W. Boehm in [3] as a special case.

4) C^2-continuity implies curvature continuity and C^3-continuity implies torsion continuity, but converses generally are not true. But if we choose the parametrization per arc length, curvature continuity implies C^2-continuity and torsion continuity implies C^3-continuity.

5) Curvature continuity implies the "second-degree geometric continuity" of Barsky and Beatty. They consider in [1] a "curvature vector" $K(t)$, which has the property

$$K(t) = \kappa \cdot e_2$$

If we have continuous curvature κ the Frenet-equations imply a continuous principal normal vector e_2 and therefore we have a continuous curvature vector.

Considering segmented curves we can use the tangent, the curvature and the torsion continuity to establish connection conditions. Let X_l: $[u_{l-1}, u_l] \to E^3$; $l = 1, \ldots, k$ be the curve segments, with $X_{l-1}(u_l) = X_l(u_l)$.

For the tangent continuity it is sufficient that

$$\dot{X}_{l-1}(u_l) = \dot{X}_l(u_l), \quad l = 2, \ldots, k, \tag{2.2.i}$$

at every node u_l.

For the curvature continuity it is sufficient that

$$\ddot{X}_{l-1}(u_l) + \lambda_{l-1}\dot{X}_{l-1}(u_l) = \ddot{X}_l(u_l) \tag{2.2.ii}$$

and

$$\dot{X}_{l-1}(u_l) = \dot{X}_l(u_l), \quad l = 2, \ldots, k,$$

at every node u_l.

For the torsion continuity it is sufficient that

$$\dddot{X}_{l-1}(u_l) + \mu_{l-1}\ddot{X}_{l-1} + \delta_{l-1}\dot{X}_{l-1} = \dddot{X}_l(u_l), \tag{2.2.iii}$$

$$\ddot{X}_{l-1}(u_l) + \lambda_{l-1}\dot{X}_{l-1} = \ddot{X}(u_l), \quad l = 2, \ldots, k,$$

and

$$\dot{X}_{l-1}(u_l) = \dot{X}_l(u_l)$$

at every node u_l.

3. Bezier-Curves with geometric continuity. In the analytic description and approximation of arbitrary shaped curves the Bezier-curves (see [4]) are of great importance.

DEFINITION 3.1. A Bezier-curve is a segmented curve. The segments

$x_\ell(u)$, $\ell = 0, \ldots, k$ of a Bezier-curve of degree m over the parameter interval $u_\ell \leqq u \leqq u_{\ell+1}$ are

$$x_\ell(u) := \sum_{i=0}^{m} b_{\ell m+i} \cdot B_i^m\left(\frac{u - u_\ell}{u_{\ell+1} - u_\ell}\right).$$

The Bernstein polynomials

$$B_i^m(t) := \binom{m}{i}(1 - t)^{m-i}\, t^i, \quad 0 \leqq t \leqq 1,$$

are used as blending functions.

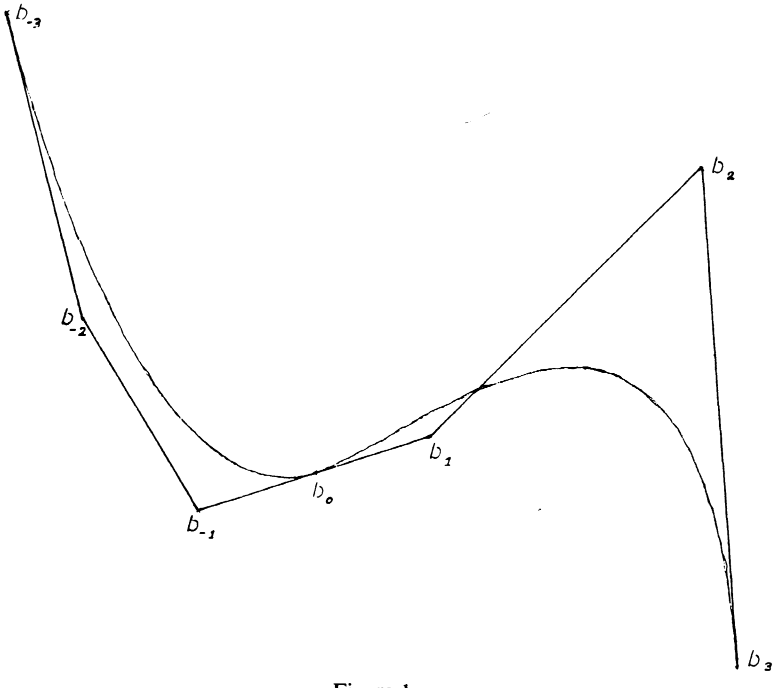


Figure 1.

REMARKS. 1) Let $\lambda_\ell := u_{\ell+1} - u_\ell$, $\ell = 0, \ldots, k$, be the length of the parameter interval belonging to the segment $x_\ell(u)$.

2) The coefficients $b_{m\ell+i}$ are called Beizer points. They form the so called Bezier polygon.

3) The edges $\overline{b_{m\ell} b_{m\ell+1}}$ and $\overline{b_{m(\ell+1)-1}\, b_{m(\ell+1)}}$ are tangents at the boundary points $b_{m\ell}$ and $b_{m(\ell+1)}$ of the segment $x_\ell(u)$. These boundary points are (in general) the only Bezier points the Bezier curve passes through.

4) Bezier-curves have the convex-hull and the variation diminishing property (see [3]).

5) As connection conditions, are usually chosen the C^1- and C^2-continuity.

Using curvature- and torsion-continuity offers more possibilities. Considering the two Bezier-segments

$$x_\ell(u) = \sum_{i=0}^{m} b_{m\ell+i} \cdot B_i^m\left(\frac{u-u_\ell}{u_{\ell+1}-u_\ell}\right), \quad u_\ell \leqq u \leqq u_{\ell+1},$$

and

$$x_{\ell+1}(u) = \sum_{j=0}^{m} b_{m(\ell+1)+j} B_j^m\left(\frac{u-u_{\ell+1}}{u_{\ell+2}-u_{\ell+1}}\right), \quad u_{\ell+1} \leqq u \leqq u_{\ell+2},$$

we get, as derivatives at the common nodes:

$$x_\ell^{(1)}(u_{\ell+1}) = \frac{m}{\lambda_\ell}(b_{m(\ell+1)} - b_{m(\ell+1)-1}),$$

$$x_\ell^{(2)}(u_{\ell+1}) = \frac{m(m-1)}{\lambda_\ell^2}(b_{m(\ell+1)} - 2b_{m(\ell+1)-1} + b_{m(\ell+1)-2}),$$

$$x_\ell^{(3)}(u_{\ell+1}) = \frac{m(m-1)(m-2)}{\lambda_\ell^3}(b_{m(\ell+1)} - 3b_{m(\ell+1)-1} + 3b_{m(\ell+1)-2} - b_{m(\ell+1)-3});$$

and

$$x_{\ell+1}^{(1)}(u_{\ell+1}) = \frac{m}{\lambda_{\ell+1}}(b_{m(\ell+1)+1} - b_{m(\ell+1)}),$$

$$x_{\ell+1}^{(2)}(u_{\ell+1}) = \frac{m(m-1)}{\lambda_{\ell+1}^2}(b_{m(\ell+1)+2} - 2b_{m(\ell+1)+1} + b_{m(\ell+1)}),$$

$$x_{\ell+1}^{(3)}(u_{\ell+1}) = \frac{m(m-1)(m-2)}{\lambda_{\ell+1}^3}(b_{m(\ell+1)+3} - 3b_{m(\ell+1)+2} + 3b_{m(\ell+1)+1} - b_{m(\ell+1)}).$$

Therefore, a Bezier curve has tangent-continuity if

$$b_{m(\ell+1)} - b_{m(\ell+1)-1} = b_{m(\ell+1)+1} - b_{m(\ell+1)}, \tag{3.1.1}$$

curvature-continuity if

$$\begin{aligned} &\| [(b_{m(\ell+1)} - b_{m(\ell+1)-1}), (b_{m(\ell+1)-2} - b_{m(\ell+1)-1})] \| \\ &= \| [(b_{m(\ell+1)} - b_{m(\ell+1)-1}), (b_{m(\ell+1)+2} - b_{m(\ell+1)})] \| \end{aligned} \tag{3.1.2}$$

and

$$b_{m(\ell+1)} - b_{m(\ell+1)-1} = b_{m(\ell+1)+1} - b_{m(\ell+1)},$$

and torsion-continuity if

$$(3.1.3)\quad \begin{aligned}&\langle [(b_{m(\ell+1)} - b_{m(\ell+1)-1}), (b_{m(\ell+1)-2} - b_{m(\ell+1)-1}], (b_{m(\ell+1)-2} - b_{m(\ell+1)-3})\rangle \\ &= \langle [(b_{m(\ell+1)} - b_{m(\ell+1)-1}), (b_{m(\ell+1)-2} - b_{m(\ell+1)-1})], (b_{m(\ell+1)+3} - b_{m(\ell+1)})\rangle\end{aligned}$$

and

$$\begin{aligned}&\| [(b_{m(\ell+1)} - b_{m(\ell+1)-1}), (b_{m(\ell+1)-2} - b_{m(\ell+1)-1})] \| \\ &= \| [(b_{m(\ell+1)} - b_{m(\ell+1)-1}), (b_{m(\ell+1)+2} - b_{m(\ell+1)})] \|\end{aligned}$$

and

$$b_{m(\ell+1)} - b_{m(\ell+1)-1} = b_{m(\ell+1)+1} - b_{m(\ell+1)}.$$

THEOREM 3.2. *Let* $X: I \to E^3$ *be a Bezier curve,*

$$I = [u_0, \ldots, u_k],$$
$$X_\ell(u) = \sum_{i=0}^{m} b_{m\ell+i} \cdot B_i^m\left(\frac{u - u_\ell}{u_{\ell+1} - u_\ell}\right)$$
$$\ell = 0, \ldots, k \text{ and } u_\ell \leqq u \leqq u_{\ell+1}.$$

(a) *A Bezier curve has tangent-continuity if*

$$(3.2.1)\quad b_{m(\ell+1)+1} = 2b_{m(\ell+1)} - b_{m(\ell+1)-1}$$

(b) *A Bezier curve has curvature-continuity if*

$$(3.2.2)\quad \begin{aligned}b_{m(\ell+1)+2} &= C_{\ell 0} \cdot (b_{m(\ell+1)} - b_{m(\ell+1)-1}) + b_{m(\ell+1)-2},\\ b_{m(\ell+1)+1} &= 2b_{m(\ell+1)} - b_{m(\ell+1)-1}.\end{aligned}$$

(c) *A Bezier curve has torsion-continuity if*

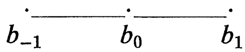
$$(3.2.3)\quad \begin{aligned}b_{m(\ell+1)+3} &= C_{\ell 1}(b_{m(\ell+1)} - b_{m(\ell+1)-1}) + C_{\ell 2}(b_{m(\ell+1)-2} - b_{m(\ell+1)-1}),\\ b_{m(\ell+1)+2} &= C_{\ell 0}(b_{m(\ell+1)} - b_{m(\ell+1)-1}) + b_{m(\ell+1)-2},\\ b_{m(\ell+1)+1} &= 2b_{m(\ell+1)} - b_{m(\ell+1)-1}.\end{aligned}$$

REMARKS. 1) Since $X: I \to E^3$ is planar if and only if its torsion vanishes identically, we consider only tangent- and curvature-continuity for planar curves.

2) Considering Bezier curves with C^2- or C^3-continuity we have only one choice for $b_{m(\ell+1)+2}$ and $b_{m(\ell+1)+3}$. In the case of curvature-continuity we have a one parameter family of alternatives and in the case of torsion-continuity we have a two parameter family of alternatives!

Theorem (3.2) implies easy constructions for the Bezier-polygons of Bezier curves with tangent-, curvature-, and torsion-continuity.

(3.3.1) Bezier-polygon-construction for tangent-continuity:



$\overline{b_{-1}b_0} = \overline{b_0b_1}$

(3.3.2) Bezier-polygon-construction for curvature-continuity:

1) $\overline{b_{-1}b_0} = \overline{b_0b_1}$

2) $\overline{b_{-2}b_{-1}} = \overline{b_{-1}d_0}$

$\overline{d_0b_1} = \overline{b_1\tilde{b}_2}$

3) $\overline{d_0b_0} = \overline{b_0\bar{b}_2}$

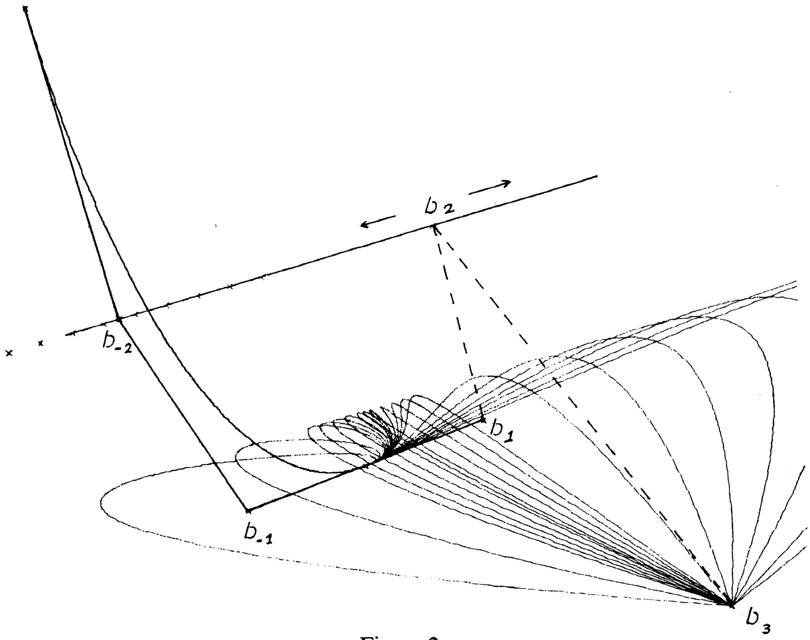


Figure 2.

Span $(\tilde{b}_2, \bar{b}_2)$ implies the one parameter family of alternatives to choose $b_{m(\ell+1)+2}$.

REMARKS. (1) The construction (3.3.2) is of course not the only one. But it is most convenient for graphic terminals, since it uses only "midpoint-constructions."

(2) Since a space curve is determined by two planar projections, the above techniques can be used to construct space curves.

REFERENCES

1. B. Barsky J. Beatty, *Local Control of Bias and Tension in Beta-splines*, Computer Graphics **17** (1983), 193–278.

2. P. Bezier, *Numerical Control, Mathematics and Applications*, Wiley, London, 1972.

3. W. Boehm, *On cubics: a survey*, Computer Graphics and Image Processing **19** (1982), 201–226.

4. R. Forrest, *Interactive interpolation and approximation by Bezier polynomials*, The Computer Journal **15** (1972), 71–79.

5. W. Gordon, R. Riesenfeld, *Bernstein-Bezier-Methods for the Computer Aided Design of Free-Form Curves and Surfaces*, Journal ACM **21** (1974), 293–310.

6. W. Klingenberg, *A Course in Differential Geometry*, New York, Springer, 1978.

7. E. Kreyszig, *Differential Geometry*, University of Toronto Press, 1959.

ABT. MATHEMATIK DER UNIVERSITÄT, POSTFACH 500500, D-46 DORTMUND 50, WEST GERMANY