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BOUNDARY VALUE PROBLEMS WITH JUMPING NONLINEARITIES

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1. Introduction. Let Ω be a bounded domain in $\mathbf{R}^n$ with sufficiently smooth boundary and consider the boundary value problem

$$(1.1) \qquad \begin{aligned} \Delta u + f(u) &= g(x), \quad x \in \Omega \\ u &= 0, \qquad x \in \partial \Omega \end{aligned}$$

where $f \in C^1(\mathbf{R})$ and g is Holder continuous on $\bar{\Omega}$. Let $\lambda_0 < \lambda_1 \leqq \lambda_2 \leqq \cdots$ denote the set of eigenvalues of the homogeneous problem

$$(1.2) \qquad \begin{aligned} \Delta u + \lambda u &= 0, \qquad x \in \Omega \\ u &= 0, \qquad x \in \partial \Omega. \end{aligned}$$

Assuming that the limits

$$\alpha = \lim_{s \to -\infty} f'(s), \qquad \beta = \lim_{s \to \infty} f'(s)$$

exist and satisfy

$$\alpha < \lambda_0 < \beta < \lambda_1,$$

Ambrosetti and Prodi [3] showed that there exists in $C^{0,\alpha}(\bar{\Omega})$ a connected C^1 manifold M separating $C^{0,\alpha}(\bar{\Omega})$ into components A_1 and A_2 such that:

(i) if $g \in A_1$, then (1.1) has no solution;

(ii) if $g \in M$, then (1.1) has a unique solution; and

(iii) if $g \in A_2$, then (1.1) has exactly two solutions.

This fundamental paper has generated much interest, and many interesting generalizations, extensions and refinements have since appeared (see, e.g., Manes/Micheletti [22], Kazdan/Warner [18], Dancer [6, 7], Amann/Hess [2], Hess [15], Fucik [10–12], Lazer/McKenna [19–21], Hofer [16], Ruf [25], and Solimini [27]).

Rewriting g as $t\theta + h$, where θ is a positive eigensolution of (1.2) corresponding to λ_0, it was Dancer [7] who showed that if $\alpha < \lambda_0 < \beta$, then there exists t_0 such that:

(i) if $t < t_0$, (1.1) has no solution; and

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(ii) if $t > t_0$, (1.1) has at least two solutions.

Further work by Lazer-McKenna **[19]** showed that if λ_2 is a simple eigenvalue and $\lambda_2 < \beta < \lambda_3$, then for sufficiently large t there are at least three solutions. They conjectured that, as β crosses higher eigenvalues, more solutions of (1.1) would appear for large t. In fact they proved this in **[21]** for the case of ordinary differential equations; more precisely, they showed that each time β crosses an eigenvalue, two more solutions of (1.1) will appear for large t. Their proof was based on shooting techniques and nodal properties of solutions of second order ordinary differential equations. In this paper we present a different proof of their result which utilizes techniques from global bifurcation theory. These arguments may also be applied if we remove the restriction that $\alpha < \lambda_0$ and we obtain results on the multiplicity of solutions for large t; in fact the multiplicity question becomes somewhat more intricate in this case and more precise restrictions on α and β yield a more precise count.

We restrict ourselves throughout to the case of ordinary differential equations precisely because the techniques used apply there. Whether such results are also valid for partial differential equations, even under the restriction that all eigenvalues of (1.1) are simple, is an open question.

We remark here that if $\lambda_i < \alpha \leqq \beta < \lambda_{i+1}$, then the equation is non-resonant and a nonlinear Fredholm-type alternative holds and (1.1) is solvable for any right hand side. Results of this type have their origin in work of Hammerstein **[13]** and were extended by Dolph **[8]** and others (for a recent survey see, e.g., **[26]**).

2. Preliminaries. We consider the following nonlinear boundary value problem

$$\begin{aligned} u'' + f(u) &= H(x), \quad -\pi < x < \pi \\ u(0) &= 0 = u(\pi), \end{aligned} \tag{2.1}$$

where f is a continuously differentiable function satisfying

$$\lim_{s\to-\infty} \frac{f(s)}{s} = \alpha, \qquad \lim_{s\to\infty} \frac{f(s)}{s} = \beta, \tag{2.2}$$

with $\alpha < \beta$. As pointed out in § 1, (2.1) is solvable for any $H \in L^2(0, \pi)$ whenever the interval $[\alpha, \beta]$ does not contain an eigenvalue of

$$\begin{aligned} u'' + \lambda u &= 0, \quad 0 < x < \pi \\ u(0) &= 0 = u(\pi), \end{aligned}$$

i.e., whenever $[\alpha, \beta]$ does not contain n^2, $n = 1, 2, \ldots$. We write $H(x)$ in the form

$$H(x) = t \sin x + h(x),$$

where $\int_0^\pi h(x)\sin x\, dx = 0$, and provide restrictions on α and β in order that (2.1) will be solvable for h fixed and t large. In addition we provide lower bounds on the number of solutions of (2.1) which will depend upon the restrictions imposed on the pair (α, β).

We first establish that (2.1) will, for $t \gg 1$, always have a positive solution as long as $\beta > 1$.

LEMMA 2.1. *Let $\beta > 1$, $\beta \neq n^2$. Then, for t sufficiently large,* (2.1) *has a solution U_t with $U_t(x) > 0$, $0 < x < \pi$.*

PROOF. Here and in what follows we let

$$E = C_0^1\,[0, \pi] = \{u : [0, \pi] \to \mathbf{R} : u(0) = 0 = u(\pi)\} \cap C^1[0, \pi]$$

with norm $\|u\| = \max_{[0,\pi]} |u(x)| + \max_{[0,\pi]} |u'(x)|$.

The problem (2.1) is equivalent to the integral equation

$$u(x) = \int_0^\pi G(x, s)\ (f(u(s)) - t \sin s - h(s))\, \mathrm{d}s, \tag{2.3}$$

where G is the Green's function determined by $d^2/(dx^2)$ and the trivial boundary conditions. We show that, for a given compact interval $[a, b]$ and $t \in [a, b]$, positive solutions of (2.2) are a priori bounded in $C\,[0, \pi]$. If the contrary were the case, there would exist a sequence $\{u_n\} \subseteq C\,[0, \pi]$ with $\max_{0 \leq x \leq \pi} u_n(x) \to \infty$ as $n \to \infty$, and a sequence $\{t_n\} \subseteq [a, b]$ such that

$$u_n = Lf(u_n) - t_n\, L\sin - Lh, \tag{2.4}$$

where the completely continous linear operator L is defined by (2.3). Thus, letting
$v_n = u_n/\|u_n\|$, we obtain

$$v_n = L(f(u_n)/\|u_n\| - \frac{1}{\|u_n\|}(t_n\, L \sin - Lh). \tag{2.5}$$

We write $f(u_n(x)) = \beta\, u_n(x) + g(u_n(x))$, where $g(s)/s \to 0$ as $s \to \infty$ and observe that $v_n = \beta\, L\, v_n + o\,(\|u_n\|)$ as $n \to \infty$.

Hence $\{v_n\}$ will have a convergent subsequence converging to, say v, which satisfies $v = \beta\, L\, v$, or equivalently

$$v'' + \beta\, v = 0, \quad v(0) = 0 = v(\pi),$$

implying, since $v\,(x) > 0$, $0 < x < \pi$, that $\beta = 1$, a contradiction.

Consider now the problem

$$v = L\beta\, v + L\frac{1}{t}g(tv) - L \sin - \frac{1}{t}\, Lh, \tag{2.6}$$

which, for $t \neq 0$, is equivalent to (2.1), and associate with it the linear problem

$$v = L\beta\, v - L \sin, \tag{2.7}$$

or equivalently

$$\begin{aligned} v'' + \beta v &= \sin x, \quad 0 < x < \pi, \\ v(0) &= 0 = v(\pi), \end{aligned} \tag{2.8}$$

which has, since $\beta \neq n^2$, the unique positive solution

$$v_\infty(x) = \frac{\sin x}{\beta - 1}.$$

Let V be a bounded open isolating neighborhood of v_∞. Then the Leray-Schauder degree

$$d(id - \beta\, L + L \sin,\, V,\, 0) = (-1)^n,$$

where $n^2 < \beta < (n+1)^2$. Hence, if t is sufficiently large, then

$$d\Big(id - \beta\, L - \frac{1}{t} Lg\,(t\,\cdot) + L \sin + \frac{1}{t} Lh,\, V, 0\Big) = (-1)^n \tag{2.9}$$

as long as V is a sufficiently small neighborhood of v_∞, satisfying that for all $v \in V$, $v(x) > 0$, $0 < x < \pi$.

Hence (2.6) will have a solution $v \in V$, and thus $U_t = t\, v$ will be a positive solution of (2.1).

We remark that no restriction was imposed by the lemma upon α.

3. The case $\boldsymbol{\alpha} < 1 < \boldsymbol{\beta}$. In this section we shall assume that

$$\alpha < 1 \leqq n^2 < \beta < (n+1)^2 \tag{3.1}$$

and shall establish that for t sufficiently large, (2.1) has at least $2n$ distinct solutions. Before establishing the result we shall present a sharpened form of part of the proof of Lemma 2.1.

LEMMA 3.1. *Let* (3.1) *hold. Then for any compact interval* $[a, b]$ *and* $t \in [a, b]$, *solutions of* (2.2) *are a priori bounded in* $C[0, \pi]$ *and hence in* E.

PROOF. Because of the proof of Lemma 2.1 we only need to consider solutions of (2.3) which assume negative values on some subinterval of $(0, \pi)$.

For given α', $\alpha < \alpha' < 1$, there exists a constant $C > 0$ such that

$$f(s) \geqq \alpha' s - C, \quad s \in \mathbf{R};$$

hence, if u is a solution of (2.1) it will satisfy the inequality

$$u'' + \alpha' u \leqq t \sin x + h(x) + C \qquad (3.2)$$
$$u(0) = 0 = u(\pi).$$

Let w be the unique solution ($\alpha' < 1$!) of

$$w'' + \alpha' w = t \sin x + h(x) + C \qquad (3.3)$$
$$w(0) = 0 = w(\pi).$$

Then, since $\alpha' < 1$, it follows that $w(x) \leqq u(x)$, $0 \leqq x \leqq \pi$. It now easily follows that, as t ranges over a compact interval, solutions of (3.3) are a priori bounded and we thus obtain a uniform lower bound for solutions of (2.1) (or (3.2)).

Now, since u assumes negative values, u is also a solution of the initial value problem

$$u'' + f(u) = t \sin x + h(x)$$
$$u(s) = \bar{u} \qquad (3.4)$$
$$u'(s) = 0,$$

where $\bar{u} = \min_{[0,\pi]} u(x) = u(s)$. Since $s \in (0, \pi)$ and $t \in [a, b]$, $\bar{u}$ is uniformly bounded. And since solutions of (3.4) depend continuously upon initial conditions and parameters (see, e.g., Hartman **[14]**), we obtain the desired a priori bound.

LEMMA 3.2. *Assume that* $\alpha < 1 < \beta$. *Then there exists* t_0 *such that, for* $t < t_0$, *problem* (2.1) *has no solutions.*

PROOF. Multiply (2.1) by $\sin x$ (recall that $H(x) = t \sin x + h(x)$ with $\int_0^\pi h(x) \sin x \, dx = 0$) and integrate between 0 and π to obtain

$$\int_0^\pi u''(x) \sin x \, dx + \int_0^\pi f(u(x)) \sin x \, dx = \frac{\pi t}{2}.$$

Hence

$$\int_0^\pi (f(u(x)) - u(x) \sin x \, dx = \frac{\pi t}{2}. \qquad (3.5)$$

On the other hand, since f satisfies (2.2) and $\alpha < 1 < \beta$, it follows that there exists C such that $f(s) - s \geqq C$, $s \in \mathbf{R}$, showing that the left hand side of (3.5) is bounded below and proving the lemma.

LEMMA 3.3. *Let* $\alpha < 1 < \beta$, $\beta \neq n^2$. *Then there exists an open set* $\mathcal{O} \subset E \times \mathbf{R}$ *with the following properties*:

(i) $\mathcal{O} \cap E \times [a, b]$ *is bounded for any compact interval* $[a, b]$;

(ii) $\mathcal{O}_t = \{(u, s) \in \mathcal{O}: s = t\} \neq \emptyset$ *for any* $t \in \mathbf{R}$;

(iii) *if* u *is a solution of* (2.1), *then* $(u, t) \in \mathcal{O}$; *and*

(iv) *the Leray-Schauder degree*

$$d(id - Lf - t\,L\,sin - Lh, \mathcal{O}_t, 0) = 0, \quad t \in \mathbf{R}.$$

Furthermore if S is the set of solutions of (2.1), *then for every $\bar{t}$ sufficiently large there exists a continuum $C \subset S$ such that: if $(u, t) \in C$, then $t \leqq \bar{t}$ and $C_{\bar{t}}$ contains $(U_{\bar{t}}, \bar{t})$ and a solution $(u, \bar{t})$ with $u \neq U_{\bar{t}}$.*

PROOF. The construction of an open set $\mathcal{O}$ with properties (i) – (iii) follows from Lemma 3.1. That (iv) holds, follows from Lemma 3.2. If $\bar{t}$ is sufficiently large, we may apply Lemma 2.1 to obtain a positive solution $U_{\bar{t}}$, and equation (2.9) implies that the global implicit function theorem may be applied relative to the open neighborhood V of (2.9). We obtain a continuum C which is defined for $t \leqq \bar{t}$. Since, by Lemma 3.2, no solutions exist for $t \leqq t_0$, the continuum C must intersect the hyperplane $t = \bar{t}$ outside V.

Hence, for t sufficiently large, problem (2.1) always will have at least two solutions. We next wish to obtain more precise information. In order to accomplish this we linearize equation (2.1) along the positive solution U_t. We first write

$$f(u) = \beta\, u + g(u),$$

where $g(s)/s \to 0$ as $s \to \infty$ and seek solutions of (2.1) of the form $u = U_t + y$. Then y must satisfy

$$\begin{gathered} y'' + \beta\, y + g(U_t + y) - g(U_t) = 0 \\ y(0) = 0 = y(\pi). \end{gathered} \tag{3.6}$$

To obtain nontrivial solutions of (3.6) we embed (3.6) into the one parameter family of problems

$$\begin{gathered} y'' + \lambda y + g(U_t + y) - g(U_t) = 0, \quad 0 < x < \pi, \\ y(0) = 0 = y(\pi), \end{gathered} \tag{3.7}$$

and count the number of solutions of (3.7) at the $\lambda = \beta$, level.

THEOREM 3.4. *Let t_1 be such that* (2.1) *has a positive solution U_{t_1}* (Lemma 2.1). *Denote by λ_i, $i = 1, 2, \ldots$ the eigenvalues of*

$$\begin{gathered} v'' + \lambda\, v + g'(U_{t_1})\, v = 0, \quad 0 < x < \pi, \\ v(0) = 0 = v(\pi), \end{gathered} \tag{3.8}$$

$\lambda_1 < \lambda_2 < \ldots$, and assume that $\lambda_j < \beta \leqq \lambda_{j+1}$. Then (2.1) *has at least $2j$ distinct solutions.*

PROOF. Consider problem (3.7) which has the trivial solution $y = 0$ for all $\lambda \in \mathbf{R}$. Linearizing (3.7) along the trivial solution we obtain (3.8). Since all eigenvalues λ_i of (3.8) are simple it follows that each λ_i is a bifurcation value and two unbounded continua C_i^+ and C_i^- of solutions of

(3.7) (a solution of (3.7) is a pair (y, λ)!) bifurcate from $(0, \lambda_i)$. These have the property that $(\lambda, y) \in C_i^+$ implies that y has precisely $i - 1$ simple zeros in $(0, \pi)$, $y'(0) > 0$, whereas $(\lambda, y) \in C_i^-$ implies that y has precisely $i - 1$ simple zeros in $(0, \pi)$ and $y'(0) < 0$ (see, e.g., Rabinowtiz **[24]**). We next examine the continua C_1^-, $C_i^\pm$, $2 \leqq i \leqq j$, and regard their intersections with $E \times (-\infty, \beta]$.

Let $\alpha' > 0$ be such that $\alpha < \alpha' < 1$. Then there exists $C > 0$ such that $f(s) \geqq \alpha' s - C$, $s \in \mathbf{R}$, and hence $g(s) \geqq (\alpha' - \beta)s - C$, $s \in \mathbf{R}$. Therefore

$$g(U_t + y) - g(U_t) \geqq (\alpha' - \beta)y + (\alpha' - \beta)U_t - g(U_t) - C$$

and y satisfies the inequality

$$\begin{aligned} y'' + (\lambda + \alpha' - \beta)y &\leqq (\beta - \alpha')\, U_t - g(U_t) + C \\ y(0) &= 0 = y(\pi). \end{aligned} \tag{3.9}$$

As long as $\lambda + \alpha' - \beta < 1$, the boundary value problem

$$\begin{aligned} w'' + (\lambda + \alpha' - \beta)w &= (\beta - \alpha'_t)U_t + g(U_t) + C, \\ w(0) &= 0 = w(\pi), \end{aligned} \tag{3.10}$$

has a unique solution w_λ and $w_\lambda(x) \leqq y(x)$, $0 \leqq x \leqq \pi$.

We note that, for λ in compact intervals, $\lambda < 1 + \beta - \alpha'$, the family $\{w_\lambda(x)\}$ will be a priori bounded in E. Further, since g' is bounded, and if y solves (3.7), then

$$y'' + (\lambda + g'(\zeta(x))y = 0,$$

for some function $\zeta(x)$. It follows that λ must be bounded away from $-\infty$, say $\lambda \geqq \Lambda > -\infty$. Thus if $(y, \lambda) \in C_1^-$ or $(y, \lambda) \in C_i^\pm$, $i \geqq 2$, then, for compact λ intervals, $\lambda < 1 + \beta - \alpha'$, minima of y are uniformly bounded below; hence, by an argument similar to the one used in the proof of Lemma 3.1 we obtain that the solutions $\{y\}$ are bounded in E. Since, on the other hand, the continuua C_1^- and $C_i^\pm$ are unbounded in $E \times \mathbf{R}$ it follows that they must reach the $\lambda = \beta$ level. This is true for all i, $2 \leqq i \leqq j$, i.e.,

$$C_1^- \cap E \times \{\beta\} \neq \emptyset \neq C_i^\pm \cap E \times \{\beta\},\ i = 2, \ldots, j.$$

We thus obtain at least $2j - 1$ distinct nontrivial solutions of (3.6) and hence at least $2j$ distinct solutions of (2.1).

COROLLARY 3.5. *Assume* (3.1) *holds. Then there exists* t_2 *such that, for* $t \geqq t_2$, *problem* (2.1) *has at least* $2n$ *distinct solutions.*

PROOF. Since, for given $\varepsilon > 0$, there exists $\bar{t}$ such that, for $t \geqq \bar{t}$, $\|U_t - tv_\infty\| < \varepsilon$, where v_∞ is given by the proof of Lemma 2.1, it follows that

$g'(U_t)$ may be made arbitrarily small on given compact subintervals $[a, b]$ of $(0, \pi)$ for t sufficiently large. Hence the eigenvalues λ_j will have the property that $\lambda_j - j^2$ may be made arbitrarily small for t large, as long as $1 \leqq j \leqq n$. Hence, for sufficiently large t, we have that $\lambda_n < \beta$. We may now apply Theorem 3.4 to obtain the desired result.

Let now t_2 be as in Corollary 3.5, let

$$A_1^- = C_1^- \cap E \times \{\beta\}$$
$$A_i^\pm = C_i^\pm \cap E \times \{\beta\},$$

and identify $A_1^-, \ldots, A_i^\pm$, $2 \leqq i \leqq n$, as subsets of E. Then

$$\{U_{t_2}\}, \{U_{t_2} + A_1^-\}, \ldots, \{U_{t_2} + A_i^\pm\}$$

are solutions of (2.1) and because of the nodal properties of the continua C_1^-, $C_i^\pm$, there will exist disjoint bounded open sets $\mathcal{O}_0$, $\mathcal{O}_1^-$, $\mathcal{O}_i^\pm$ containing no solutions of (2.1) in their respective boundaries, such that

$$U_{t_2} \in \mathcal{O}_0 t$$
$$\{U_{t_2} + A_1^-\} \subseteq \mathcal{O}_i^-$$
$$\{U_{t_2} + A_i^\pm\} \subseteq \mathcal{O}_i^\pm, \quad 2 \leqq i \leqq n$$

and furthermore

$$d(id - Lf - t_2 L \sin - Lh, \mathcal{O}_0, 0) = (-1)^n \qquad \text{(see Lemma 2.1)}$$
$$d(id - Lf - t_2 L \sin - Lh, \mathcal{O}_1^-, 0) = 1$$
$$d(id - Lf - t_2 L \sin - Lh, \mathcal{O}_i^\pm, 0) = (-1)^{i+1},$$

where the last two degree calculations follow from an argument similar to the one used by Rabinowitz **[24]**.

We may now employ the global implicit function theorem (with respect to t) (see the proof of Lemma 3.3) and find continua of solutions $\mathcal{S}_0$, $\mathcal{S}_1^-$, $\mathcal{S}_i^\pm$ of (2.1) defined for $t \leqq t_2$ such that

$$\mathcal{S}_0 \cap \mathcal{O}_0 \neq \varnothing \neq \mathcal{S}_0 \cap \text{comp}\, \mathcal{O}_0^-$$
$$\mathcal{S}_1^- \cap \mathcal{O}_1 \neq \varnothing \neq \mathcal{S}_1^- \cap \text{comp}\, \bar{\mathcal{O}}_1^-$$
$$\mathcal{S}_i^\pm \cap \mathcal{O}_i^\pm \neq \varnothing \neq \mathcal{S}_i^\pm \cap \mathcal{O}_i^\pm$$

and each of these continua is bounded below in the t-direction (by Lemma 3.2).

If it is the case that for $t = t_2$ there exist precisely $2n$ solutions, then it will be the case that each of the continua above equals another one in the listing.

4. Varying α. In this section we shall consider the case where $\alpha = \lim_{s\to\infty}(f(s)/s$, may exceed 1. We shall first discuss in detail the situation where

$$1 \leqq \alpha < 4 \leqq n^2 < \beta < (n+1)^2 \tag{4.1}$$

and then consider the general case. Without further restrictions on the pair (α, β) we have the following result.

THEOREM 4.1. *Let* (4.1) *hold. Then there exists* t_1 *such that, for all* $t \geqq t_1$, *problem* (2.1) *has at least* $2n - 4$ *solutions.*

PROOF. Lemma 2.1 is still valid in the present situation, i.e., for t sufficiently large, problem (2.1) has a positive U_t. We again consider problem (3.6) and embed it into the family of problems (3.7), We choose t_3 so large that the eigenvalue λ_n of (3.8) satisfies $\lambda_n < \beta$ (see the proof of corollary 3.5). Let $C_i^{\pm}$, $i = 1, \ldots, n$, denote (as in the proof of Theorem 3.4) the continua of solutions of (3.7) bifurcating from the trivial solution at $(0, \lambda_i)$. As in the proof of Theorem 3.4, one may show that all these continua are bounded below in the λ-direction. We next show that if λ varies over a compact interval, $\lambda < 4 + \beta - \alpha'$, and if $(\lambda, y) \in C_3^-$ or $(\lambda, y) \in C_j^{\pm}$, $j \leqq 4$, then there exists a constant $R > 0$ such that $\|y\| < R$. In any of the mentioned cases $y(x)$ has at least two negative humps, i.e., there exist $0 \leqq x_1 < x_2 < x_3 < x_4 \leqq \pi$ such that

$$y(x) < 0, \quad x_1 < x < x_2, \quad x_3 < x < x_4,$$

and it must be the case that at least one of $x_2 - x_1$ or $x_4 - x_3$ does not exceed $\pi/2$.

Choose α' such that $\lambda \leqq \alpha < \alpha' < 4$. Then, as in the proof of Theorem 3.4, $y(x)$ will satisfy the inequality

$$y'' + (\lambda + \alpha' - \beta)\, y \leqq (\beta - \alpha')U_t + g(U_t) + C,$$

where C is chosen so that $f(s) \geqq \alpha' s - C$, $s \in R$. Assume that $x_2 - x_1 \leqq \pi/2$ and let $w_\lambda(x)$ be the unique solution of (note that $\lambda + \alpha' - \beta < 4$)

$$w'' + (\lambda + \alpha' - \beta)w = (\beta - \alpha')U_t + g(U_t) + C$$
$$w(x_1) = 0 = w(x_2).$$

Then $w_2(x) \leqq y(x)$, $x_1 \leqq x \leqq x_2$. Again, for λ in compact intervals, $\lambda + \alpha' - \beta < 4$, and $0 \leqq x_1 < x_2 \leqq \pi$, $x_2 - x_1 \leqq \pi/2$, the family $\{w_\lambda\}$ is a priori bounded; hence, we obtain an a priori bound on $\|y\|$, proving the claim. Therefore, all the continua C_3^- and $C_j^{\pm}$, $4 \leqq j \leqq n$ may be continued to the $\lambda = \beta$ level, and we obtain solutions U_t and $U_t + y$, $(y, \beta) \in C_3^-$, $(y, \beta) \in C_j^{\pm}$, $4 \leqq j \leqq n$ of (2.1). This yields at least $2n - 4$ solutions for large $t > 0$.

In what is to follow we shall show that under additional restrictions imposed upon the pair (α, β) more solutions actually will exist.

In order to obtain these results we rewrite the equation (2.1) in the form

$$(4.2)\qquad \begin{gathered} u'' + \beta u^+ + \alpha u^- + r(u) = t \sin x + h(x), \quad 0 < x < \pi, \\ u(0) = 0 = u(\pi), \end{gathered}$$

where $u^+ = \max\{u, 0\}$, $u^- = \min\{u, 0\}$. Then $u = u^+ + u^-$ and

$$(4.3)\qquad \lim_{s\to+\infty} \frac{r(s)}{s} = 0.$$

Let U_t ,as before, denote positive solutions of (4.2) for $t \gg 1$. We again seek solutions of u of (4.2) of the form $U_t + y$, where y is a solution of (3.6) or of (3.7) for $\lambda = \beta$.

Let us again assume that t_1 has been chosen so large that $\lambda_n < \beta$, where λ_n is the nth eigenvalue of (3.8). We already know that C_3^- and $C_j^\pm$, $4 \leqq j \leqq n$ may be continued to the $\lambda = \beta$ level. Let us now examine $C_2^\pm$ and C_3^+.

Since $g(s) = f(s) - \beta s$ and $f(s) = \beta s^+ + \alpha s^- + r(s)$ it follows that if y solves (3.7), then it must satisfy

$$(4.4)\qquad y'' + \lambda y + (\alpha - \beta)(U_t + y)^- + r(U_t + y) - r(U_t) = 0.$$

If C is one of the above continua which becomes unbounded at a finite λ value, say $\lambda = \lambda_*$, then there exists a sequence $\{(y_n, \lambda_n)\} \subseteq C$ with $\lambda_n \to \lambda_*$ and $\|y_n\| \to \infty$. Letting $v_n = y_n/\|y_n\|$, we obtain

$$(4.5)\qquad \begin{aligned} v_n'' + \lambda_n v_n + (\alpha - \beta)\left(\frac{U_t}{\|y_n\|} + v_n\right)^- & \\ + \frac{1}{\|y_n\|} r(U_t + y_n) - \frac{1}{\|y_n\|} r(U_t) &= 0. \end{aligned}$$

The properties of r imply that

$$\lim_{n\to\infty} \frac{1}{\|y_n\|} r(U_t + y_n) = 0.$$

Using the integral equation equivalent to (5.6) (subject to 0 boundary conditions), we obtain that $\{v_n\}$ has a subsequence converging to a nontrivial solution v of

$$\begin{gathered} v'' + \lambda_* v + (\alpha - \beta)v^- = 0 \\ u(0) = 0 = u(\pi), \end{gathered}$$

or equivalently

$$(4.6)\qquad \begin{gathered} v'' + \lambda_* v^+ + (\lambda_* + \alpha - \beta)v^- = 0 \\ v(0) = 0 = u(\pi). \end{gathered}$$

If, now, $C = C_2^+$ or C_2^-, then v must have precisely one zero in $(0, \pi)$, and hence

$$\frac{1}{\sqrt{\lambda_*}} + \frac{1}{\sqrt{\lambda_* + \alpha - \beta}} = 1 \tag{4.7}$$

or

$$\sqrt{\lambda_* + \alpha - \beta} = \frac{\sqrt{\lambda_*}}{\sqrt{\lambda_*} - 1}. \tag{4.8}$$

Equation (4.8) has a unique solution λ_*. If now $\lambda_* > \beta$, then it follows that both C_2^+ and C_2^- may be continued to the $\lambda = \beta$ level. This motivates the following lemma.

LEMMA 4.2. *Let* (4.1) *hold and assume that*

$$\sqrt{\alpha} < \frac{\sqrt{\beta}}{\sqrt{\beta} - 1}. \tag{4.9}$$

Then the continua $C_2^{\pm}$ may be continued to the $\lambda = \beta$ level.

PROOF. If (4.9) holds, then the unique solution λ_* of (4.8) satisfies $\lambda_* > \beta$, hence both C_2^+ and C_2^- may be continued to the $\lambda = \beta$ level.

We note that if (4.1) holds, then (4.9) is not necessarily satisfied; however, the solution set of (4.9) subject to (4.1) is nonempty.

Next let us consider the continuum C_3^+. We employ a similar consideration and obtain a solution v of (4.6) having two zeros interior to $(0, \pi)$ and two positive humps. Hence we obtain that

$$\frac{2}{\sqrt{\lambda_*}} + \frac{1}{\sqrt{\lambda_* + \alpha - \beta}} = 1,$$

or equivalently

$$\sqrt{\lambda_* + \alpha - \beta} = \frac{\sqrt{\lambda_*}}{\sqrt{\lambda_*} - 2}. \tag{4.10}$$

Again, (4.10) has a unique solution λ_*, and if $\lambda_* > \beta$, then C_3^+ may be continued to the $\lambda = \beta$ level. We thus have the following Lemma.

LEMMA 4.3. *Let* (4.1) *hold, and let*

$$\sqrt{\alpha} < \frac{\sqrt{\beta}}{\sqrt{\beta} - 2}. \tag{4.11}$$

Then C_3^+ may be continued to the $\lambda = \beta$ level.

We again note that if (4.1) holds, then (4.11) is not necessarily satisfied unless $n = 3$, i.e., $9 < \beta < 16$. Thus, in this special case, we obtain an improvement of Theorem 4.1.

Let us summarize the above considerations in the following table.

$1 \leqq \alpha < 4 \leqq n^2 < \beta < (n+1)^2$

restrictions	# solutions for $t \gg 1$
none	$2n-4$
$\dfrac{\sqrt{\beta}}{\sqrt{\beta}-1} \leqq \sqrt{\alpha} < \dfrac{\sqrt{\beta}}{\sqrt{\beta}-2}$	$2n-3$
$\sqrt{\alpha} < \dfrac{\sqrt{\beta}}{\sqrt{\beta}-1}$	$2n-1$

TABLE 1.

We now examine the general case. To this end we must examine how far in the λ- direction the continua $C_i^{\pm}$ may be continued. We distinguish between the cases i even and i odd.

If C is such a continuum which may not be continued beyond λ_*, then λ_* again satisfies (4.6).

Case $i = 2k$. In this case each solution in C_i^+ and C_i^- has k positive and k negative "humps", i.e., λ_* must satisfy

$$\frac{k}{\sqrt{\lambda_*}} + \frac{k}{\sqrt{\lambda_* + \alpha - \beta}} = 1 \tag{4.12}$$

or equivalently

$$\sqrt{\lambda_* + \alpha - \beta} = \frac{k\sqrt{\lambda_*}}{\sqrt{\lambda_*} - k}; \tag{4.13}$$

consequently, C_i^+ may be continued to the $\lambda = \beta$ level whenever

$$\sqrt{\alpha} < \frac{k\sqrt{\beta}}{\sqrt{\beta} - k}. \tag{4.14}$$

Case $i = 2k - 1$. We consider first C_i^+. Any solution $(y, \lambda) \in C_i^+$ has the property that y has k positive humps and $k - 1$ negative humps; hence, λ_* must satisfy

$$\frac{k}{\sqrt{\lambda_*}} + \frac{k-1}{\sqrt{\lambda_* + \alpha - \beta}} = 1, \tag{4.14}$$

i.e.,

$$\sqrt{\lambda_* + \alpha - \beta} = \frac{(k-1)\sqrt{\lambda_*}}{\sqrt{\lambda_*} - k},$$

Therefore, if

$$\sqrt{\alpha} < \frac{(k-1)\sqrt{\beta}}{\sqrt{\beta}-k}, \tag{4.15}$$

then C_i^+ may be continued to the $\lambda = \beta$ level. If we consider C_i^-, then similar calculations show that C_i^- may be continued to the $\lambda = \beta$ level whenever

$$\sqrt{\alpha} < \frac{k\sqrt{\beta}}{\sqrt{\beta}-(k-1)}. \tag{4.16}$$

These calculations may be now used to determine a lower bound on the number of solutions in a particular situation. The following lemma simplifies such considerations.

LEMMA 4.4. *Let $i < n$ be such that C_i^- may be continued to the $\lambda = \beta$ level by means of the above criteria; then $C_j^\pm$, for $i < j \leqq n$, may also be continued to that level.*

PROOF. Let $i = 2k$ and assume that (4.14) holds. Then, since

$$\frac{k\sqrt{\beta}}{\sqrt{\beta}-k} < \frac{k-\sqrt{\beta}}{\sqrt{\beta}-(k+1)} < \frac{(k+1)\sqrt{\beta}}{\sqrt{\beta}-k},$$

it follows that (4.15) and (4.16) hold with k replaced by $k+1$; an induction argument completes the proof. A similar argument may be employed in case i is odd.

The general case, where

$$m^2 \leqq \alpha < (m+1)^2 \leqq n^2 < \beta < (n+1)^2, \tag{4.17}$$

may now be treated by employing formulas (4.14), (4.15) and (4.16). We consider one more special case which illustrates how the general case may be treated. Namely, we shall consider the case

$$4 \leqq \alpha < 9 \leqq n^2 < \beta < (n+1)^2$$

and obtain Table 2.

5. Bibliographical remarks and open questions. If it is the case that $\alpha > 1$, $\alpha \neq n^2$, $n = 2, 3, \ldots$, then an analysis similar to the one used in the proof of Lemma 2.1 shows that if $t \ll -1$, then problem (2.1) has a solution $U_t(x)$ with $U_t(x) < 0$, $0 < x < \pi$ (a similar result is true for equation (1.1) for appropriate g). In this case the following questions arise:

(i) Depending upon the location of β, how many solutions are there for $t \ll -1$?

(ii) Do there exist solution continua which are defined for all $t \in \mathbf{R}$?

In §3 we indicated the existence of various solution continua which arise by applying the implicit function theorem in a neighborhood of

β	Number of solutions.
$9 < \beta < 16$	
$\sqrt{\alpha} < \dfrac{\sqrt{\beta}}{\sqrt{\beta} - 2}$	3
$\dfrac{\sqrt{\beta}}{\sqrt{\beta} - 2} \leqq \sqrt{\alpha} < \dfrac{2\sqrt{\beta}}{\sqrt{\beta} - 1}$	2
no other restrictions on α	1
$16 < \beta < 25$	
$\sqrt{\alpha} < \dfrac{2\sqrt{\beta}}{\sqrt{\beta} - 1}$	4
no other restrictions on α	3
$25 < \beta < 36$	
$\sqrt{\alpha} < \dfrac{\sqrt{2\beta}}{\sqrt{\beta} - 1}$	6
no other restrictions on α	5
$n^2 < \beta < (n + 1)^2, \quad n = 6, 7, 8$	
$\sqrt{\alpha} < \dfrac{2\sqrt{\beta}}{\sqrt{\beta} - 1}$	$2n - 4$
$\dfrac{2\sqrt{\beta}}{\sqrt{\beta} - 1} \leqq \sqrt{\alpha} < \dfrac{2\sqrt{\beta}}{\sqrt{\beta} - 2}$	$2n - 5$
no other restrictions on α	$2n - 7$
$n^2 < \beta < (n + 1)^2, \quad n \geqq 9$	
$\sqrt{\alpha} < \dfrac{2\sqrt{\beta}}{\sqrt{\beta} - 1}$	$2n - 4$
$\dfrac{2\sqrt{\beta}}{\sqrt{\beta} - 1} \leqq \sqrt{\alpha} < \dfrac{2\sqrt{\beta}}{\sqrt{\beta} - 2}$	$2n - 5$
$\dfrac{2\sqrt{\beta}}{\sqrt{\beta} - 2} \leqq \sqrt{\alpha} < \dfrac{2\sqrt{\beta}}{\sqrt{\beta} - 3}$	$2n - 7$
no other restrictions on α	$2n - 8$

TABLE 2.

$t = \infty$, the projections of these continua onto the t-axis are not understood and it is of interest to know how these projections depend upon the perturbation term h. Some results in this direction are available (see, e.g., **[10]**, **[11]**, **[1]**, **[5]**, **[17]**.

Many additional results for other classes of problems (such as parabolic and hyperbolic equations) have recently become available; the interested reader is referred to **[4]**, **[12]**, **[16]**, **[20]**, **[23]** and the references in these papers.

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