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SOLID FUEL COMBUSTION-SOME MATHEMATICAL PROBLEMS

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1. Introduction. The initiation of a combustion process involves a myriad of complex physical phenomena which are fascinating to observe and challenging to describe in quantitative terms. In general one is concerned with the time-history of a spatially varying process occurring in a deformable material in which there is a strong interaction between chemical heat release, diffusive effects associated with the transport properties, bulk material motion as well as several types of propagating wave phenomena. Mathematical models capable of describing these combustion systems incorporate not only familiar reaction-diffusion effects associated with rigid materials, but those arising from material compressibility as well. For a combustible gas, the complete reactive Navier-Stokes equations are required to describe the phenomena involved.

In this paper, we shall focus on the initiation and evolution of thermal explosion processes in rigid materials. In this situation the physical processes are determined by a pointwise balance between chemical heat addition and heat loss by conduction.

The mathematical system which describes a thermal reaction event for a gaseous fuel in a bounded container is given in §2. Also in this section, we show how the complete system (c) can be simplified for a rigid fuel to a reactive diffusive system (2.1)–(2.2), and to the ignition model (2.3)–(2.4) by activation energy asymptotics. Closely related to the ignition model are the steady-state problems (2.5)–(2.6) and (2.7)–(2.8), referred to here as the Gelfand problem [7] and the perturbed Gelfand problem, respectively.

In §3, we survey some known results for such steady state problems for rather general domains Ω. Then in §4 we give more precise multiplicity results for the case $\Omega = B_1$, a ball in $\mathbf{R}^n$. In §5, we study the solution profiles for these steady-state models and in §6 we return to the classical ignition model to analyze the problem of thermal runaway.

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2. Simplification of the complexity of the system. If one considers a heat-conductive, viscous reactive chemical fuel in a bounded container $\Omega \subset \mathbf{R}^n$ assuming simple one-step chemistry, its behavior is described by the system of equations (see **[11]**) which in Euler coordinates has the form:

$$\rho_t + \nabla\cdot(\rho u) = 0$$

$$\rho(u_t + u\cdot\nabla u) = -\frac{1}{\gamma}\nabla p + \frac{4}{3} MP_r\Delta u$$

(c)

$$\rho(T_t + u\cdot\nabla T) = \varepsilon M\gamma\delta\rho^m Y^m e^{(T-1)/\varepsilon T} + M\gamma\Delta T - (\gamma - 1)p\nabla u + \frac{4}{3} M\gamma(\gamma - 1)P_r\nabla u\cdot\nabla u$$

$$\rho(Y_t + u\cdot\nabla Y) = -\varepsilon M\gamma\delta\rho^m Y^m\Gamma e^{(T-1)/\varepsilon T} + \frac{M}{L_e}\nabla\cdot(\rho\nabla Y),$$

where

ρ - density
u - velocity
p - pressure
T - temperature
Y - fuel mass fraction
ε - activation energy
δ - Frank-Kamenetski parameter
Γ - thermal energy
P_r - Prandl number
L_e - Lewis number
m - order of the reaction
$\gamma \geqq 1$ - gas constant
M - ratio of acoustic time to conduction time

If the one chemical species is a solid, then $u = 0$, $\rho = 1$, $\gamma = 1$, and $M = 1$ and (c) reduces to the parabolic system

(2.1)
$$T_t - \Delta T = \varepsilon\delta Y^m e^{(T-)1/\varepsilon T}$$
$$Y_t - \beta\Delta Y = -\varepsilon\delta\Gamma Y^m e^{(T-1)/\varepsilon T}$$

with initial-boundary conditions:

(2.2)
$$T(x, 0) = T_0(x), \qquad Y(x, 0) = 1, \quad x \in \Omega$$
$$T(x, t) = 1, \qquad \frac{\partial T}{\partial n(x)}(x, t) = 0, \quad (x, t) \in \partial\Omega \times [0, \infty).$$

To further simplify the complexity of IBVP (2.1)–(2.2), one method is to identify and restrict the range of certain parameters, then use an asymptotic analysis. In our case, the (reciprocal of) activation energy

ε is a parameter which, for solid fuels of interest, is assumed small ($\ll 1$).

By using the method of activation energy asymptotics (AEA), as a first order approximation setting $T = 1 + \varepsilon\theta$ and $Y = 1 - \varepsilon y$, IBVP (2.1)–(2.2) can be rewritten as

$$\text{(2.1')} \qquad \begin{aligned} \theta_t - \Delta\theta &= \delta(1 - \varepsilon y)^m e^{\theta/(1+\varepsilon\theta)} \\ y_t - \beta\Delta y &= \delta\Gamma(1 - \varepsilon y)^m e^{\theta/(1+\varepsilon\theta)} \end{aligned}$$

with

$$\text{(2.2')} \qquad \begin{aligned} &\theta(x, 0) = \theta_0(x), \quad y(x, 0) = 0, \quad x \in \Omega \\ &\theta(x, t) = 0, \qquad \frac{\partial y}{\partial n(x)}(x; t) = 0, \quad (x, t) \in \partial\Omega \times (0, \infty). \end{aligned}$$

For $\varepsilon \ll 1$, the AEA method has essentially decoupled IBVP (1)–(2) and we need only consider the ignition model

$$\text{(2.3)} \qquad \begin{aligned} \theta_t - \Delta\theta &= \delta e^{\theta} \\ \theta(x, 0) &= \theta_0(x), \quad x \in \Omega \end{aligned}$$

$$\text{(2.4)} \qquad \theta(x, t) = 0, \qquad x \in \partial\Omega, \ t \in (0, \infty),$$

the associated steady state model (or Gelfand problem)

$$\text{(2.5)} \qquad -\Delta\psi = \delta e^{\psi}, \quad x \in \Omega$$

$$\text{(2.6)} \qquad \psi(x) = 0, \quad x \in \partial\Omega,$$

and the closely related small fuel loss model (or perturbed Gelfand problem)

$$\text{(2.7)} \qquad -\Delta\phi = \delta e^{\phi/1+\varepsilon\phi}$$

$$\text{(2.8)} \qquad \phi(x) = 0, \quad x \in \partial\Omega.$$

3. Existence-arbitrary domains. For rather arbitrary domains Ω, there are many existence results for a wide variety of nonlinearities f (e.g., P.L. Lions **[14]** and K. Schmitt **[15]**). In this section, we collect together some of those results which pertain to the Gelfand and the perturbed Gelfand problem.

Let Ω be an n-dimensional bounded domain with boundary $\partial\Omega$ and closure $\bar{\Omega}$. Assume $\partial\Omega$ belongs to class $C^{2+\alpha}$ which means, for every $x \in \partial\Omega$, there exists a neighborhood N of x such that $\partial\Omega \cap N$ may be represented in the form $x^i = h(x^1, \ldots, x^{i-1}, x^{i+1}, \ldots, x^n)$, for some i where h belongs to class $C^{2+\alpha}$. Assume f: $\bar{\Omega} \times \mathbf{R} \to \mathbf{R}$ is locally Holder continuous and consider

$$(3.1)\qquad \begin{aligned} -\Delta u &= f(x, u), \quad x \in \Omega \\ u(x) &= 0, \quad x \in \partial\Omega. \end{aligned}$$

A continuous function $\alpha(x)\colon \bar{\Omega} \to \mathbf{R}$ is a lower solution of (3.1) if $\alpha(x) \in C^2(\Omega)$, $-\Delta\alpha(x) \leqq f(x, \alpha(x))$ on Ω, and $\alpha(x) \leqq 0$ on $\partial\Omega$. An upper solution $\beta(x)$ is similarly defined.

The now classical existence result (see [**15**; Theorem 3.2, p. 276]) for (3.1) is

THEOREM 3.1. *If there exists a lower solution $\alpha(x)$ and an upper solution $\beta(x)$ for* (3.1) *with $\alpha(x) \leqq \beta(x)$ on $\bar{\Omega}$, then* (3.1) *has a solution $u(x) \in [\alpha, \beta]$.*

We are interested in the parameterized version of (3.1), that is,

$$(3.1)_\lambda\qquad \begin{aligned} -\Delta u &= \lambda f(x, u), \quad x \in \Omega \\ u(x) &= 0, \quad x \in \partial\Omega, \end{aligned}$$

where $f\colon \bar{\Omega} \times \mathbf{R} \to [0, \infty)$ is nonnegative. Then obviously $\alpha(x) = 0$ is a lower solution of $(3.1)_\lambda$, for all $\lambda \geqq 0$.

Define the spectrum Σ of $(3.1)_\lambda$ to be the set of all $\lambda \in \mathbf{R}$ such that $(3.1)_\lambda$ has a nonnegative solution. With this definition, we immediately have

LEMMA 3.2. *If λ_1 is positive and $\lambda_1 \in \Sigma$, then $[0, \lambda_1] \subset \Sigma$.*

PROOF. Let $\beta(x)$ be a solution of $(3.1)_\lambda$. Then $-\Delta\beta(x) = \lambda_1 f(x, \beta(x)) \geqq \lambda f(x, \beta(x))$ for any $\lambda \in [0, \lambda_1]$ and $\beta(x) \equiv 0$ on $\partial\Omega$. Thus $\beta(x)$ is an upper solution and, by Theorem 3.1, $(3.1)_\lambda$ has a nonnegative solution.

LEMMA 3.3. *Assume there exist nonzero nonnegative functions $g(x)$, $r(x) \in C^\alpha(\bar{\Omega})$ such that*

$$f(x, u) \geqq g(x) + r(x)u, \quad x \in \bar{\Omega},\ u \geqq 0.$$

Then $(3.1)_\lambda$ has no nonnegative solutions for $\lambda \geqq \lambda_1(r)$, where $\lambda_1(r)$ is the first eigenvalue of

$$(3.2)\qquad \begin{aligned} -\Delta u &= \lambda r(x)u, \quad x \in \Omega \\ u(x) &= 0, \quad x \in \partial\Omega. \end{aligned}$$

PROOF. Assume, for some $\lambda \geqq \lambda_1(r) > 0$, there exists a nonnegative solution $v(x)$ of $(3.1)_\lambda$. Then $-\Delta v = \lambda f(x, v) \geqq \lambda g(x) + \lambda r(x)v$, for $x \in \Omega$ with $v(x) = 0$ on $\partial\Omega$. Since $\alpha(x) = 0$ is a lower solution and $v(x)$ is an upper solution of $(3.1)_\lambda$ with $0 \leqq v(x)$, there exists, by Theorem 3.1, a solution $u(x)$ of $-\Delta u = \lambda(g(x) + r(x)u)$ with $u = 0$ on $\partial\Omega$ with $0 \leqq u(x) \leqq v(x)$. By the maximum principle, $u(x) > 0$ on Ω.

Let $w(x)$ be a nonnegative eigenfunction corresponding to $\lambda_1(r)$. Integrating $w(-\Delta u) - u(-\Delta w)$ over Ω, we have $(\lambda_1 - \lambda)\int_\Omega r(x)u(x)w(x)dx = \lambda\int_\Omega w(x)g(x)dx > 0$ which contradicts $\lambda \geqq \lambda_1(r)$.

As a consequence of this lemma, we see that the Gelfand problem (2.5)–(2.6) has no solutions for $\delta \geqq \lambda_1$, where λ_1 is the first eigenvalue of

$$(3.3)\qquad \begin{aligned} -\Delta u &= \lambda u \\ u(x) &= 0, \quad x \in \partial\Omega, \end{aligned}$$

since $e^u \geqq 1 + u$ for all u.

For $(3.1)_\lambda$ with nonlinearity $f(x, u)$ satisfying $f(x, 0) > 0, f_u(x, u) > 0$, and $f_{uu}(x, u) \geqq 0$, for $x \in \bar{\Omega}$, $u \geqq 0$, we immediately can conclude by Lemma 3.3 that if $\lambda \geqq \lambda_1(f_u(\cdot, 0))$, then $\lambda \notin \Sigma$ and we have an upper bound for the spectrum of $(3.1)_\lambda$.

Define $\lambda^* = \sup \Sigma$, then $\lambda^* \in [0, \infty]$ for nonnegative $f(x, u)$. If f is positive, increasing, and convex, then $\lambda^* < \lambda_1(f_u(\cdot, 0))$.

Bandle [1] used symmetrization techniques to get lower bounds on λ^*. The following lemma whose proof can be found in [1] is the key.

LEMMA 3.4. *The solution $w(x)$ of*

$$(3.4)\qquad \begin{aligned} -\Delta w &= 1, \quad x \in \Omega \\ w &= 0, \quad x \in \partial\Omega \end{aligned}$$

satisfies

$$0 \leqq w(x) \leqq (2n)^{-1}\left(\frac{V_n}{S_n}\right)^{2/n}$$

where V_n and S_n are the n-dimensional volumes of Ω and the unit ball, respectively.

As a consequence, we have

THEOREM 3.5. *Assume there exists a nondecreasing function $f_0 \in C^\alpha$ $[0, \infty)$ such that $f_0(u) > 0$ for $u \geqq 0$ and $f(x, u) \leqq f_0(u)$ for $x \in \bar{\Omega}$, $u \geqq 0$. Assume the function $m/f_0(m)$, $m \geqq 0$, assumes its maximum at m . Then*

$$\left[0,\ 2n \cdot \frac{m_0}{f(m_0)} \cdot \left(\frac{S_n}{V_n}\right)^{2/n}\right] \subset \Sigma.$$

PROOF. Let $\lambda \in [0, 2n(m_0/f(m_0)) \cdot (S_n/V_n)^{2/n}]$ and consider

$$(3.5)\qquad \begin{aligned} -\Delta\beta &= \lambda f_0(m_0), \quad x \in \Omega \\ \beta(x) &= 0, \quad x \in \partial\Omega. \end{aligned}$$

The function $\beta(x) = \lambda f_0(m_0)\, w(x)$ where $w(x)$ is the solution of (3.4) is a solution of (3.5). In addition $\beta(x) \geqq 0$ on $\bar{\Omega}$ and

$$\beta(x) = \lambda f_0(m_0) w(x) \leqq \lambda f_0(m_0)(V_n/S_n)^{2/n}(2n)^{-1} \leqq m_0.$$

Since $-\Delta\beta = \lambda f_0(m_0) \geqq \lambda f_0(\beta(x)) \geqq \lambda f(x, \beta(x))$, $\beta(x)$ is an upper solution for $(3.1)_\lambda$. Clearly, $\alpha(x) = 0$ is a lower solution and, by Theorem 3.1, $(3.1)_\lambda$ has a solution $u(x)$. Thus, $\lambda \in \Sigma$.

The following result due to Kazdan-Warner [12] gives upper bounds on λ^*.

THEOREM 3.6. *If* $f(x, u) > 0$ *for* $x \in \bar{\Omega}$, $u \geqq 0$, *then* $\lambda^* \in (0, \infty]$. *If in addition*

a) $\liminf_{s\to\infty} f(x, s)/s > 0$, *then* $\lambda^* < \infty$

b) $\lim_{s\to\infty} f(x, s)/s = 0$, *then* $\lambda^* = \infty$.

PROOF. By the maximum principle, all solutions for $\lambda > 0$ are positive on Ω. Also, $(3.1)_\lambda$ has a solution for $\lambda > 0$ sufficiently small. To see this, observe that the solution $\bar{u}(x)$ of (3.4) is positive and $-\Delta\bar{u} \geqq \lambda f(x, \bar{u}(x))$, for all $x \in \bar{\Omega}$ and $\lambda > 0$ sufficiently small. Thus $\bar{u}(x)$ is an upper solution and $(3.1)_\lambda$ has a solution. Hence Σ is nonempty and $\lambda^* = \sup \Sigma$ exists.

a) We now show that $\lambda^* < \infty$ if $\liminf_{s\to\infty} f(x, s)/s > 0$. In this case there exist $a \geqq 0$, $b > 0$ such that $f(x, s) > a + bs$. If $u(x)$ is a positive solution of $(3.1)_\lambda$ and if $\psi \geqq 0$ is an eigensolution of (3.3) associated with the first eigenvalue λ_1 normalized so that $\|\psi\|_2 = 1$, then

$$0 = \int_\Omega \psi(x)(-\Delta u - \lambda_1 u(x))dx \geqq \int_\Omega \psi(\lambda a + (\lambda b - \lambda_1)u(x))dx$$

which is impossible if $\lambda b \geqq \lambda_1$. Thus, $\lambda < \lambda_1/b$ and $\lambda^* \leqq \lambda_1/b < \infty$.

b) If $\lim_{s\to\infty} f(x, s)/s = 0$, then one can construct an upper solution $\bar{u}(x)$ for any $\lambda > 0$. Thus, $\lambda^* = +\infty$.

REMARKS. The last two theorems give us the following information.

1. For the Gelfand problem (2.5)–(2.6),

$$2n \cdot e^{-1}(S_n/V_n)^{2/n} < \delta^* < \lambda_1/e.$$

2. For the perturbed Gelfand problem (2.7)–(2.8), $\delta^* = \infty$ and solutions exist for any $\delta > 0$, $\varepsilon > 0$.

3. For $\Omega = B_1$, a ball in $\mathbf{R}^n$ of radius 1, the lower bound for δ^* for (2.5)–(2.6) given by Theorem 3.5 is $(2n)/e$. De Figueiredo and Lions [5] improve this lower bound to get

$$\delta^* > \max\left\{\frac{\ln(1 + \alpha M_\alpha)}{M_\alpha} : 0 \leqq \alpha < \lambda_1\right\},$$

where λ_1 is first eigenvalue of (3.3) and, for $n \geqq 3$,

$$M_\alpha = \frac{1}{\alpha}\left(\frac{\alpha^{p/2}}{2^p\Gamma(p + 1)J_p(\sqrt{\alpha})} - 1\right)$$

where $p = (n - 2)/2$, J_p is the Bessel function of order p. For $n = 3$, this gives $\delta^* > 2.865$ instead of $\delta^* > 6/e = 2.21$.

There are some uniqueness and multiplicity results for $(3.1)_\lambda$ with arbitrary Ω (e.g., [**5**, **16**, **17**]) where the nonlinearity is general enough to give information about (2.5)–(2.6) or (2.7)–(2.8). For example, Schuchman [**16**] proves

THEOREM 3.7. *Consider* $(3.1)_\lambda$. *If*
1) $f(x, 0) > 0$, for all $x \in \bar{\Omega}$,
2) *f is continuously differentiable in u, for $u \geqq 0$, and*
3) $0 \leqq f_u(x, u) \leqq K(1 + u)^{-(1+\alpha)}$, for $x \in \bar{\Omega}$, $u \geqq 0$,
then $\lambda^* = \infty$ *and there exists* $\lambda_u > 0$ *such that* $(3.1)_\lambda$ *has a unique solution for* $\lambda > \lambda_u$.

Thus, the perturbed Gelfand problem has a unique solution for large δ.

4. Existence and multiplicity-spherical domains. For $\Omega = B_1 \subset \mathbf{R}^n$, very precise multiplicity results are known for both the Gelfand problem and the perturbed Gelfand problem. In this section we summarize these results for (2.5)–(2.6) and (2.7)–(2.8).

By the maximum principle, any solution $u(x) \in C^2(B_1, \mathbf{R})$ of either the Gelfand problem or the perturbed Gelfand problem is positive on B_1. By the result of Gidas-Ni-Nirenberg [**8**], all solutions are radially symmetric, that is, $u = u(r)$ where $r = |x|$.

For (2.5)–(2.6), one can hence equivalently look for solutions $u(r) \in C^2[0, 1]$ of

(4.1)
$$u'' + \frac{n-1}{r}u' + \delta e^u = 0, \quad 0 < r < 1$$
$$u'(0) = 0, \quad u(1) = 0$$

or

(4.2)
$$(r^{n-1}u')' + \delta r^{n-1}e^u = 0$$
$$u(0) = \alpha \quad u(1) = 0$$

or

(4.3)
$$u'' + \frac{n-1}{r}u' + \delta e^u = 0, \quad 0 < r < 1$$
$$u(1) = 0, \quad u'(1) = c = -\beta.$$

The oft-quoted multiplicity result due to Joseph-Lundgren [**9**];

THEOREM 4.1. *Consider* (2.5)–(2.6)
a). $n = 1$. *There exists* $\delta^* > 0$ *such that*
i) *for* $\delta \in (0, \delta^*)$, *there exist two solutions,*
ii) *for* $\delta = \delta^*$, *there exists a unique solution, and*

iii) *for* $\delta > \delta^*$, *no solution exists.*

b) $n = 2$. *Let* $\delta^* = 2$, *then*

i) *for each* $\delta \in (0, \delta^*)$, *there exist two solutions,*

ii) *for* $\delta = \delta^*$, *there exists a unique solution, and*

iii) *for* $\delta > \delta^*$, *no solution exists.*

c) $3 \leqq n \leqq 9$. *Let* $\bar{\bar{\delta}} = 2(n-2)$. *Then there exists* $\delta^* > \bar{\bar{\delta}}$ *such that*

i) *for* $\delta = \delta^*$, *there exists a unique solution,*

ii) *for* $\delta > \delta^*$, *there are no solutions,*

iii) *for* $\delta = \bar{\bar{\delta}}$, *there exist a countable infinity of solutions, and*

iv) *for* $\delta \in (0, \delta^*) - \{\bar{\bar{\delta}}\}$, *there exists a finite number of solutions.*

d) $n \geqq 10$. *Let* $\delta^* = 2(n-2)$. *Then*

i) *for* $\delta \geqq \delta^*$, *there are no solutions, and*

ii) *for* $\delta \in (0, \delta^*)$, *there exists a unique solution.*

PROOF. Let $\alpha = u(0)$, $\beta = -c = -u'(1)$. For $n = 1$, (4.1) can be solved by integration to obtain

$$\beta = (\beta^2 + 2\delta)\tanh((\beta^2 + 2\delta)/2),$$

$$\delta = \frac{1}{2}e^{-\alpha}\ln\left(\frac{1 + (1 - e^{-\alpha})^{1/2}}{1 - (1 - e^{-\alpha})^{1/2}}\right)^2, \tag{4.4}$$

$$u(r) = \alpha - 2\ln\cosh\left(\frac{1}{2}(2\delta e^{\alpha})^{1/2}r\right).$$

For $n = 2$, (4.1) can also be solved by making the change of variables $r = e^{-t}$, $w(t) = u(r) - 2t$ to obtain $\ddot{w} + \delta e^{w} = 0$. Then

$$\beta^2 - 4\beta + 2\delta = 0,$$

$$\delta = 8(e^{-\alpha/2} - e^{-\alpha}), \tag{4.5}$$

$$u(r) = \alpha - 2\ln\left(1 + \frac{1}{8}\delta e^{\alpha}r^2\right).$$

For $n \geqq 3$, let $t_1 = 1/2\ln((2(n-2))/(\delta e^2))$, $r = e^{-(t-t_1)}$, and $u(r) = \alpha + 2t + z(t)$. Then (4.2) becomes

$$\frac{\ddot{z}}{n-2} - \dot{z} + 2e^{z} - 2 = 0, \quad t_1 < t < \infty \tag{4.6}$$

$$z(\infty) = -\infty, \qquad \dot{z}(\infty) = -2$$

with compatibility condition $z(t_1) = -\alpha - 2t_1$. Let $y(t) = \dot{z}(t) + 2$ and $x(t) = 2(n-2)e^{z(t)}$, then

$$\dot{x} = x(y-2)$$
$$\dot{y} = (n-2)y - x, \quad t_1 < t < \infty \tag{4.7}$$

with $x(\infty) = y(\infty) = 0$ and compatibility condition $t_1 = 1/2\ln(2(n-2))/(\delta e^{\alpha})$. Thus, $\delta = x(t_1)$ and $\beta = y(t_1)$.

The two-dimensional system (4.7) has critical points at (0, 0) and $(2(n-2), 2)$. If $3 \leqq n \leqq 9$, $(2(n-2), 2)$ is an unstable spiral and (0, 0) is a saddle. For $n \geqq 10$, $(2(n-2), 2)$ is an unstable mode and (0, 0) is a saddle. One can prove [4] that there exists a heteroclinic orbit $D = \{(x(t), y(t)) : t \in \mathbf{R}\}$ connecting these critical points.

The orbit segment $(x(t), y(t))$, $t_1 \leqq t < \infty$, corresponds to a pair$(\delta_1, \beta_1) = (x(t_1), y(t_1))$ and a function $u(r)$ on (0, 1) such that $u(r)$ is a solution of (4.1) with $\beta_1 = u'(1)$ and $\delta = \delta_1$.

These observations can be summarized in terms of (δ, β) bifurcation diagrams, (see Figure 1).

For the perturbed Gelfand problem (2.7)–(2.8) with domain $\Omega = B_1 \subset \mathbf{R}^n$, Dancer [4] proved

THEOREM 4.2. *For any* $\varepsilon > 0$, $\delta > 0$, (2.7)–(2.8) *has at least one and at most finitely many solutions.*

A more precise description is given by the following bifurcation diagrams in Figure 2.

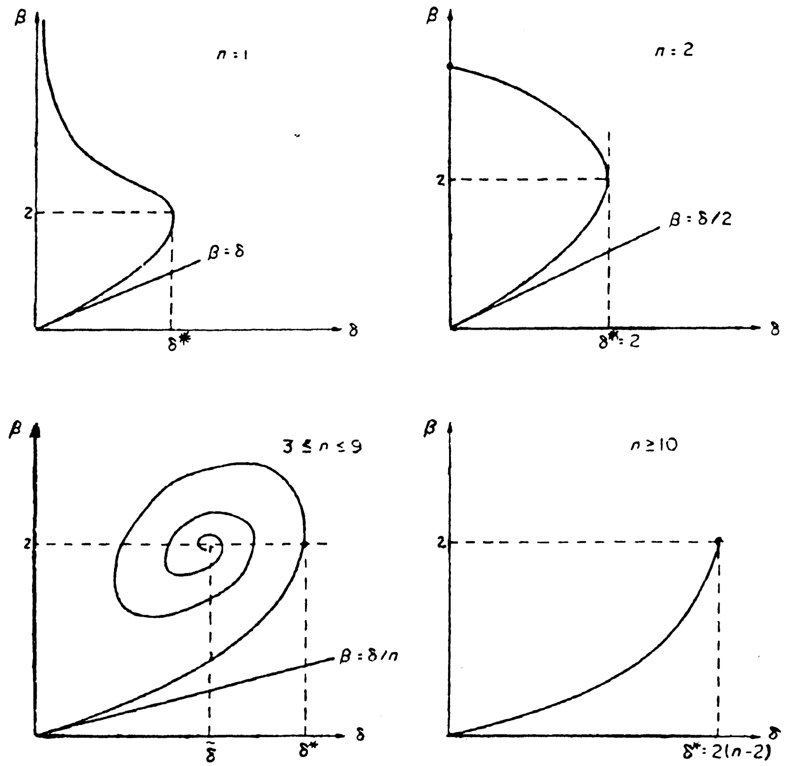

Figure 1

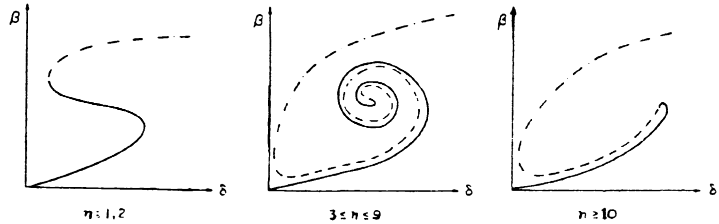

Figure 2

5. Solution Profiles. We first consider the Gelfand problem (2.5)–(2.6). We will define a solution $u(x)$ of this problem to be *bell-shaped* if the corresponding solution $u(r)$ of (4.1) has a unique point of inflection for $r \in (0, 1)$. We can now prove the following theorem which gives very precise information about the shape of solutions to the Gelfand problem.

THEOREM 5.1.
a) *For* $n = 1$, *all solutions are concave down on* $[0, 1]$.
b) *For* $n = 2$.
i) *if* $\delta \in (0, \delta^*)$, *the minimal solution is concave on* $[0, 1]$ *and the maximal solution is bell-shaped*; *and*
ii) *if* $\delta = \delta^* = 2$, *the solution is concave on* $[0, 1)$ *with* $u''(1) = 0$.
c) *For* $n \geqq 3$, *there exists* $\bar{\delta} < \delta^*$ *such that*:
i) *if* $\delta = \bar{\delta}$, *then the minimal solution is concave on* $[0, 1)$ *with* $u''(1) = 0$;
ii) *if* $\bar{\delta} < \delta \leqq \delta^*$, *then all solutions are bell-shaped*; *and*
iii) *if* $0 < \delta < \bar{\delta}$, *then the minimal solution is concave down on* $[0, 1]$ *and all other solutions are bell-shaped.*

PROOF. (a). For $n = 1$, $u''(r) = -\delta e^{u(r)} < 0$ on $[0, 1]$ and concavity is obvious.

(b) and (c). For $n \geqq 2$, note that $u''(0) = -(\delta/n)e^{\alpha} < 0$ and that $u''(1) = (n - 1)\beta - \delta$ so $\operatorname{sgn} u''(1) = \operatorname{sgn}((n - 1)\beta - \delta)$. Thus if the points of inflection are unique (if they exist) and if the bifurcation curve D intersects $\beta = L(\delta) = \delta/(n - 1)$ uniquely on the minimal branch, then our assertions (b) and (c) hold. For if $(\delta, \beta) \in D$ satisfies $\beta > \delta/(n - 1)$, then $u''(1) > 0$ and $u''(0) < 0$ imply there exists $R \in (0, 1)$ such that $u''(R) = 0$ and $(R, u(R))$ is a point of inflection. By the uniqueness of inflection points, the solution $u(r)$ corresponding to (δ, β) is therefore bell-shaped. If $(\delta, \beta) \in D$ satisfies $\beta < \delta/(n - 1)$, then $u''(1) < 0$ and $u''(0) < 0$ imply no inflection points or more than one. Uniqueness (see Lemma 5.5) rules out this latter case.

For $n = 2$, since $D = \{(\delta, \beta): \delta > 0, \beta^2 - 4\beta + 2\delta = 0\}$ the result is immediate assuming uniqueness of points of inflection since D obviously intersects $\beta = \delta$ at (2, 2).

For $n \geqq 3$, to show that D intersects $\beta = L(\delta) = \delta/(n-1)$ uniquely, we prove a sequence of lemmas. By Theorem 3.6, we have $\delta^* < \lambda_1/e < 2(n-1)$.

LEMMA 5.2. *The heteroclinic orbit D and the graph of* $L(\delta) = \delta/(n-1)$ *intersect in at least one point where* $n-1 < \delta < 2(n-1)$, $1 < \beta < 2$.

PROOF. Let $\beta(\delta)$ be the arc of D which originates at (0, 0) and terminates at $(\delta^*, 2)$. From (4.7),

$$\beta'(\delta) = \frac{d\beta}{d\delta} = \frac{(n-2)\beta - \delta}{\delta(\beta - 2)}. \tag{5.1}$$

For any $(\delta, \beta) \in D$ with $\beta = 2$, we have $\delta < 2(n-1)$ and thus $\beta(\delta)$ reaches $\beta = 2$ at $\delta < 2(n-1)$. Since $L(2(n-1)) = 2$, $\beta(\delta)$ intersects $L(\delta)$ at $\delta < 2(n-1)$.

If $\beta(\delta_0) = L(\delta_0) = \beta_0$ for $\beta_0 \in (0, 1]$, $\delta_0 \in (0, n-1]$, then $\beta'(\delta_0) = ((n-1)(2-\beta_0))^{-1} < (n-1)^{-1} = L'(\delta_0)$. Thus, if there are any points of intersection for $\beta_0 \in (0, 1]$, then there is only one. This implies $\beta'(0) \geqq (n-1)^{-1}$. But $\beta'(0) = n^{-1} < (n-1)^{-1}$. Thus there are no points of intersection for $\beta \leqq 1$.

Since $\beta'(\delta) = ((n-1)(2-\beta))^{-1} > (n-1)^{-1}$ for any (δ, β) with $\beta = L(\delta)$ and $n-1 < \delta < 2(n-1)$ the intersection is unique.

REMARK. For $n \geqq 10$, it is clear from the geometry that this point of intersection is unique. It remains to be shown that, for $3 \leqq n \leqq 9$, there are no points of intersection other than the one just constructed.

LEMMA 5.3. *For* $3 \leqq n \leqq 9$, *D intersects L uniquely.*

PROOF. By Lemma 5.2, $D \cap L \neq \varnothing$. Other than the point of intersection on the lower branch of D we will now show that there are no other intersections as D spirals toward $(2(n-2), 2)$.

Let R be the region bounded by:

a) $n = 3$,

$$L_1 = \{(\delta, \beta): \delta = 4,\ 2 \leqq \beta \leqq 3\},$$
$$L_2 = \{(\delta, \beta): \beta = 3,\ 3 \leqq \delta \leqq 4\},$$
$$L_3 = \{(\delta, \beta): \beta = -\frac{1}{4}\delta^2 + \frac{3}{2}\delta + \frac{3}{4},\ 1 \leqq \delta \leqq 3\},$$
$$L_4 = \{(\delta, \beta): \delta = 1,\ 1 \leqq \beta \leqq 2\},$$
$$L_5 = \{(\delta, \beta): \beta = 1,\ 1 \leqq \delta \leqq 2\},$$

and

$$L_6 = \{(\delta, \beta): \beta = \frac{1}{2}\delta, 2 \leqq \delta \leqq 4\}.$$

b) $n \geqq 4$,

$$L_1 = \left\{(\delta, \beta): \delta = 2(n-1), 2 \leqq \beta \leqq 2\frac{n-1}{n-2}\right\},$$

$$L_2 = \{(\delta, \beta): \beta = 2\frac{n-1}{n-2}, n \leqq \delta \leqq 2(n-1)\},$$

$$L_3 = \{(\delta, \beta): \beta = \frac{\delta}{n-1} + 1, n-2 \leqq \delta \leqq n\},$$

$$L_4 = \{(\delta, \beta): \delta = n-2, 1 \leqq \beta \leqq 2\},$$

$$L_5 = \{(\delta, \beta): \beta = 1, n-2 \leqq \delta \leqq n-1\},$$

and

$$L_6 = \{(\delta, \beta): \beta = \frac{\delta}{n-1}, n-1 \leqq \delta \leqq 2(n-1)\}.$$

With the observation that $\beta'(\delta) < 0$ on $S = \{(\delta, \beta): \delta > 2(n-2), 2 < \beta < \delta/(n-2)\} \cup \{(\delta, \beta): 0 < \delta < 2(n-2), \delta/(n-2) < \beta < 2\}$ and $\beta'(\delta) > 0$ in $\{(\delta, \beta): \beta > 0, \delta > 0\} - S$, we see that the heteroclinic orbit cannot leave R through L_1 or L_2. The orbit cannot leave R through L_3, L_4, L_5, or L_6 since the slope at such a crossing would not agree with $\beta'(\delta)$ evaluated on these sets. Thus the first point of intersection of D with L_6 is the only such point.

We now show that points of inflection for the graph $u(r)$ on $[0, 1]$ are unique.

LEMMA 5.4. *Consider* (4.1) *with* $(n-1)\beta - \delta = 0$, *for* $n \geqq 2$. *There exists one and only one solution* $u(r)$ *of* (4.1) *with* $\delta = (n-1)\beta$.

REMARK. This shows that there is a unique solution of (4.1) with $u''(1) = 0$.

PROOF. For $n \geqq 3$, D intersects L at the unique point $(\bar{\delta}, \bar{\beta})$. This gives the unique solution $u(r)$. For $n = 2$, D is given by $\beta^2 - 4\beta + 2\delta = 0$ which intersects $\beta - \delta = 0, \delta > 0$, uniquely at $\delta = 2, \beta = 2$.

LEMMA 5.5. *Let* $u(r) \in C^2([0, 1])$ *be a solution of* (4.1) *for* $n \geqq 2$. *Then* u *has at most one inflection point.*

PROOF. Let $R \in (0, 1)$ be the first such that $u''(R) = 0$. Define $m = u'(R)$, then $u(R) = \ln((-m(n-1))/(\delta R))$. In (4.1), let $r = sR$, $v(s) = u(r) - u(R)$. Then, for $s \in [0, 1]$, we have

$$(5.1)\quad \begin{gathered} v'' + \frac{n-1}{s} v' + \bar{\delta} e^{v} = 0 \\ v'(0) = 0, \qquad v(1) = 0, \quad v'(1) = -\bar{\beta} \\ (n-1)\bar{\beta} - \bar{\delta} = 0 \end{gathered}$$

where $\bar{\delta} = -(n-1)mR > 0$ and $\bar{\beta} = -mR > 0$.

By Lemma 5.4, there exists a unique $(\bar{\delta}, \bar{\beta})$ and a unique solution $v(s)$ satisfying $(n-1)\bar{\beta} - \bar{\delta} = 0$. Thus, $v''(1) = 0$. Since $u''(r) < 0$ on $0 \leqq r < R$, $v''(s) < 0$ for $0 \leqq s < 1$.

Suppose there exists P, $0 < R < P \leqq 1$ such that $u''(R) = u''(P) = 0$. Set $l = u'(P)$. Then $u(P) = \ln((-l(n-1))/(\delta P))$. Make a change of variables $r = sP$ and $v(s) = u(r) - u(P)$. Restricting $s \in [0, 1]$ we have that $v(s)$ satisfies (5.1) with $\delta = -(n-1)lP$ and $\beta = -lP > 0$. By Lemma 5.4, we must have $v''(s)$ on $[0, 1)$. But $v''(R/P) = P^2 u''(R) = 0$ with $0 < R/P < 1$ is a contradiction.

With this sequence of lemmas, the proof of Theorem 5.1 is now complete.

For the perturbed Gelfand problem on $\Omega = B_1$, some information about the solution profiles can be given. Consider

$$(5.2)\quad \begin{gathered} u'' + \frac{n-1}{r} u' + \delta \exp\left(\frac{u}{1+\varepsilon u}\right) = 0, \qquad 0 < r < 1 \\ u'(0) = 0, \qquad u(1) = 0. \end{gathered}$$

Set, as before, $\alpha = u(0)$, $\beta = -u'(1)$.

The following theorem is proven in **[2]**.

THEOREM 5.6.

a) *For $n = 1$, every solution of* (5.2) *is concave down.*

b) *For $n = 2$, all solutions are bell-shaped or concave down.*

c) *For $n \geqq 3$ and $\varepsilon > 0$ sufficiently small, there exist $\delta_1(\varepsilon) < \delta_2(\varepsilon)$ such that the minimal solution is concave for $0 < \delta < \delta_1(\varepsilon)$ and not concave down for $\delta_2(\varepsilon) < \delta < \delta^*(\varepsilon)$.*

6. Blow-up for the ignition model. In this final section, we discuss the problem of blow-up (or thermal runaway) for the ignition model (2.3) (2.4) for a solid fuel in a bounded container $\Omega \subset \mathbf{R}^n$. For simplicity in our discussion, we assume $\theta_0(x) \equiv 0$.

It is now well-known (e.g., **[3]**) that:

THEOREM 6.1.

a) *For $0 < \delta < \delta^*$, where δ^* is as in* §3, *the problem* (2.3)–(2.4) *has a unique solution $\theta(x, t)$ on $\bar{\Omega} \times [0, \infty)$ with $0 \leqq \theta(x, t) \leqq u_{\min}(x)$ where $u_{\min}$ is the minimal solution of* (2.5)–(2.6).

b) *For* $\delta > \delta^*$, (2.3)–(2.4) *has a unique solution* $\theta(x, t)$ *on* $\bar{\Omega} \times [0, t^*)$ *where* $1/\delta < t^* \leqq \infty$ *and* $\lim_{t \to t^*} \sup_x u(x, t) = +\infty$.

Thus, nonexistence occurs by having blow-up in the L_∞ – norm and thermal runaway or blow-up occurs at t^*. If t^* is finite, we say that we have ignition and such behavior may characterize a thermal explosion. A natural problem therefore is to determine values of δ which result in finite time blow-up.

In **[3]**, we showed that the solution $\theta(x, t)$ of (2.3)–(2.4) blows up in finite time t^* if $\delta > \delta_B \equiv \lambda_1/e$, where λ_1 is the first eigenvalue of (3.3) and

$$\frac{1}{\delta} < t^* < T \equiv \int_0^\infty \frac{dz}{\delta e^z - \lambda_1 z} < \infty. \tag{6.1}$$

The parameter value δ_B gives an upper bound for the classical Frank-Kamenetski critical value δ^* (see §3).

This leaves open the question: does thermal runaway occur in finite time for $\delta \in (\delta^*, \delta_B]$? This was answered positively by Lacey **[13]** if δ^* belongs to the spectrum of (2.5)–(2.6), the Gelfand problem which means (2.5)–(2.6) has a positive solution for $\delta = \delta^*$ or if $\Omega = B_1 \subset R^n$. We include here a proof of the first result which is a slight improvement of that found in **[13]**. Both this result and that in **[3]** are proven by a comparison argument using an essential idea of Kaplan **[10]**.

THEOREM 6.1 *If* $\delta > \delta^*$ *and if* δ^* *belongs to the spectrum of* (2.5)–(2.6), *then the solution* $\theta(x, t)$ *of* (2.3)–(2.4) *blows up in finite time* t^* *where* $t^* < (2/\delta^*)^{1/2} \cdot \pi \cdot (\delta - \delta^*)^{1/2}$.

PROOF. Let $w^*(x)$ be the solution of (2.5)–(2.6) for $\delta = \delta^*$. Then the first variational problem

$$\begin{aligned} -\Delta\phi &= (\delta^* e^{w^*(x)})\,\phi, \quad x \in \Omega \\ \phi(x) &= 0, \quad x \in \partial\Omega \end{aligned} \tag{6.2}$$

has a solution $\phi(x) > 0$ on Ω (see **[1]**) which can be normalized to $\int_\Omega \phi(x)dx = 1$.

Define $v(x, t) = \theta(x, t) - w^*(x)$, Then

$$\begin{aligned} v_t &= \theta_t = \delta e^\theta + \Delta\theta = (\delta - \delta^*)e^\theta + \delta^* e^{w^*+v} + \Delta w^* + \Delta v \\ v_t &= (\delta - \delta^*)e^\theta + \delta^*(e^v - 1 - v)e^{w^*} + \delta^* v e^{w^*} + \Delta v \end{aligned} \tag{6.3}$$

Let $a(t) = \int_\Omega \phi(x)v(x, t)dx$. Then $a(t) \leqq \max_{x \in \Omega} \theta(x, t)$ and $a(0) = \int_\Omega \phi(x)v(x, 0)dx \geqq -\max_{\bar{\Omega}} w^*(x)$. Multiplying (6.3) by ϕ and integrating over Ω, we have

$$a'(t) = (\delta - \delta^*)\int_\Omega \phi\, e^\theta\, dx + \delta^* \int_\Omega \phi(ev - 1 - v)e^{w^*}\, dx. \tag{6.4}$$

Since $(e^v - 1 - v)e^{w^*} \geqq (\delta^* v^2)/2$ and by Jensen's inequality, we have $\delta^* \int_\Omega \phi(e^v - 1 - v)e^{w^*}\,dx \geqq (\delta^*/2)a^2$. Hence, $a(t)$ satisfies the differential inequality:

$$a'(t) \geqq (\delta - \delta^*) + \delta^* \frac{a^2}{2} \tag{6.5}$$
$$a(0) \geqq -\max_\Omega w^*(x) = -w_m^*.$$

The solution of

$$\phi' = (\delta - \delta^*) + \frac{\delta^*}{2}\phi^2 \tag{6.6}$$
$$\phi(0) = -w_M^*$$

is

$$\phi(t) = \left(\frac{2(\delta - \delta^*)}{\delta^*}\right)^{1/2} \tan\left(\left(\frac{\delta^*(\delta - \delta^*)}{2}\right)^{1/2} t + \tan^{-1} K\right)$$

where $K = -(\delta^*/(2\cdot(\delta - \delta^*)))^{1/2}\, w_m^*$. The function $\phi(t)$ blows up before $t_A = (2/\delta^*)^{1/2}\cdot\pi\cdot(\delta - \delta^*)^{-1/2}$. Thus, $\sup \theta(x, t) \geqq a(t) \geqq \phi(t)$ and $t^* < t_A$.

If δ^* does not belong to the spectrum of the Gelfand problem and if $\Omega \neq B_1$, then we still do not know if t^* is finite or infinite for $\delta \in (\delta^*, \delta_B]$. Another closely related problem is that of determining blow-up in different norms. For example, if $b(t) = \int_\Omega \theta(x, t)dx$ where $\theta(x, t)$ is the solution of of (2.3)–(2.4), what happens as time advances? This is the problem of L_1-blow up. This problem is motivated by the following

THEOREM 6.2. *If $\theta(x, t)$ blows up in the L_1 – sense as $t \to t^{**} \leqq \infty$, then $t^* < \infty$, i.e., $\theta(x, t)$ blows up in the L_∞-sense in finite time.*

PROOF. Since $b(t) \to \infty$ as $t \to t^{**} \leqq \infty$, we have that $\int_\Omega \theta(x, t)\psi(x)dx \to \infty$ as $t \to t^{**}$ where $\psi(x)$ is the solution of (3.3) associated with the first eigenvalue λ_1 with $\int_\Omega \psi\,dx = 1$.

Choose $M > 0$ sufficiently large so that $\int_M^\infty dz/(\delta e^z - \lambda_1 z) < \infty$. Let $a(t) = \int_\Omega \psi(x)\theta(x, t)dx$ and let t_M be the first time that $M = a(t_M)$. Let $\Psi(t)$ be the solution of

$$z' = \delta e^z - \lambda_1 z \tag{6.7}$$
$$z(t_M) = M.$$

Then $0 \leqq t - t_M = \int_M^{\Psi(t)} dz/(\delta e^z - \lambda_1 z) < \infty$. But $\Psi(t) \leqq a(t) \leqq \max_\Omega\theta(x, t)$ on $t_M \leqq t \leqq t_\Psi < \infty$ with $\Psi(t) \to \infty$ as $t \to t_{\overline{\Psi}}$. Hence $t^* < t_\Psi < \infty$.

The new problem is to determine those δ for which $b(t)$ becomes infinite as $t \to t^{**} \leqq \infty$. By using an argument similar to the proof of Theorem 6.1, one can show that if $\delta > \lambda_1$, the first eigenvalue of (3.3), then $t^{**} < \infty$.

The solution $\theta(x, t)$ of (2.3)-(2.4) can be expressed as

$$\theta(x, t) = \delta \int_0^t d\tau \int_\Omega G(x, y, t - \tau) e^{\theta(y,\tau)} dy \tag{6.8}$$

where G is the Green's function for

$$\begin{aligned} \theta_t - \Delta\theta &= 0 \\ \theta(x, t) &= 0 \end{aligned} \tag{6.9}$$

on the parabolic boundary of $\Omega \times [0, T]$. Integrating (6.8) over Ω, we have

$$b(t) = \delta \int_0^t d\tau \int \gamma(y, t - \tau) e^{\theta(y,\tau)} dy \tag{6.10}$$

where $\gamma(y, t) = \int_\Omega G(x, y, t)dx$. We note that $\gamma(y, t)$ is the solution of

$$\begin{aligned} u_t - \Delta u &= 0 \\ u(x, 0) &= 1, \quad x \in \Omega \\ u(x, t) &= 0, \quad x \in \Omega, t > 0. \end{aligned} \tag{6.11}$$

Applying the Tchebecheff inequality [6] to (6.10), we get

$$b(t) \geqq \frac{1}{\operatorname{vol}\Omega} \int_0^t d\tau \left(\int_\Omega \gamma(y, t - \tau)dy\right)\left(\int_\Omega e^{\theta(y,\tau)} d\tau\right) \tag{6.12}$$

Set

$$m(t) = \frac{1}{\operatorname{vol}\Omega} \int_\Omega \gamma(y, t)dy. \tag{6.13}$$

Applying Jensen's inequality to the last integral in (6.12), we have

$$b(t) \geqq \delta \int_0^t m(t - \tau) e^{b(\tau)} d\tau \tag{6.14}$$

Thus, again applying the Tchebycheff inequality,

$$b(t) \geqq \frac{\delta}{t} \int_0^t m(\tau)d\tau \int_0^t b(\tau)d\tau \geqq \delta\left(\frac{1}{T} \int_0^t m(\tau)d\tau\right) \int_0^t e^{b(\tau)} d\tau,$$

for $0 \leqq t \leqq T$. Set $C_T = (1/T) \int_0^T m(\tau)d\tau$; then

$$b(t) \geqq \delta\, C_T \int_0^T e^{b(\tau)} d\tau. \tag{6.15}$$

If $r(t) = \int_0^t e^{b(\tau)} d\tau$, then $r'(t) = e^{b(t)}$ and $\ln r'(t) \geqq \delta\, C_T\, r(t)$. Thus,

$$b(t) \geqq \delta\, C_t\, r(t) = -\ln\left(1 - \delta \int_0^t m(\tau)d\tau\right), \tag{6.16}$$

for all $t \geqq 0$, and $b(t)$ blows up for $\delta \geqq (\int_0^\infty m(\tau)d\tau)^{-1}$. Thus, we have proven.

THEOREM 6.3. *If* $\delta \geqq (\int_0^\infty m(\tau)d\tau]^{-1}$, *then* $b(t) \to \infty$ *as* $t \to t^{**}$ *where* $m(t)$ *is given by* (6.13).

Many open problems remain. For example, does $\theta(x, t)$ blow-up in the L_∞-sense in finite time t^* for $\delta > \delta^*$ and any domain Ω? Can one improve the estimate on δ given by Theorem 6.3 for L_1-blow up? Can one describe how blow up occurs? Weissler **[18]** has shown for a one-dimensional problem with polynomial nonlinearity u^α that blow up in L_∞-sense occurs at a single point. Is the same true for the ignition model?

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