ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 16, Number 2, Spring 1986

A SIMPLE PROBLEM FOR THE SCALAR WAVE EQUATION ADMITTING SURFACE-WAVE AND AH-WAVE SOLUTIONS

DAVID S. GILLIAM AND JOHN R. SCHULENBERGER

It has recently been found that the surface- and AH-wave solutions of a classical problem for Maxwell's equations [1] are generated by solutions of a simple problem for the scalar wave equation in R^3, namely,

$$(1)\qquad \begin{aligned} &(\partial_t^2 - c_0^2\Delta)\phi(x, t) = 0,\ x_3 > 0,\ t > 0 \\ &(\partial_t^2 + \kappa\partial_t - c^2\Delta)\phi(x, t) = 0,\ x_3 < 0,\ t > 0, \\ &\phi(x, 0^+) = f(x),\ \partial_t\phi(x, 0^+) = F(x), \\ &c^2\partial_3\phi(x', 0^-, t) = c_0^2\partial_3\phi(x', 0^+, t), \\ &\partial_t\phi(x', 0^-, t) + \kappa\phi(x', 0^-, t) = \partial_t\phi(x', 0^+, t), \end{aligned}$$

where $c_0 > c > 0$ and $\kappa > 0$ are constants and $x' = (x_1, x_2)$. In the present note we show that problem (1) is uniquely solvable for a certain class of intial data (f, F) and present the explicit form of the surface- and AH-wave solutions to (1). We further show how to construct the corresponding solutions to the classical problem for Maxwell's equation (2) from the solutions to (1). Surface-wave solutions of (1) are superpositions of modes with frequencies having nonzero real and imaginary parts which decay exponentially in space away from the interface $\{x_3 = 0\}$. AH-wave solutions are super-positions of modes having this same spatial decay, but their frequencies have no real part, so they simply decay in time without propagating—rather peculiar wave-like behavior.

Denoting by $\chi_\pm = \chi_\pm(x_3)$ the characteristic functions of the half spaces $R^3_\pm = \{x \in R^3\colon \pm\ x_3 > 0\}$ and defining the 6×6 diagonal matrices $E_\pm = \mathrm{diag}[\varepsilon_\pm I_3, \mu_\pm I_3]$, $B = \mathrm{diag}(\sigma I_3, 0_{3\times 3})$, where I_3 is the 3×3 identity matrix, the Cauchy problem for Maxwell's equations in two semi-infinite media (the lower of which is conducting) separated by the plane boundary $\{x_3 = 0\}$ can be written

$$(2)\qquad \begin{aligned} \partial_t f(x, t) = \{&\chi_+(x_3)E_+^{-1} A(\partial) \\ &+ \chi_-(x_3)E_-^{-1}[A(\partial) + B]\}f(x. t),\ x_3 \neq 0,\ t > 0, \\ f(x, 0^+) = &f_0(x) \in L_2(R^3, C^6), \end{aligned}$$

Received by the editors on October 10, 1984

where

$$A(\partial) = \begin{pmatrix} 0 & \mathrm{rot} \\ -\mathrm{rot} & 0 \end{pmatrix}, \quad \mathrm{rot} = \begin{pmatrix} 0 & -\partial_3 & \partial_2 \\ \partial_3 & 0 & -\partial_1 \\ -\partial_2 & \partial_1 & 0 \end{pmatrix}$$

and $f = \binom{E}{H}$ is the six-vector consisting of the electric (E) and magnetic (H) fields. If we now suppose that $f = f(x_2, x_3)$ is independent of x_1, then the system (2) splits into two 3×3 systems for TE (transverse electric) and TM (transverse magnetic) waves. The latter system has the form

$$\partial_t \begin{pmatrix} H_1 \\ E_2 \\ E_3 \end{pmatrix} = \chi_+ \operatorname{diag}(\mu_+^{-1}, \varepsilon_+^{-1}, \varepsilon_+^{-1}) \begin{pmatrix} 0 & \partial_3 & -\partial_2 \\ \partial_3 & 0 & 0 \\ -\partial_2 & 0 & 0 \end{pmatrix} \begin{pmatrix} H_1 \\ E_2 \\ E_3 \end{pmatrix} +$$
$$\chi_- \operatorname{diag}(\mu_-^{-1}, \varepsilon_-^{-1}, \varepsilon_-^{-1}) \begin{pmatrix} 0 & \partial_3 & -\partial_2 \\ \partial_3 & -\sigma & 0 \\ -\partial_2 & 0 & -\sigma \end{pmatrix} \begin{pmatrix} H_1 \\ E_2 \\ E_3 \end{pmatrix}.$$

If we seek ${}^t(H_1, E_2, E_3)$ in the form

$${}^t(H_1, E_2, E_3) = {}^t(\partial_t, \phi, \varepsilon_+^{-1}\partial_3\phi, -\varepsilon_+^{-1}\partial_2\phi), \; x_3 > 0,$$
$${}^t(H_1, E_2, E_3) = {}^t(\partial_t\phi + \sigma\varepsilon_-^{-1}\phi, \varepsilon_-^{-1}\partial_3\phi, -\varepsilon_-^{-1}\partial_2\phi), \; x_3 < 0,$$

then for the scalar function ϕ we obtain precisely problem (1) with Δ replaced by $\Delta_2 = (\partial_2^2 + \partial_3^2)$, (x', x_3) by (x_2, x_3) and $c_0^2 = (\varepsilon_+\mu_+)^{-1}$, $c^2 = (\varepsilon_-\mu_-)^{-1}$, $\kappa = \varepsilon_-^{-1}\sigma$ where in order to simplify notation, here and throughout the rest of the paper, we have assumed $\mu_+ = \mu_- = 1$ so that $c_0^2 = \varepsilon_+^{-1}$ and $c^2 = \varepsilon_-^{-1}$. If f is not independent of one of the coordinates then it is no longer possible to reduce problem (2) to a problem for the scalar wave equation, but it is remarkable that the surface-wave and AH-wave components of the solution of (2) continue to be generated by solutions of Eq. (1): they are obtained by applying simple (vector) differential operators to solutions of (1) (see [1] and Theorem 3 below). This bears witness to the degenerate (essentially two-dimensional) structure of these components of the solution of (2).

We now formulate problem (1) in a form which is more suitable for establishing the existence and uniqueness result, namely, as an evolution equation in Hilbert space. It will be shown that the spatial operator is maximal dissipative and hence generates a (C_0) contraction semigroup which delivers the solutions to (1) for arbitrary initial data in the domain of the infinitesimal generator of the semigroup.

Defining

$$H = i\left\{\chi_+\begin{pmatrix} 0 & 1 \\ c_0^2\Delta & 0\end{pmatrix} + \chi_-\begin{pmatrix} -\kappa & 1 \\ c^2\Delta & 0\end{pmatrix}\right\} \equiv \chi_+h_+ + \chi_-h_-, \tag{3}$$

$$f(x, t) = {}^t(f_1(x, t), f_2(x, t)),$$

we can write problem (1) in the form

$$\begin{aligned} i\partial_t f(x, t) &= Hf(x, t),\ t > 0,\ x_3 \neq 0, \\ c^2\partial_3 f_1(x', 0^-, t) &= c_0^2\partial_3 f_1(x', 0^+, t), f_2(x', 0^-, t) = f_2(x', 0^+, t), \\ f(x, 0^+) &= f_0(x), \end{aligned} \tag{4}$$

where $f_0(x) = {}^t(f, F + \kappa f)$ and $f_1(x, t) \equiv \phi(x, t)$. (The factor "$i$" here is a nuisance, but we retain it in order that the spectrum of H coincide with that of the operator in [1].)

We denote by $\mathscr{D}(R^3_\pm)$, $\mathscr{D}(\bar{R}^3_\pm)$ the spaces of smooth functions with bounded support in $R^3_\pm$, $\bar{R}^3_\pm$ and define

$$\begin{aligned} \mathscr{D}(\mathring{H}) = \{f = {}^t(f_1, f_2) &: f_1, f_2 \in \mathscr{D}(\bar{R}^3_\pm), \\ c^2\partial_3 f_1(x', 0^-) &= c_0^2\partial_3 f_1(x', 0^+), \\ f_2(x', 0^-) &= f_2(x', 0^+)\}. \end{aligned}$$

We define $\mathscr{H}$ to be the completion of $\mathscr{D}(\mathring{H})$ in the norm

$$|f|^2 = \int_{R^3_+} [c_0^2|\nabla f_1|^2 + |f_2|^2] + \int_{R^3_-} [c^2|\nabla f_1|^2 + |f_2|^2];$$

the inner product in $\mathscr{H}$ is

$$\begin{aligned} \langle f, g\rangle &= \int_{R^3_+} [c_0^2\nabla\bar{f}_1 \cdot \nabla g_1 + \bar{f}_2 g_2] + \int_{R^3_-} [c^2\nabla\bar{f}_1 \cdot \nabla g_1 + \bar{f}_2 g_2] \\ &\equiv c_0^2(\nabla f_1, \nabla g_1)_+ + (f_2, g_2)_+ + c^2(\nabla f_1, \nabla g_1)_- + (f_2, g_2)_-. \end{aligned}$$

We define the operator $\mathring{H}$ in $\mathscr{H}$ as the differential operator (3) on $\mathscr{D}(\mathring{H})$, i.e., for $f \in \mathscr{D}(\mathring{H})$ and any $g \in \mathscr{H}$

$$\begin{aligned} &-i\langle g, \mathring{H}f\rangle \\ &= c_0^2(\nabla g_1, \nabla f_2)_+ + c_0^2(g_2, \Delta f_1)_+ + c^2(\nabla g_1, \nabla(f_2 - \kappa f_1))_- + c^2(g_2, \Delta f_1)_-. \end{aligned}$$

It is clear that $\mathring{H}$ is densely defined.

THEOREM 1. *The operator $\mathring{H}$ on $\mathscr{D}(\mathring{H})$ is dissipative, and its closure H is maximal dissipative. For any $f \in \mathscr{D}(H)$ problem* (4) *is thus solved by $S(t)f$ where $S(t)$ "$=$" $\exp(-iHt)$ is a contractive semigroup in $\mathscr{H}$.*

PROOF. To prove that $\mathring{H}$ is dissipative we must show [4] that for any $f \in \mathscr{D}(\mathring{H})$, $-i\langle f, \mathring{H}f\rangle + i\langle \mathring{H}f, f\rangle \leqq 0$. Let $f \in \mathscr{D}(\mathring{H})$; integrating by parts, we compute

$$-i\langle f, Hf\rangle_- + i\langle Hf, f\rangle_- = -2\kappa\int_{R^3_-}|\nabla f_1|^2 + c^2\int_{R^2}[\partial_3\bar{f}_1(x',0^-)f_2(x',0^-) + \partial_3 f_1(x',0^-)\bar{f}_2(x',0^-)],$$

$$-i\langle f, Hf\rangle_+ + i\langle Hf, f\rangle_+ = -c_0^2\int_{R^3_+}[\partial_3\bar{f}_1(x',0^+)f_2(x',0^+) + \partial_3 f_1(x',0^+)\bar{f}_2(x',0^+)].$$

Adding these two expressions and using the first condition of (4), we obtain

$$\begin{aligned}-i\langle f, Hf\rangle + i\langle Hf, f\rangle &= -2\kappa\int_{R^3_-}|\nabla f_1|^2 \\ &\quad + c_0^2\int_{R^2}\{\partial_3\bar{f}_1(x',0^+)[f_2(x',0^-) - f_2(x',0^+)] \\ &\quad + \partial_3 f_1(x',0^+)[\bar{f}_2(x',0^-) - \bar{f}_2(x',0^+)]\} \\ &= -2\kappa\int_{R^3_-}|\nabla f_1|^2 \leqq 0,\end{aligned}$$

since $f_2(x', 0^-) = f_2(x', 0^+)$. Thus, $-i\langle f, Hf\rangle + i\langle Hf, f\rangle \leqq 0$ for $f \in \mathscr{D}(\mathring{H})$. Since H is dissipative, the range of $H - ikI$, $k > 0$, on $\mathscr{D}(H)$ is closed **[4]**. To show that H is maximal, we must show that this range is also dense. This we do in the appendix.

One way to obtain a representation of solutions of (4) is to construct the resolvent $(H - \zeta I)^{-1}$ of H and integrate it around the spectrum as in **[2]**. To do this we first write down the resolvents of the operators h_+ and h_- of (3), add terms to each corresponding to reflected and transmitted waves, and match these two expressions at the interface $\{x_3 = 0\}$ to satisfy condition (4). This involves solving a system of linear equations with determinant $D(\xi, \zeta)$, $\xi \in R^2$. The construction of $(H - \zeta I)^{-1}$ is thus possible for all ζ not in the spectrum of h_+ or h_- for which $D(\xi, \zeta) \neq 0$. The zeros of this function give rise to the surface and AH modes. We now describe this function and its zeros.

We define two families of analytic functions with nonnegative imaginary parts which depend on the parameter $|\xi|^2 > 0$, $\xi = (\xi_1, \xi_2)$, by

$$\text{(5)}\qquad \begin{aligned}\mathring{\tau}(\xi,\zeta) &= c_0^{-1}\sqrt{[\zeta^2 - c_0^2|\xi|^2]} \\ \tau(\xi,\zeta) &= c^{-1}\sqrt{[\zeta(\zeta + i\kappa) - c^2|\xi|^2]}\end{aligned}$$

where for $c|\xi| < \kappa/2$ the branch cut of the latter function is finite. The function $D(\xi, \zeta)$ is now

$$\text{(6)}\qquad D(\xi,\zeta) = c_0^{-2}\zeta\tau(\xi,\zeta) + c^{-2}(\zeta + i\kappa)\mathring{\tau}(\xi,\zeta).$$

THEOREM 2 **[1]**. *For each $|\xi| \neq 0$ the function $D(\xi, \zeta)$ has three roots*

$$\text{(7)}\qquad \mathring{m}(s) = -im_0(s),\ m_+(s) = -\bar{m}_-(s) = m_1(s) + im_2(s),\ s = |\xi|^2 > 0,$$

with m_0, $m_1 > 0$ and $m_2 < 0$. The function $m_0(s)$ decreases monotonically from $\kappa = m_0(0^+)$ to $m_0(\infty) = \kappa c_0^2(c_0^2 + c^2)^{-1}$. The real part of $m_+(s)$ has the representation $m_1(s) = \alpha(s)\sqrt{s}$, where $\alpha(s)$ increases monotonically from $c_0 = \alpha(0^+)$ to $\sqrt{(c_0^2 + c^2)} = \alpha(\infty)$. The imaginary part of $m_+(s)$ is $m_2(s) = 1/2(m_0(s) - \kappa)$ which thus decreases monotonically from $0 = m_2(0^+)$ to $m_2(\infty) = -1/2\kappa c^2(c_0^2 + c^2)^{-1}$.

We sketch the proof of the theorem; full details can be found in **[1]**. From (5), (6)

$$\zeta D(\xi, \zeta) = c_0^{-2}\zeta^2\tau + c^{-2}\zeta(\zeta + i\kappa)\mathring{\tau}$$
$$= \tau(\mathring{\tau}^2 + |\xi|^2) + \mathring{\tau}(\tau^2 + |\xi|^2) = (\mathring{\tau} + \tau)(\mathring{\tau}\tau + |\xi|^2),$$

and hence $D(\xi, \zeta)$ admits a root $\zeta = m(\xi) \neq 0$ if and only if $\tau_m\mathring{\tau}_m \equiv \tau(\xi, m)\mathring{\tau}(\xi, m) = -|\xi|^2$. This relation and the equation $D(\xi, m) = 0$ imply that $m = m(|\xi|^2)$ is a root of the polynomial

$$q(\zeta) = \zeta^3 + i\kappa\zeta^2 - (c_0^2 + c^2)|\xi|^2\zeta - i\kappa c_0^2|\xi|^2. \tag{8}$$

We observe that $q(m) = 0$ if and only if $q(-\bar{m}) = 0$, so the roots have the form shown in (7). For the root $m(s) = -im_0(s)$ the function $m_0(s)$ is the positive root of

$$p(x) = x^3 - \kappa x^2 + (c_0^2 + c^2)sx - \kappa c_0^2 s, \quad s = |\xi|^2.$$

We note that $p(\kappa) = sc^2\kappa > 0$, $p(1/2\kappa) = -2^{-3}\kappa^3 - 1/2(c_0^2 - c^2)\kappa s < 0$, so $m_0(s) \in (1/2\kappa, \kappa)$ for all $s > 0$. Using the fact that $m_0(s) \in (1/2\kappa, \kappa)$ and letting $s \to 0$ in $0 = p(m_0(s))$, we see that $m_0(0^+) = \kappa$. Dividing $0 = p(m_0(s))$ by s, letting $s \to \infty$, and again using the fact that $m_0(s) \in (1/2\kappa, \kappa)$, we find $m_0(\infty) = \kappa c_0^2(c_0^2 + c^2)^{-1} > \kappa/2$. Differentiating $0 = p(m_0(s))$ with respect to s, we see that $m_0(s)$ decreases monotonically. The expression for $m_2(s)$ follows from (8), since $m_+ + m_- - im_0 = -i\kappa$. It likewise follows from (8) that $m_0|m_+|^2 = \kappa c_0^2 s$, and from this we obtain the representation for $m_1(s)$ with $\alpha^2(s) = \kappa c_0^2[m_0(s)]^{-1} - 4^{-1}s^{-1}[m_0(s) - \kappa]^2$.

With $\mathring{\tau}(\xi, m) = \mathring{\tau}_m$, $\tau(\xi, m) = \tau_m$, $m = m_\pm, \mathring{m}$, we now know that $c_0^{-2}m\tau_m + c^{-2}(m + i\kappa)\mathring{\tau}_m = 0$. Using this fact and (5), it is easy to verify that if $\ell(\xi)$ is compactly supported, say, then the following functions are solutions of problem (1) with initial data obtained by setting $t = 0$ in $L_m(x, t)$ and $\partial_t L_m(x, t)$:

$$L_m(x, t) = c_0^{-2}\chi_+(x_3)(2\pi)^{-1}\int_{R^2}\exp[ix'\xi + i\mathring{\tau}_m x_3 - im(\xi)t]\ell(\xi)\,d\xi -$$
$$c^{-2}\chi_-(x_3)(2\pi)^{-1}\int_{R^2}\exp[ix'\xi - i\tau_m x_3 - im(\xi)t]\mathring{\tau}_m\tau_m^{-1}\ell(\xi)d\xi,$$

where, of course, the $\chi_\pm$ are not to be differentiated ($\mathring{\tau}_m\tau_m^{-1}$ is a bounded function of $\xi \in R^2$ [1]). For $m = m_\pm$ and $m = \mathring{m}$ respectively these are the surface-wave and AH-wave solutions of (1) described at the beginning.

We now show that the surface wave and AH wave solutions to (2) (see [1]) are essentially scalar waves: they consist of elementary (vector) differential operators applied to solutions of (1). We further show how to construct data giving rise to pure surface and AH waves and the corresponding solutions to (2) from solutions of the scalar wave equation (1).

Let

$$\hat{f}(\xi, x_3) \equiv \Phi_{x'} f(\xi, x_3) = (2\pi)^{-1} \int e^{-ix'\xi} f(x', x_3)\, dx'$$

be the two dimensional Fourier transform in x' with inverse $\Phi^*_{x'} f(x', x_3) = \Phi_{x'} f(-x', x_3)$. For $m = m_\pm(|\xi|)$, $\mathring{m}(|\xi|)$ define the operators

$$m(\Delta_2) = \Phi^*_{x'} m(\cdot)\Phi_{x'},\ \Delta_2 = (\partial_1^2 + \partial_2^2)$$

$${}^t m_+(\partial) = \varepsilon_+^{-1}(\partial_1\partial_3, \partial_2\partial_3, -\Delta_2, -i\varepsilon_+ m(\Delta_2)\partial_2, i\varepsilon_+ m(\Delta_2)\partial_1, 0)$$

$${}^t m_-(\partial) = \varepsilon_-^{-1}(\partial_1\partial_3, \partial_2\partial_3, \Delta_2, \varepsilon_-[\varepsilon_-^{-1}\sigma - im(\Delta_2)]\partial_2, -\varepsilon_-[\varepsilon_-^{-1}\sigma - im(\Delta_2)]\partial_1, 0)$$

(${}^t m$ is transpose of m).

Then we have the following result which is easily checked by direct computation.

THEOREM 3[1]. *If* $(1 + |\xi|^3)\,\swarrow(\xi) \in L_2(\mathbf{R}^2, \mathbf{C})$, *say, then for all* $t \geqq 0$, $m = m_\pm, \mathring{m}$, $P_m(x, t) = \chi_+ m_+(\partial)\, p_+(x, t) + \chi_- m_-(\partial)\, p_-(x, t)$ *is the solution to* (2) *with initial data* $P_m(x, 0+)$ *where* $p(x, t) = \chi_+ p_+(x, t) + \chi_- p_-(x, t)$ *is the solution to* (1) *with initial data* $p(x, 0+)$, $\partial_t p(x, 0+)$.

In conclusion, we note that functions

$$\tilde{L}_m(x, t) = \chi_+(x_3)(2\pi)^{-1} \int_{R^2} \exp[ix'\xi + i\mathring{\tau}_m x_3 - im(\xi)\, t]\,\swarrow(\xi)\, d\xi + \chi_-(x_3)(2\pi)^{-1} \int_{R^2} \exp[ix'\xi - i\tau_m x_3 - im(\xi)\, t]\,\swarrow(\xi)\, d\xi$$

are solutions of the same differential equations as in (1) with initial data $\tilde{L}_m(x, 0)$, $\partial_t \tilde{L}_m(x, 0)$, but they satisfy the second-order interface conditions

$$\phi(x', 0^+, t) = \phi(x', 0^-, t),$$

$$\partial_3\partial_t\phi(x', 0^+, t) + \kappa\partial_3\phi(x', 0^+, t) = c_0^{-2}c^2\, \partial_3\partial_t\, \phi(x', 0^-, t),$$

so we have yet another problem for the scalar wave equation admitting surface- and AH-wave solutions.

APPENDIX

Just as in [3], p. 95 we verify that functions f_1 of $f = {}^t(f_1, f_2) \in \mathscr{H}$ are in

$L_2^{\mathrm{loc}}(R^3)$, and they thus define distributions in $\mathscr{D}'(R^3)$. We suppose that for all $g \in \mathscr{D}(\mathring{H})$.

$$(9)\qquad \begin{aligned} 0 = -i\langle f, (H-ik)g\rangle = c_0^2(\nabla f_1, \nabla(kg_1+g_2))_+ + (f_2, c_0^2\Delta g_1 - kg_2)_+ \\ + c^2(\nabla f_1, \nabla[-(\kappa+k)g_1+g_2])_- + (f_2, c^2\Delta g_1 - kg_2)_-, \end{aligned}$$

where $k > 0$. We shall show that $f = 0$, First, we suppose that $\chi_- g_i = 0$, $i = 1, 2$. If $\chi_+ g_1 = 0$ and $g_2 \in \mathscr{D}(R^3_+)$, then (9) implies that $c_0^2 \Delta f_1 + kf_2 = 0$ in $\mathscr{D}'(R^3_+)$; if $\chi_+ g_2 = 0$ and $g_1 \in \mathscr{D}(R^3_+)$, then (9) implies that $\Delta f_1 + k^{-1}\Delta f_2 = 0$ in $\mathscr{D}'(R^3_+)$. These two equations imply first that $[\partial_3^2 - (|\xi|^2 + c_0^{-2}k^2)]f_2 = 0$, and hence that

$$(10)\quad \chi_+\hat{f}_2(\xi, x_3) = \chi_+(x_3)\mu_+(\xi)\exp[-\ell_0(\xi)x_3],\ \ell_0(\xi) = \sqrt{[|\xi|^2 + c_0^{-2}\kappa^2]} > 0,$$

where $\hat{f}(\xi, x_3) = \Phi_2 f(\xi, x_3)$ denotes the two-dimensional Fourier transform in the tangential variables. The same two equations now imply that $(\partial_3^2 - |\xi|^2)\hat{f}_1 = -c_0^{-2}\,k\hat{f}_2$, and hence that

$$(11)\quad \chi_+\hat{f}_1(\xi, x_3) = \chi_+(x_3)[\nu_+(\xi)\exp(-|\xi|x_3) - k^{-1}\mu_+(\xi)\exp(-\ell_0 x_3)].$$

In a similar way, supposing that $\chi_+ g_i = 0$, $i = 1, 2$, we obtain

$$(12)\qquad \begin{aligned} \chi_-\hat{f}_2(\xi, x_3) &= \chi_-(x_3)\mu_-(\xi)\exp(\ell x_3),\ \ell = \sqrt{[|\xi|^2 + c^{-2}k(k+\kappa)]} > 0, \\ \chi_-\hat{f}_1(\xi, x_3) &= \chi_-(x_3)[\nu_-(\xi)\exp(|\xi|x_3) - (k+\kappa)^{-1}\,\mu_-(\xi)\exp(\ell x_3)]. \end{aligned}$$

Because of the form of f in (10)–(12), we can integrate by parts and use condition (4) to obtain

$$(13)\qquad \begin{aligned} -i\langle \Phi_2 f, \Phi_2(H-ik)g\rangle = \int_{R^2} \{ & c_0^2[k\partial_3\bar{\hat{f}}_1(\xi, 0^+) + \partial_3\bar{\hat{f}}_2(\xi, 0^+)]\hat{g}_1(\xi, 0^+) \\ & - c^2[(\kappa+k)\partial_3\bar{\hat{f}}_1(\xi, 0^-) + \partial_3\bar{\hat{f}}_2(\xi, 0^-)]\hat{g}_1(\xi, 0^-) \\ & + [\bar{\hat{f}}_2(\xi, 0^-) - \bar{\hat{f}}_2(\xi, 0^+)]c^2\partial_3 g_1(\xi, 0^-) \\ & + [c^2\partial_3\bar{\hat{f}}_1(\xi, 0^-) - c_0^2\partial_3\bar{\hat{f}}_1(\xi, 0^+)]g_2(\xi, 0^+)\}. \end{aligned}$$

If now $\chi_- g_1 = \chi_- g_2 = 0$, then by condition (4) also $g_2(\xi, 0^+) = 0$, and from (10), (11), (12) we obtain

$$0 = k\partial_3\hat{f}_1(\xi, 0^+) + \partial_3\hat{f}_2(\xi, 0^+) = -\nu_+(\xi)|\xi|k,$$

and hence $\nu_+(\xi) = 0$ a.e. If now $\chi_+ g_1 = \chi_+ g_2 = 0$, then from (12),

$$(13)\qquad 0 = (\kappa + k)\partial_3\hat{f}_1(\xi, 0^-) + \partial_3\hat{f}_2(\xi, 0^-) = (\kappa + k)|\xi|\nu_-(\xi),$$

and hence $\nu_-(\xi) = 0$ a.e. The expressions (10)–(12) for f now have the form

$$(14)\qquad \begin{aligned} f &= {}^t(-k^{-1}, 1)\mu_+(\xi)\exp(-\ell_0 x_3), \\ f &= {}^t(-(k + \kappa)^{-1}, 1)\mu_-(\xi)\exp(\ell x_3). \end{aligned}$$

Supposing now in (13) first that $\partial_3 g_1(\xi, 0^-) \neq 0$, $g_1(\xi, 0^-) = g_1(\xi, 0^+) = g_2(\xi, 0^+) = 0$ and then that $g_2(\xi, 0^+) \neq 0$, $\partial_3 g_1(\xi, 0^-) = g_1(\xi, 0^+) = g_1(\xi, 0^-) = 0$, from (13), (14) we obtain the system of equations

$$\mu_-(\xi) - \mu_+(\xi) = 0,$$
$$c^2(k + \kappa)^{-1} \ell \mu_-(\xi) + c_0^2 k^{-1} \ell_0 \mu_+(\xi) = 0,$$

with determinant equal to $c^2(k + \kappa)^{-1}\ell + c_0^2 k^{-1} \ell_0 > 0$ for all $\xi \in R^2$, and hence $\mu_-(\xi) = \mu_+(\xi) = 0$ a.e. Thus, $f = 0$, and the range of $H - ik$ is dense in $\mathscr{H}$.

REFERENCES

1. D.S. Gilliam and J.R. Schulenberger, *The propagation of electromagnetic waves through, along, and above a three dimensional half space*, to appear in Methoden und Verfahren der Mathematischen Physik.

2. D.S. Gilliam and J.R. Schulenberger, *Electromagnetic waves in a three-dimensional half space with a dissipative boundary*, J. Math. Anal. Appl., **88** (1982).

3. P.D. Lax and R.S. Phillips, *Scattering Theory*, Academic Press, New York (1967).

4. R.S. Phillips, *On dissipative operators*, Lecture Series in Differential Equations, Vol. 2 (A.K. Aziz, Ed.), Van Nostrand-Reinhold, Princeton, N.J. (1969).

DEPARTMENT OF MATHEMATICS, TEXAS TECH UNIVERSITY, LUBBOCK, TEXAS 79409
AN AH CORPORATION, TUCSON, ARIZONA