ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 16, Number 2, Spring 1986

OSCILLATORY AND ASYMPOTOTIC BEHAVIOR OF CERTAIN FOURTH ORDER DIFFERENCE EQUATIONS

B. SMITH AND W. E. TAYLOR, JR.

Introduction. In several recent papers the oscillatory and asymptotic behavior of solutions of second order difference equations have been discussed, e.g., [1], [2], [3]. However, when compared to differential equations, the study of the oscillation properties of difference equations has received little attention, especially for orders greater than two.

This note is concerned with the solutions of the fourth order linear difference equation

$$\Delta(\Delta^3 u_n + p_n u_{n+2}) + p_n \Delta u_{n+1} + q_n u_{n+2} = 0 \tag{1}$$

where Δ denotes the differencing operator, i.e., $\Delta x_n = x_{n+1} - x_n$. While no sign conditions are explicitly stated for the real sequence $\{p_n\}$, it will be assumed that $q_n > 0$ for each n.

By a solution of (1) we will mean a real sequence $\{u_n\}$ defined on the set of nonnegative integers which satisfies (1). A nontrivial solution of (1), say $\{u_n\}$, is called *nonoscillatory* if there exists $n_0 \geqq 0$ such that $u_m u_{m+1} > 0$ for all $m \geqq n_0$; otherwise it is said to be oscillatory.

The results established herein are extensions to difference equations of certain results obtained by Taylor in [6]. It is clear that (1) is a discrete analogue of the equation

$$(y''' + p(x)y)' + p(x)y' + q(x)y = 0.$$

Moreover, we shall show herein that certain techniques developed to study this differential equation can be used to great advantage in the study of (1).

Main Results. Let V denote the solution space of (1). To begin our study of (1) we consider the following operator defined on V: For each $\{u_n\} \in V$, define

$$F_n = F[u_n] = u_{n+1}[\Delta^3 u_n + p_n u_{n+2}] - \Delta u_n \Delta^2 u_n. \tag{2}$$

Computing the difference of F_n and making appropriate substitutions we find that

$$\Delta F_n = -(\Delta^2 u_n)^2 - q_n u_{n+2}^2.$$

Hence we have the following result.

THEOREM 1. *If* $\{u_n\} \in V$, *then the operator* F_n *defined by* (2) *is nonincreasing. Moreover, for a nontrivial solution* $\{u_n\}$ *of* (1) *there cannot exist two nonconsecutive values of* n *such that* $u_n = u_{n+1} = 0$.

The proof of Theorem 1 is easy and will be left to the reader to verify.

Following [6], we define a solution $\{u_n\}$ to be type I if and only if $F[u_n] \geqq 0$ for all n. While a type II solution is one where $F[u_m] < 0$ for some m.

THEOREM 2. *There exists a nontrivial type* I *solution of* (1).

PROOF. Let $\{y_n^1\}$, $\{y_n^2\}$, $\{y_n^3\}$ be three linearly independent solutions of (1) which vanish at $n = 0$, i.e., $y_0^1 = y_0^2 = y_0^3 = 0$. For each positive integer κ define $u_n^\kappa = t_{1\kappa} y_n^1 + t_{2\kappa} y_n^2 + t_{3\kappa} y_n^3$ where the $t_{i\kappa}$ are chosen in such a way that

$$u_\kappa^\kappa = u_{\kappa+1}^\kappa = 0$$

and

$$t_{1\kappa}^2 + t_{2\kappa}^2 t + t_{3\kappa}^2 = 1$$

Note that $F[u_\kappa^\kappa] = 0$ and $F[u_n^\kappa] \geqq 0$ if $0 \leqq n < \kappa$. Let $T_\kappa = (t_{1\kappa}, t_{2\kappa}, t_{3\kappa})$ where the $t_{i\kappa}$ are as above. Then $\|T_\kappa\| = 1$ for each κ and it follows that there exists a subsequence $u_n^{\kappa i}$ which converges to a nontrivial solution $\{V_n\}$ of (1). It is easy to see that $\{V_n\}$ is a type I solution.

Clearly, if $\{u_n\}$ is a nontrivial type I solution, then $F[u_n] > 0$ for $n \geqq 0$. Also note that the type I solution constructed in the previous proof vanished at $n = 0$; however one could in a similar fashion construct a type I solution which vanishes at $n = 1$. Hence we see that (1) will always have at least two independent type I solutions.

THEOREM 3. *Let* $\{u_n\}$ *be a type* I *solution. Then*

$$\text{(i)} \sum^{\infty} (\Delta^2 u_n)^2 < \infty, \textit{ and}$$

$$\text{(ii)} \sum^{\infty} q_n u_n^2 < \infty.$$

PROOF. Since $\{u_n\}$ is type I, $F[u_n] \geqq 0$ for all n. Differencing $F[u_n]$, making appropriate substitutions from (1) and summing from 0 to $m - 1$, we obtain

$$0 \leqq F[u_m] = F_0 - \sum^{m-1} (\Delta^2 u_n)^2 - \sum^{m-1} q_n u_{n+2}^2.$$

Hence $\sum^{m-1}(\Delta^2 u_n)^2 + \sum^{m-1} q_n u_{n+2}^2 \leqq F_0$. Letting m tend to infinity establishes both (i) and (ii) since F_0 is independent of m.

COROLLARY. *Suppose* $\lim_{n\to\infty} \inf q_n > 0$. *If* $\{u_n\}$ *is a type* I *solution of* (1), *then* $\{u_n\} \in \ell_2$, *i.e.*, $\sum u_n^2 < \infty$.

EXAMPLES. Consider the equations

$$\Delta\left(\Delta^3 u_n + \frac{1}{n} u_{n+2}\right) + \frac{1}{n} \Delta u_{n+1} + \left(\frac{1}{n} - \frac{1}{n+1}\right) u_{n+2} = 0 \tag{3}$$

and

$$\Delta(\Delta^3 V_n + V_{n+2}) + \Delta V_{n+1} + \frac{5}{4} V_{n+2} + 0. \tag{4}$$

The sequence $u_n = 1$ and $V_n = (1/2)^n$ are type I solutions of (3) and (4) respectively. Note that $u_n \notin \ell_2$ but $V_n \in \ell_2$. Hence the condition $\lim_{n\to\infty} \inf q_n > 0$ is necessary for the type I solution to belong to ℓ_2.

Finally we show that (1) has an oscillatory solution provided p_n satisfies a certain "growth" condition.

THEOREM 4. *Assume* $\sum_{n=0}^{\infty} p_n = \infty$ *and let* $\{u_n\}$ *be a type* II *solution. Then* $\{u_n\}$ *is oscillatory.*

PROOF. Suppose the contrary, i.e., suppose $\{u_n\}$ is a nonoscillatory type II solution. We can assume without loss of generality that there exists n_0 such that $u_n > 0$ for all $n > n_0$ and $F[u_n] < 0$ for all $n \geqq n$.

Consider the function

$$J_n = \frac{\Delta^2 u_n}{u_{n+1}} + \sum_{\kappa=n_0}^{n-1} p_k$$

Differencing J_n, we find that

$$\Delta J_n = \frac{F[u_n] - (\Delta^2 u_n)^2}{u_{n+1} u_{n+2}} < 0 \text{ for } n \geqq n_0.$$

So J_n is decreasing for $n \geqq n_0$. An easy argument shows that the function

$$\sigma_n = \frac{\Delta^2 u_n}{u_{n+1}}$$

must be negative for large n and in fact $\sigma_n \to -\infty$. But $u_n > 0$ for large n and $\Delta^2 u_n < 0$ for large n implies that $\Delta u_n > 0$ for all n sufficiently large. Thus u_n is increasing for all $n \geqq n_1 \geqq n_0$. Let $\beta < 0$ be a number such that $\sigma_n < \beta$ for $n \geqq n_1$. Then

$$\Delta^2 u_n < \beta u_{n+1} < \alpha < 0 \tag{5}$$

for some $\alpha < 0$. Such an α exists because u_n is increasing. But (5) implies $\Delta u_n \to -\infty$ as $n \to \infty$. This contradiction proves the theorem.

Type II solutions of (1) always exist since any nontrivial solution of (1) vanishing at two consecutive values of n is type II, hence initial values can be used to construct these solutions.

References

1. J.W. Hooker and W.T. Patula, *Riccati Type Transformations for Second Order Linear Difference Equations*, J. Math. Anal. Appl., **82** (1981), 451–462.

2. P.J. McCarthy, *Note on the Oscillation of Solutions of Second Order Difference Equations*, Port, Math. **18** (1959), 203–205.

3. W.T. Patula, *Growth and Oscillation Properties of Second Order Linear Difference Equations*, SIAM, J. Math. Anal. **10** (1979), 55–61.

4. ———, *Growth, Oscillation and Comparison theorems for Second Order Linear Difference Equations*, pp. 1272–1279.

5. B. Szmanda, *Oscillation of Solutions of Second Order Difference Equations*, Port. Math. **37** (1978), 251–254.

6. W.E. Taylor Jr., *Asymptotic Behavior of Solutions of a Fourth Order Nonlinear Differential Equation*, PAMS. **65** (1977), 70–72.

Department of Mathematics, Texas Southern University Houston, Texas, 77004