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ANOTHER FAMILY OF q-LAGRANGE INVERSION FORMULAS

IRA GESSEL AND DENNIS STANTON

ABSTRACT. A q-analog of Lagrange inversion is stated for $(x/(1 - x^r)^b)$. Applications to basic hypergeometric series, identities of the Rogers-Ramanujan type, and orthogonal polynomials are given.

1. Introduction. The generalized Lagrange inversion problem is: given

$$G_k(x) = \sum_{n=k}^{\infty} B_{nk}x^n,\ k = 0, 1, \ldots, \tag{1.1}$$

for some lower triangular non-singular matrix B_{nk}, and a formal power series

$$f(x) = \sum_{n=0}^{\infty} f_n x^n, \tag{1.2}$$

find constants a_k such that

$$f(x) = \sum_{k=0}^{\infty} a_k G_k(x). \tag{1.3}$$

It is clear that

$$f_n = \sum_{k=0}^{n} B_{nk}a_k. \tag{1.4}$$

Thus to find a_k it is sufficient to find the inverse matrix $B_{k\ell}^{-1}$:

$$a_k = \sum_{\ell=0}^{k} B_{k\ell}^{-1} f_\ell. \tag{1.5}$$

The usual Lagrange inversion formula takes $G_k(x) = y^k$, where $y(x)$ is a formal power series in x such that $y(0) = 0$ and $y'(0) \neq 0$.

In a recent paper **[10]** we gave a q-analog of B_{nk}, $B_{k\ell}^{-1}$, and $G_k(x)$ for $G_k(x) = x^k/(1 - x)^{a+(b+1)k}$. In this paper we similarly find a q-Lagrange inversion formula for a q-analog of $G_k(x) = x^k/(1 - x^r)^{a+(b+1)k}$ for $r = 1, 2, \cdots$. Our main theorem is stated as Theorem 2.3. Just as in **[10]**,

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we give some applications of Theorem 2.3. These include transformations for basic hypergeometric series, identities of Rogers-Ramanujan type, and connections with orthogonal polynomials.

We use the standard notation for q-binomial coefficients and basic hypergeometric series [9]. When not specified, the base is always q, but sometimes we will use other bases. Thus

$$(A)_n = \prod_{j=0}^{n-1}(1 - Aq^j), \tag{1.6}$$

$$(B; p)_n = \prod_{j=0}^{n-1}(1 - Bp^j). \tag{1.7}$$

At the beginning of section 2 we momentarily consider the $q = 1$ case; there

$$(a)_k = a(a + 1) \cdots (a + k - 1) \tag{1.8}$$

is the usual shifted factorial.

Recall that the q-difference operator D_q

$$(D_q f)(x) = \frac{f(qx) - f(x)}{(q - 1)x} \tag{1.9}$$

satisfies [11, Eq. [2.6.]])

$$(q-1)^N(-x)^N(D_q^N f)(x) = \sum_{k=0}^{N} \begin{bmatrix} N \\ k \end{bmatrix}_q (-1)^k q^{\binom{k+1}{2}-Nk} f(xq^k). \tag{1.10}$$

Given a formal Laurent series $f(x)$, we let $\mathrm{Res}_x f(x)$ denote the coefficient of $1/x$ in $f(x)$.

2. The main theorem. In this section we give our q-Lagrange inversion formula for a q-analog of $G_k(x) = x^k(1 - x^r)^{-a-(b+1)k}$. First we state the matrices B_{nk} and $B_{k\ell}^{-1}$ for the $q = 1$ case. According to (1.1) here we have

$$\tilde{B}_{nk} = \begin{cases} \dfrac{(a + (b + 1)k)_{(n-k)/r}}{((n - k)/r)!}, & n - k \equiv 0 (\mathrm{mod}\ r) \\ 0 & \text{, otherwise.} \end{cases} \tag{2.1}$$

The Lagrange inversion formula implies

$$\tilde{B}_{k\ell}^{-1} = \begin{cases} \dfrac{(-a-(b+1)k+1)_{(k-\ell)/r-1}}{((k-\ell)/r)!}(-a-\ell(b+1)), & k-\ell \equiv 0(\mathrm{mod}\ r) \\ 0 & \text{, otherwise.} \end{cases} \tag{2.2}$$

Our object is to give a q-analog of (2.1) and (2.2). This is accomplished by Theorem 2.3.

THEOREM 2.3. *(q-Lagrange inversion for $x/(1 - x^r)^{b+1}$) Let B_{nk} be the lower triangular matrix*

$$B_{nk} = \begin{cases} \dfrac{(Ap^{k/r}q^{k/r};\, p)_{(n-k)/r}}{(q)_{(n-k)/r}}\, q^{-nk/r^2}, & n - k \equiv 0 (\mathrm{mod}\ r) \\ 0 & \text{, otherwise.} \end{cases}$$

Then

$$B^{-1}_{k\ell} = \begin{cases} \dfrac{(Ap^{k/r-1}q^{k/r};\, p^{-1})_{(k-\ell)/r-1}}{(q)_{(k-\ell)/r}}(1 - Ap^{\ell/r}q^{\ell/r})(-1)^{(k-\ell)/r} q^{\binom{(k-\ell)/r+1}{2}} + \ell k/r^2 \\ & , k - \ell \equiv 0 (\mathrm{mod}\ r) \\ 0 & \text{, otherwise.} \end{cases}$$

In order to derive (2.1) and (2.2) from Theorem 2.3 put $q = p^{rb+r-1}$ and $A = p^a$. Then

$$\begin{aligned} \lim_{p\to 1} B_{nk} &= \lim_{p \to 1} \frac{(p^{a+k+bk};\, p)_{(n-k)/r}}{(p^{rb+r-1};\, p^{rb+r-1})_{(n-k)/r}} \\ &= \frac{(a + (b + 1)k)_{(n-k)/r}}{((n - k)/r)!} (rb + r - 1)^{(k-n)/r}. \end{aligned} \tag{2.4}$$

This agrees with (2.1) except for the scaling factor $(rb + r - 1)^{(k-n)/r}$. In this case

$$G_k(x) = \sum_{n=k}^{\infty} (\lim_{p\to 1} B_{nk})x^n = x^k(1 - x^r/(rb + r - 1))^{-a-(b+1)k}. \tag{2.5}$$

So it is more accurate to call Theorem 2.3 a q-analog of Lagrange inversion for $x(1 - x^r/(rb + r - 1))^{-b-1}$. The same type of scaling factors occurred in [**10**, Th. 3.7]. In this case we cannot put $b = (1 - r)/r$ so we state the theorem for $x/(1 - x^r)^{1/r}$ separately.

THEOREM 2.6. *(q-analog of Lagrange inversion for $x/(1 - x^r)^{1/r}$). Let B_{nk} be the lower triangular matrix*

$$B_{nk} = \begin{cases} \dfrac{(Aq^{k/r})_{(n-k)/r}}{(q)_{(n-k)/r}}\, q^{-nk/r^2}, & n - k \equiv 0 (\mathrm{mod}\ r) \\ 0 & \text{, otherwise} \end{cases}$$

Then

$$B^{-1}_{k\ell} = \begin{cases} \dfrac{(Aq^{k/r-1};\, q^{-1})_{(k-\ell)/r}}{(q)_{(k-\ell)/r}}(-1)^{(k-\ell)/r} q^{\binom{(k-\ell)/r+1}{2}} + \ell k/r^2, & k - \ell \equiv 0 (\mathrm{mod}\ r) \\ 0 & \text{, otherwise.} \end{cases}$$

Theorem 2.3 has the disadvantage that it is not in general possible to evaluate

$$(2.7)\quad G_k(x) = \sum_{n=k}^{\infty} B_{nk}x^n = x^k q^{-k^2/r^2} \sum_{n=0}^{\infty} \frac{(Ap^{k/r}q^{k/r}; p)_n}{(q)_n} (x^r q^{-k/r})^n.$$

However, there are special cases which can be evaluated by the q-binomial theorem [**14**; Eq. (3.2.2.11)]

$$(2.8)\qquad G_k(x) = x^k q^{-k^2/r^2}(x^r q^{-k/r}(1 - Aq^{k/r}))_\infty, \quad (p = 1)$$

$$(2.9)\qquad G_k(x) = x^k q^{-k^2/r^2}(Ax^r q^{k/r})_\infty/(x^r q^{-k/r})_\infty, \quad (p = q)$$

and

$$(2.10)\qquad G_k(x) = x^k q^{-k^2/r^2}(x^r q^{-k/r})_\infty, \quad (A = 0).$$

For Theorem 2.6 we see that

$$(2.11)\qquad G_k(x) = x^k q^{-k^2/r^2}(Ax^r)_\infty/(x^r q^{-k/r})_\infty.$$

Proof of Theorem 2.3. We need to show that

$$(2.12)\qquad (*) = \sum_{k=\ell}^{n} B_{nk}B_{k\ell}^{-1} = \delta_{n\ell}.$$

If $n - \ell \not\equiv 0 \pmod r$, then clearly this sum is zero. So we assume $n - \ell \equiv 0 \pmod r$. Then, after replacing k by $\ell + rk$

$$(2.13)\quad (*) = q^{\ell(\ell-n)/r^2}\frac{(1 - Ap^{\ell/r}q^{\ell/r})}{(q)_{(n-\ell)/r}} \sum_{k=0}^{(n-\ell)/r} \begin{bmatrix} (n-\ell)/r \\ k \end{bmatrix}_q (-1)^k q^{\binom{k+1}{2}+k(\ell-n)/r} \cdot (Ap^{\ell/r+1}q^{\ell/r+k}; p)_{(n-\ell)/r-1}.$$

Note that (2.13) can be rewritten with the q-difference operator D_q, and (1.10) implies

$$(2.14)\qquad (*) = q^{\ell(\ell-n)/r^2}\frac{(1 - Ap^{\ell/r}q^{\ell/r})}{(q)_{(n-\ell)/r}} (D_q^{(n-\ell)/r}f)(x)\Big|_{x=1}$$

where

$$(2.15)\qquad f(x) = (Ap^{\ell/(r+1)}q^{\ell/r}x; p)_{(n-\ell)/r-1}.$$

If $n - \ell > 0$, $f(x)$ is a polynomial in x of degree $(n - \ell)/r - 1$. Thus $D_q^{(n-\ell)/r} f \equiv 0$. The proof of Theorem 2.3 is completed by checking the $n = \ell = k$ case.

In fact, Theorem 2.3 can be shown to be equivalent to Theorem 3.7 of [**10**], which is q-Lagrange inversion for $x/(1 - x)^{b+1}$. This is not surprising because Theorem 2.3 should correspond to an r-section of Theorem 3.7 of [**10**]. Carlitz [**8**] and Al-Salam and Verma [**1**, p. 416] had previously stated theorems equivalent to Theorem 3.7 of [**10**]. Carlitz used the q-binomial theorem, while Al-Salam and Verma used the q-difference operator for their proof. They did not refer to q-Lagrange inversion, and

[10] did not prove Theorem 3.7. So we have included the simple proof. The proof of Theorem 2.6 is similar, and it corresponds to an r-section of Theorem 3.3 of [10].

Krattenthaler [12, p. 343] has given four examples of inverse relations. His first, second, and fourth examples are equivalent to Theorem 3.3, Theorem 3.2, and the $p = 1$ case of Theorem 3.7 in [10]. His second example is equivalent to the $r = 2$ and $p = q$ case of Theorem 2.3. We could not derive our Theorem 2.3 from his Theorem 2. However, Theorem 2.6 does follow in this way. In Theorem 2 of [12], put $h_n(z) = H_k(z) = 1$, $\alpha_n = \beta_n = 1$,

$$g_n(z) = (az^r)_{n/r}, \tag{2.16}$$

$$G_k(z) = (az^r)_{k/r+1}, \tag{2.17}$$

and

$$F(z) = \frac{(z^r)_\infty}{(az^r)_\infty}, \tag{2.18}$$

where

$$(y)_\alpha = \frac{(yq^\alpha)_\infty}{(y)_\infty} \tag{2.19}$$

(In Krattenthaler's notation (2.4), $(y)_\alpha = P_\alpha(1, y)$.) Then the matrix d_{nk} given by his Theorem 2 is equivalent to B_{nk} in Theorem 2.6.

In Theorem 2.3 we can clearly switch the factor $(1 - Ap^{\ell/r}q^{\ell/r})$ from $B_{k\ell}^{-1}$ to $(1 - Ap^{n/r}q^{n/r})$ in B_{nk}. This changes the function $G_k(x)$ to the "derivative form", as in [10].

3. Applications. We give some applications of Theorem 2.3 and Theorem 2.6 which are similar in spirit to those in [10].

Our first application is the orthogonality relation

$$\sum_{\ell=m}^{k} B_{k\ell}^{-1} B_{\ell m} = \delta_{km}. \tag{3.1}$$

We see that this is

$$\sum_{\ell=0}^{(k-m)r} \frac{(q^{(m-k)/r}; q)_\ell (Ap^{m/r}q^{m/r}; p)_\ell (Ap^{m/r+1}q^{m/r+1}; pq)_\ell}{(Ap^{m/r+1}q^{k/r}; p)_\ell (q; q)_\ell (Ap^{m/r}q^{m/r}; pq)_\ell} q^{\ell(k-m)/r} = \delta_{km}. \tag{3.2}$$

Equation (3.1) is a bibasic well poised evaluation.

Next we concentrate on the $p = q$ case of Theorem 2.3 with $r = 2$. We need to choose a_k so that (by (1.4))

$$f_n = q^{-n^2/4} (A)_n \sum_{k=0}^{n/2} \frac{q^{nk/2} a_{n-2k}}{(q)_k (Aq^{n-1}; q^{-1})_k (A)_{n-2k}} \tag{3.3}$$

is evaluable. The choice

$$a_k = \frac{(A)_k}{(q^{1/2}; q^{1/2})_k} q^{\binom{k}{2}/4}(iq^{1/8}A^{-1/2})^k \tag{3.4}$$

gives

$$f_n = q^{-n^2/8}(iA^{-1/2})^n \frac{(A)_n}{(q^{1/2}; q^{1/2})_n} {}_2\varphi_1\left[\begin{matrix} q^{n/2}, q^{(n-1)/2} \\ Aq^{n-1} \end{matrix} \Big| q^{-1}; Aq^{-1/2}\right]. \tag{3.5}$$

The q-analog of Vandermonde's theorem [**14**; p. 247] implies

$$f_n = q^{-n^2/8}(iA^{-1/2})^n \frac{(A; q^{1/2})_n}{(q^{1/2}; q^{1/2})_n}. \tag{3.6}$$

Finally, these values for a_k and f_n and (1.2), (1.3), and (2.9) imply

$$\begin{aligned} &\sum_{n=0}^{\infty} q^{-n^2/8} \frac{(A; q^{1/2})_n}{(q^{1/2}; q^{1/2})_n} (ixA^{-1/2})^n \\ &= \sum_{\ell=0}^{\infty} \frac{(A)_\ell (Ax^2q^{\ell/2})_\infty}{(q^{1/2}; q^{1/2})_\ell (x^2q^{-\ell/2})_\infty} q^{-\ell^2/8}(ixA^{-1/2})^\ell. \end{aligned} \tag{3.7}$$

This is a q-analog of

$$(1 - x^2)^{-a}(1 - 2xi/(1 - x^2))^{-a} = (1 - xi)^{-2a} \tag{3.8}$$

An interesting special case of (3.7) is $ixA^{1/2} = y$, $A \to \infty$ and $q \to q^8$

$$\sum_{n=0}^{\infty} \frac{q^{n^2-2n}y^n}{(q^4; q^4)_n} = \sum_{\ell=0}^{\infty} \frac{(-y^2q^{4\ell}; q^8)_\infty}{(q^4; q^4)_\ell} q^{3\ell^2-4\ell}y^\ell. \tag{3.9}$$

If $y = q^2$, the left-hand side of (3.9) can be evaluated by [**15**, Eq. (20)] to obtain

$$\sum_{\ell=0}^{\infty} \frac{(-q^{4\ell+4}; q^8)_\infty}{(q^4; q^4)_\ell} q^{3\ell^2-2\ell} = \frac{1}{(-q^2; q^2)_\infty (q; q^5)_\infty (q^4; q^5)_\infty}, \tag{3.10}$$

or, if $y = q^4$ we use [**15**, Eq. (16)] to find

$$\sum_{\ell=0}^{\infty} \frac{(-q^{4\ell+8}; q^8)_\infty}{(q^4; q^4)_\ell} q^{3\ell^2} = \frac{1}{(-q^2; q^2)_\infty (q^2; q^5)_\infty (q^3; q^5)_\infty}. \tag{3.11}$$

Both (3.10) and (3.11) appear to be new.

We can relate (3.10) and (3.11) to evaluations on Slater's list [**15**]. Equate even and odd powers of y in (3.9) and then set $y = q^2$. For even powers (3.9) implies

$$\sum_{n=0}^{\infty} \frac{q^{4n^2}}{(q^4; q^4)_{2n}} = \sum_{\substack{\ell=0 \\ \ell \text{ even}}}^{\infty} \frac{(-q^{4\ell+4}; q^8)_\infty}{(q^4; q^4)_\ell} q^{3\ell^2-2\ell}, \tag{3.12}$$

Applying [**15**, Eq. (98)], we see that

$$(3.13)\quad \sum_{\substack{\ell=0\\ \ell \text{ even}}}^{\infty} \frac{(-q^{4\ell+4}; q^8)_\infty}{(q^4; q^4)_\ell} q^{3\ell^2-2\ell} = \frac{1}{(q^4; q^8)_\infty (q^{16}; q^{80})_\infty (q^{64}; q^{80})_\infty}.$$

For odd powers, [15, Eq. (94)] implies

$$(3.14)\quad \begin{aligned} &\sum_{\substack{\ell=0\\ \ell \text{ odd}}}^{\infty} \frac{(-q^{4\ell+4}; q^8)_\infty}{(q^4; q^4)_\ell} q^{3\ell^2-2\ell} \\ &= \frac{q(q^{12}; q^{40})_\infty (q^{28}; q^{40})_\infty (q^{16}; q^{80})_\infty (q^{64}; q^{80})_\infty (q^{40}; q^{40})_\infty}{(q^4; q^4)_\infty}, \end{aligned}$$

If we multiply (3.10) by $(q)_\infty(-q)_\infty = (q^2; q^2)_\infty$, we see that (3.13) and (3.14) imply

$$(3.15)\quad \begin{aligned} &(-q; q^2)_\infty (q^2; q^5)_\infty (q^3; q^5)_\infty (q^5; q^5)_\infty \\ &= (q^2; q^4)_\infty (q^8; q^{16})_\infty (q^{32}; q^{80})_\infty (q^{48}; q^{80})_\infty (q^{80}; q^{80})_\infty \\ &\quad + q(q^2; q^4)_\infty (q^{12}; q^{40})_\infty (q^{28}; q^{40})_\infty (q^{64}; q^{80})_\infty (q^{16}; q^{80})_\infty (q^{40}; q^{40})_\infty. \end{aligned}$$

This identity is also new. We could apply the same steps to (3.11).

For our choice of a_k and f_n the dual evaluation (1.5) is a special case of the very-well poised ${}_5\varphi_4$ evaluation.

Next we give an application of Theorem 2.6 with $r = 2$. We need to choose a_k such that

$$(3.16)\quad f_n = q^{-n^2/4} \sum_{k=0}^{n/2} \frac{(Aq^{n/2}; q^{-1})_k}{(q)_k} q^{nk/2} a_{n-2k}$$

is evaluable. If

$$(3.17)\quad a_k = \frac{(B; q^{1/2})_k}{(q^{1/2}; q^{1/2})_k} q^{(k^2+2k)/8} i^k$$

then

$$(3.18)\quad f_n = \frac{(B; q^{1/2})_n}{(q^{1/2}; q^{1/2})_n} q^{(2n-n^2)/8} i^n \, {}_3\varphi_2\left[\begin{matrix} Aq^{(n-2)/2},\ q^{n/2},\ q^{(n-1)2} \\ Bq^{(n-1)/2},\ Bq^{(n-2)/2} \end{matrix} \Big| q^{-1}; q^{-1}\right].$$

We see that f_n can be evaluated by the q-analog of Vandermonde's theorem for $B = A$ or $B = Aq^{-1/2}$.

For $B = A$, we have

$$(3.19)\quad f_n = \frac{(1-A)(Aq^{n/2}; q^{-1})_n}{(1-Aq^{n/2})(q^{1/2}; q^{1/2})_n} q^{n^2/8} i^n$$

so

$$
\begin{aligned}
&{}_2\varphi_1\left[\begin{matrix} A,\ A^{-1} \\ q^{1/2} \end{matrix} \middle| q;\ Ax^2q^{1/2}\right] \\
(3.20)\qquad &+ \frac{1-A}{1-q^{1/2}}\, ixq^{1/8}\ {}_2\varphi_1\left[\begin{matrix} Aq^{1/2},\ A^{-1}q^{1/2} \\ q^{3/2} \end{matrix} \middle| q;\ Ax^2q^{1/2}\right] \\
&= \sum_{\ell=0}^{\infty} \frac{(A;\ q^{1/2})_\ell (Ax^2)_\infty}{(q^{1/2};\ q^{1/2})_\ell (x^2q^{-\ell/2})_\infty}\, q^{(2\ell-\ell^2)/8}(ix)^\ell.
\end{aligned}
$$

To find of what (3.20) is the q-analog, put $A = q^{a/2}$ and let $q \to 1$. The result is

$$
\begin{aligned}
&{}_2F_1\left[\begin{matrix} a/2,\ -a/2 \\ 1/2 \end{matrix} \middle| x^2\right] + aix\ {}_2F_1\left[\begin{matrix} (a+1)/2,\ (1-a)/2 \\ 3/2 \end{matrix} \middle| x^2\right] \\
(3.21)\qquad &= (1-x^2)^{-a/2}\left(1 - \frac{ix}{\sqrt{1-x^2}}\right)^{-a} = (\sqrt{1-x^2} - ix)^{-a}.
\end{aligned}
$$

The substitution $x = \sin z$ and [**9**, p. 101, Eq. (11), (12)] show that (3.21) is $\cos az + i \sin az = (\cos z - i \sin z)^{-a}$. In this sense (3.20) is a q-analog of DeMoivre's theorem.

For $B = Aq^{-1/2}$ we have

$$
(3.22)\qquad f_n = \frac{(Aq^{n/2-1};\ q^{-1})_n}{(q^{1/2};\ q^{1/2})_n}\, q^{n^2/8}\, i^n
$$

so

$$
\begin{aligned}
&{}_2\varphi_1\left[\begin{matrix} A,\ qA^{-1} \\ q^{1/2} \end{matrix} \middle| q;\ Ax^2\, q^{-1/2}\right] \\
(3.23)\qquad &+ \frac{1-Aq^{-1/2}}{1-q^{1/2}}\, ixq^{1/8}\ {}_2\varphi_1\left[\begin{matrix} Aq^{1/2},\ q^{3/2}A^{-1} \\ q^{3/2} \end{matrix} \middle| q;\ Ax^2q^{-1/2}\right] \\
&= \sum_{\ell=0}^{\infty} \frac{(Aq^{-1/2};\ q^{1/2})_\ell (Ax^2)_\infty}{(q^{1/2};\ q^{1/2})_\ell (x^2\, q^{-\ell/2})_\infty}\, q^{(2\ell-\ell^2)/8}(ix)^\ell.
\end{aligned}
$$

Again [**9**; p. 101, Eq. (11), Eq. (12)] imply that (3.23) is the q-analog of

$$
\frac{\cos(1-a)z}{\cos z} - i\,\frac{\sin(1-a)z}{\cos z} = \frac{1}{\cos z}(\cos z - i \sin z)^{1-a}.
$$

In fact, the q-analog of Euler's transformation shows that (3.23) is equivalent to (3.20).

4. Connections with orthogonal polynomials. In [**10**] we showed that q-Lagrange inversion for $x/(1-x)^2$ is related to the little q-Jacobi polynomials $p_\ell(x;\ a,\ b;\ q)$. Put

$$
(4.1)\qquad r_k(x) = (x;\ q)_k.
$$

Then an explicit formula can be given for the connection coefficients $A_{k\ell}$ in

$$r_k(x) = \sum_{\ell=0}^{k} A_{k\ell} p_\ell(x; a, b; q). \tag{4.2}$$

This is equivalent to finding $B_{k\ell}^{-1}$ for the q-analog of Lagrange inversion for $x/(1 - x^2)$. In this section we show that this idea applies to a set of q-ultraspherical polynomials.

The discrete q-ultraspherical polynomials [3] are defined by

$$P_n(x, a; q) = {}_3\varphi_2\left[\begin{matrix} q^{-n}, a^2q^{n+1}, x \\ aq, \quad -aq \end{matrix} \middle| q; q\right]. \tag{4.3}$$

However, there is another way to write $P_n(x, a; q)$ which shows that $P_n(x, a; q)$ is either even or odd [3]

$$\begin{aligned} &\frac{(a^2q)_n}{(q)_n}(iq^{-1/2}a^{-1})^n q^{-n^2/4} P_n(x, a; q) \\ &= \sum_{k=0}^{n/2} \frac{(qa^2q^{2n-4k}; q^2)_k}{(q^2; q^2)_k} q^{-n(n-2k)/2} \frac{(iq^{-1/2}a^{-1})^{n-2k} q^{(n-2k)^2/4} x^{n-2k}}{(q)_{n-2k}(a^2q; q^2)_{n-2k}}. \end{aligned} \tag{4.4}$$

Let $\tilde{P}_n(x, a; q)$ be the left-hand side of (4.4), and put

$$\tilde{r}_k(x) = \frac{(iq^{-1/2}a^{-1})^k q^{k^2/4} x^k}{(q)_k(a^2q; q^2)_k}. \tag{4.5}$$

Then we have

$$\tilde{P}_n(x, a; q) = \sum_{k=0}^{n} B_{nk}(a)\tilde{r}_k(x), \tag{4.6}$$

where $B_{nk}(a)$ is given by Theorem 2.3 with q replaced by q^2, $p = q^2$, and $A = aq^2$. Thus

$$\tilde{r}_k(x) = \sum_{\ell=0}^{k} B_{k\ell}^{-1}(a)\tilde{P}_\ell(x; a, q). \tag{4.7}$$

The connection coefficient problem for $\tilde{P}_\ell(x; a, q)$ can now be solved. Let

$$\tilde{P}_n(x; b, q) = \sum_{\ell=0}^{n} a_{n\ell}\, \tilde{P}_\ell(x; a, q) \tag{4.8}$$

so that (4.7) implies

$$a_{n\ell} = \sum_{k=\ell}^{n} B_{nk}(b)B_{k\ell}^{-1}(a). \tag{4.9}$$

Clearly (4.9) implies that $a_{n\ell} = 0$ if $n - \ell$ is odd and

$$a_{n,n-2\ell} = \frac{(bq^{1+2n-4\ell}; q^2)_\ell}{(q^2; q^2)_\ell} q^{-\ell(n-2\ell)} \tag{4.10}$$
$$\sum_{k=0}^{\ell} \frac{(aq^{1+2n-4\ell}; q^2)_{2k}(bq^{1+2n-2\ell}; q^2)_k(q^{-2\ell}; q^2)_k}{(bq^{1+2n-4\ell}; q^2)_{2k}(aq^{3+2n-4\ell}; q^2)_k(q^2; q^2)_k} q^{2k}.$$

For $q = 1$ the right-hand side of (4.10) is summable by a theorem of Bailey [5, Eq. (3.42)]. A q-analogue of Bailey's result can be given (see [2] and [10]), but it will not evaluate (4.10).

There is another set of orthogonal polynomials which is related to Theorem 2.3. Rogers [13] defined

$$A_n(\cos\theta) = \sum_{m=0}^{n} \begin{bmatrix} n \\ m \end{bmatrix} \cos(n - 2m)\theta \tag{4.11}$$

so $A_n(\cos\theta)$ is a polynomial of degree n in $\cos\theta$. Let $T_k(\cos\theta) = \cos k\theta$ be the Chebyshev polynomial of the first kind. If $r_k(x) = 2T_k(x)$, $k \geqq 1$, $r_0(x) = 1$, then (4.11) implies

$$A_n(x) = \sum_{m=0}^{n/2} \begin{bmatrix} n \\ m \end{bmatrix} r_{n-2m}(x) \tag{4.12}$$

so $A_n(x) = \sum_{m=0}^{n} C_{nm} r_m(x)$, where $C_{nm} = 0$ if $n - m$ is odd and $C_{n,n-2m} = \left[\begin{smallmatrix} n \\ m \end{smallmatrix}\right]$. In Theorem 2.3 put $A = 1$, $p = q$, and $r = 2$. Then, if $n - m$ is even,

$$B_{nm}^{-1} = \frac{(1 - q^m)}{(1 - q^n)} C_{nm}(-1)^{(n-m)/2} q^{\binom{n/2+1}{2} + \binom{m/2}{2}} \tag{4.13}$$

so

$$C_{kn}^{-1} = \frac{(1 - q^k)}{(1 - q^n)} B_{kn}(-1)^{(n-k)/2} q^{\binom{n/2+1}{2} + \binom{k/2}{2}} \tag{4.14}$$

which implies

$$r_k(x) = \sum_{n=0}^{k/2} (1 - q^k) \frac{(q)_{k-n-1}}{(q)_{k-2n}(q)_n} (-1)^n q^{\binom{n}{2}} A_{k-2n}(x). \tag{4.15}$$

Equation (4.15) was a key ingredient in Rogers' proof of the Rogers-Ramanujan identities (see [6]).

Bressoud [7] has given a matrix inverse which is equivalent to the connection coefficients problem for a certain set of discrete q-Jacobi polynomials.

5. Krattenthaler's Theorem. Krattenthaler [12] gave a q-Lagrange inversion formula (his Theorem 1) for a certain class of expansions. As we remarked in section 2, he found several examples which correspond to our matrix inversion problems. In this section we show that our general results—Theorem 3.7 of [10] and Theorem 2.3 can be put into this form.

Roughly speaking, Krattenthaler took a formal power series $f(x)$ and asked for the coefficients a_k in the expansion

$$f(x) = \sum_{k=0}^{\infty} a_k x^k \phi_k(x) \tag{5.1}$$

where $\phi_k(x)$ satisfies a certain q-difference equation. He gave two solutions for a_k (Theorem 1(A) and (B)) for two different classes of functions $\phi_k(x)$. We shall use our matrices to define functions $\phi_k(x)$. Then we shall give the solution a_k in forms that correspond to Krattenthaler's Theorem (A) and (B).

For Theorem 3.7 of [10], it is clear that we put

$$\phi_k(x, a, p, q) = \sum_{n=0}^{\infty} \frac{(q^k p^{a+k}; p)_n}{(q)_n} (xq^{-k})^n. \tag{5.2}$$

Clearly, if (5.1) holds, then Theorem 3.7 implies

$$a_k = \sum_{\ell=0}^{k} \frac{(q^k p^{a+k-1}; p^{-1})_{k-\ell-1}}{(q)_{k-\ell}} (1 - p^{a+\ell} q^{\ell}) q^{-k(k-\ell)+\binom{k-\ell+1}{2}} (-1)^{k-\ell} f_\ell. \tag{5.3}$$

Defining

$$\tilde{\phi}_k(x, a, p, q) = \sum_{n=0}^{\infty} \frac{(q^k p^{a+k}; p^{-1})_n}{(q)_n} q^{\binom{n+1}{2}} (-x)^n, \tag{5.4}$$

we see that (5.3) is equivalent to

$$a_k = \frac{1}{1 - q^k p^{a+k}} \operatorname*{Res}_x \{(f(x) - p^a f(pqx))\tilde{\phi}_k(xq^{-k}, a, p, q)/x^{k+1}\}. \tag{5.5}$$

Equation (5.5) corresponds to Theorem 1(B). Note also that

$$\tilde{\phi}_k(x, a, p, q) = \phi_{-k}(xq^k, -a, p^{-1}, q^{-1}). \tag{5.6}$$

A form that corresponds to Theorem 1(A) is

$$a_k = \frac{1}{1 - q^k p^{a+k}} \operatorname*{Res}_x \{f(x)[\tilde{\phi}_k(xq^{-k}, a, p, q) - p^{a+k} q^k \tilde{\phi}_k(xq^{-k-1}p^{-1}, a, p, q)]/x^{k+1}\}. \tag{5.7}$$

The q-binomial theorem implies that ϕ_k and $\tilde{\phi}_k$ are summable if $p = 1$ or $p = q$. These two cases are Examples 2 and 4 in [12].

For Theorems 2.3 and 2.6 we offer an example instead of a general theorem. In this example the function ϕ_k is summable. In Theorem 2.3 take $r = 2$, $A = q^a$, and $p = q$. By (2.9), let

$$f(x) = \sum_{k=0}^{\infty} a_k x^k / (x^2 q^{-k/2})_{a+k}. \tag{5.8}$$

Then Theorem 2.3 implies

(5.9) $$a_k = \frac{1}{1-q^{a+k}} \operatorname*{Res}_x \{(x^2q^{1-k/2})_{a+k}(f(x) - q^a f(qx))/x^{k+1}\}$$

or, equivalently

(5.10) $$a_k = \operatorname*{Res}_x \{(x^2q^{1-k/2})_{a+k-2}(1 - q^{a-1}x^4)f(x)/x^{k+1}\}.$$

For $a = 0$, the difference quotient in (5.9) is a q-difference operator.

We do not have a general theorem such as Krattenthaler's which will give (5.5), (5.7), (5.9), and (5.10). It is clear that ϕ_k satisfies a nice q-difference equation though.

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Department of Mathematics, Massachusetts Institute of Technology, Cambridge, MA 02139

School of Mathematics, University of Minnesota, Minneapolis, MN 55455