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REFLECTION GROUPS AND MULTIPLICATIVE INVARIANTS

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Introduction. Given a lattice M (i.e., a finitely generated torsion free abelian group), one can form the group algebra $\mathbf{C}[M]$. The operation for M, usually thought of as addition, must then be regarded as multiplication. An automorphism of M extends to an algebra automorphism of $\mathbf{C}[M]$ in a unique way. We refer to $GL(M)$ as inducing a "multiplicative action" on $\mathbf{C}[M]$.

The semi-expository paper **[2]** is devoted to such actions. One of the theorems proved there was a multiplicative analogue of the Shephard-Todd-Chevalley Theorem.

THEOREM. *Assume M is a lattice and G is a finite subgroup of $GL(M)$. Then the fixed ring $\mathbf{C}[M]^G$ is a polynomial ring over $\mathbf{C}$ if and only if G is a reflection group and, for some choice of root system, M is isomorphic as a module to a weight lattice over its Weyl group G.*

Subsequently, I was led to a paper of Steinberg **[6]** in which a related theorem appears. Indeed, it is fair to say that the theorem above is implicit in Steinberg's work. Apparently, it has been valuable, for general ring theorists, to bring the invariant theoretic statement into relief. My arguments are naive in the sense that they use no algebraic geometry and employ only the rudiments of root systems.

This note is an elaboration of the second half of **[2]**. The theorem stated above says that, even for reflection groups, it is rare that the fixed ring of the group algebra is a polynomial ring. This distinction among G-module structures for M disappears once we pass to the rational function field of fractions $\mathbf{C}(M)$.

THEOREM 10. *Assume M is a lattice and $G \subset GL(M)$ is a finite reflection group. Then $\mathbf{C}(M)^G$ is always a rational function field.*

The same techniques prove a generalization of the invariant theorem.

COROLLARY 13. *Assume M is a lattice and $G \subset GL(M)$ is a finite re-*

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flection group. Then $\mathbf{C}[M]^G$ *is a polynomial ring over* $\mathbf{C}[M^G]$ *if and only if, for some choice of root system,* M/M^G *can be realized as its weight lattice and the group induced by* G *as its Weyl group.*

Once again, the last result should be credited to Steinberg. In conjunction with §3 of **[2]**, one obtains the final word on the subject.

COROLLARY. *Assume* M *is a lattice and* G *is a finite subgroup of* $GL(M)$. *Then* $\mathbf{C}[M]^G$ *is the tensor product of a group algebra and a polynomial algebra (over* $\mathbf{C}$*) if and only if* G *is a reflection group and, for some root system,* M/M^G *is isomorphic (as a module over the group induced by* G*) to a weight lattice (over the Weyl group).*

The reader will find that §1 of this note is a review of expected and/or well known facts about an action of a reflection group which may not be effective. §2 is, I hope, a clarification of the paragraph about exponential invariants in Bourbaki **[1]**. The last section contains the main theorems as applications.

I am indebted to L. Solomon and R. Steinberg for patiently guiding this neophyte through the literature.

1. Root systems. In this section we collect folklore and trivia about finite reflection groups. For the most part, we will adopt Humphreys' notation **[4]** for root systems.

DEFINITION. Assume M is a lattice and $G \subset GL(M)$ is a finite group generated by reflections. Let $V = \mathbf{R} \otimes_{\mathbf{Z}} M$. A rooting section for M is an ordered pair (π, Φ) such that

RS 1. $\pi: V \to V$ is an idempotent $\mathbf{R}[G]$-module map;
RS 2. $\operatorname{Ker}\pi = V^G$;
RS 3. Φ is a root system for $\pi(V)$;
RS 4. The restriction of G to $\pi(V)$ is the Weyl group of Φ; and
RS 5. $\Phi \subset M \subset \pi^{-1}(\Lambda)$ where Λ is the weight lattice for Φ.

For the rest of this paper M denotes a lattice and $G \subset GL(M)$ is a finite reflection group. We first prove that rooting sections always exist by constructing a "maximal" one for M.

Average an arbitrary inner product on $V = \mathbf{R} \otimes M$ over the group G to obtain a G-invariant one. Each member of G then becomes an orthogonal transformation. We shall denote this inner product $(\cdot, \cdot)$. Define π to be the orthogonal projection on $(V^G)^\perp$. It is easy to check that π satisfies RS 1 and RS 2.

Suppose that σ is a reflection in G. The eigenspace in V corresponding to the eigenvalue -1 is one-dimensional and must meet M. Thus $M(\sigma) = \{m \in M | \sigma(m) = -m\}$ is a cyclic subgroup of M. Call its two possible generators α and $-\alpha$ and rename the reflection $\sigma = \sigma_\alpha$.

Notice that α and $-\alpha$ are the nonzero vectors of smallest length in $M(\sigma_\alpha)$. Let Φ be the collection of all $\pm\alpha$ as σ_α ranges over all reflections in G.

LEMMA 1. *Φ spans $\pi(V)$.*

PROOF. Let W be the span of Φ in V. Then

$$W^\perp = \{v \in V \mid (v, \alpha) = 0, \text{ for all } \alpha \in \Phi\}.$$

The basic formula for reflections states that if $x \in V$, then

$$\sigma_\alpha(x) = x - \frac{2(x, \alpha)}{(\alpha, \alpha)}\alpha.$$

Thus $W^\perp = \{v \in V \mid \sigma_\alpha(v) = v, \text{ for all } \alpha \in \Phi\}$. Since G is generated by $\{\sigma_\alpha \mid \alpha \in \Phi\}$, we have $W^\perp = V^G$. Therefore, $W = (V^G)^\perp$.

We will use the shorthand $\langle v, \alpha\rangle = 2(v, \alpha)/(\alpha, \alpha)$ for $\alpha \in \Phi$.

LEMMA 2. *Φ is a root system for $\pi(V)$.*

PROOF. We already know Φ spans $\pi(V)$. We next argue that each σ_α stabilizes Φ. If $\alpha, \beta \in \Phi$, then a calculation shows that

$$\sigma_\alpha(M(\sigma_\beta)) = M(\sigma_\alpha\sigma_\beta\sigma_\alpha^{-1}).$$

Because σ_α preserves length, it sends a vector of minimal length in $M(\sigma_\beta)$ to one of minimal length in $M(\sigma_\alpha\sigma_\beta\sigma_\alpha^{-1})$. Hence, $\sigma_\alpha(\beta) \in \Phi$.

A similar minimal length argument shows that the only multiples of $\alpha \in \Phi$ which lie in Φ are $\pm\alpha$.

Finally, suppose $x \in M$ and $\alpha \in \Phi$. Then $\sigma_\alpha(x) \in M$, i.e., $x - \langle x, \alpha\rangle\alpha \in M$. Thus $\langle x, \alpha\rangle\alpha \in M$. Since $M(\sigma_\alpha)$ is generated by α, we find that $\langle x, \alpha\rangle$ is an integer. In particular, $\langle\beta, \alpha\rangle \in \mathbf{Z}$ for $\alpha, \beta \in \Phi$.

It is obvious that G restricted to $\pi(V)$ is the Weyl group for Φ. We finish the verification that (π, Φ) is a rooting section by checking RS 5. By construction, $\Phi \subset M$. Recall that $\Lambda = \{y \in \pi(V) \mid \langle y, \alpha\rangle \in \mathbf{Z}, \text{ for all } \alpha \in \Phi\}$. We require

LEMMA 3. *$\pi(M) \subset \Lambda$.*

PROOF. If $x \in M$ and $\alpha \in \Phi$, then the last paragraph of Lemma 2 tells us that $\langle x, \alpha\rangle$ is an integer of the form

$$\langle x, \alpha\rangle = \frac{2(x, \alpha)}{(\alpha, \alpha)} = \frac{2(\pi(x), \alpha)}{(\alpha, \alpha)} = \langle\pi(x), \alpha\rangle.$$

2. Invariants. For the next two sections, we assume that a rooting section (π, Φ) for M is given.

Specify a base Δ for the root system. The base gives rise to a very versatile partial order on the real span of πM. Say that $x \leqq y$ if $y - x$ is a non-negative linear combination of the simple roots in Δ. There is a second candidate for a partial order which does not coincide with the first. To avoid confusion, we shall only describe its positive cone. Set $\Lambda^+ = \{\omega \in \Lambda \mid (\omega, \alpha) \geqq 0$, for all $\alpha \in \Delta\}$. We quote the crucial properties.

THEOREM 4. (i) ([**4**, p. 70] or [**1**, p. 187]) *The partial order* $\leqq$, *when restricted to* Λ^+, *satisfies the minimum condition.*

(ii) (**4**, p. 68]) *If* $x \in \Lambda^+$ *and* g *lies in the Weyl group, then* $g(x) \leqq x$.

We shall say that a finite subset F of M (or of its span V) is peaked if there is a member $y \in F$ such that $\pi(y) > \pi(z)$, for all other $z \in F$. In this case we call y the peak of F.

THEOREM 5. *The orbit under* G *of each vector in* M *is peaked and its peak lies in* $\pi^{-1}(\Lambda^+)$.

PROOF. It is known ([**4**, p. 68]) that the orbit of a weight under the Weyl group contains exactly one dominant weight. Moreover, this dominant weight is a maximum element of the orbit, by Theorem 4 (ii). These results can be pulled back once we show that $\pi(y) = \pi(gy)$, for $g \in G$ and $y \in M$, implies $y = gy$.

Choose $\xi \in M^G$ such that $g(y) = y + \xi$. By induction, $g^k(y) = y + k\xi$, for all positive integers k. In particular, if k is the order of g, then $y = y + k\xi$. It follows that $\xi = 0$, i.e., $g(y) = y$.

We are finally ready to study the group algebra $\mathbf{C}[M]$. Since the operation of addition in M becomes multiplication in $\mathbf{C}[M]$, we will write λ^* for the canonical image of $\lambda \in M$ inside the group algebra $\mathbf{C}[M]$. For example, $(\lambda - \mu)^* = \lambda^*(\mu^*)^{-1}$. If $a = \sum a_\lambda \lambda^*$, then the support of a is the finite set $\{\lambda \in M \mid a_\lambda \neq 0\}$. We say that $a \in \mathbf{C}[M]$ is peaked if its support is peaked and its peak has coefficient 1. Let $\hat{}a$ denote its peak. It is easy to check that if a_1 and a_2 in $\mathbf{C}[M]$ are peaked, then so is a_1a_2. In that case, $\hat{}(a_1a_2) = \hat{}a_1\hat{}a_2$.

Certain peaked elements merit special attention. If $\lambda \in M$, then the support of $\sum_{g \in G}(g \cdot \lambda)^*$ is the orbit of λ under G and each member of the support has the same coefficient. Multiply the sum by the reciprocal of that number to obtain $X(\lambda) \in \mathbf{C}[M]^G$. Then Theorem 5 implies that $X(\lambda)$ is peaked. In fact, $\{X(\lambda) \mid \lambda \in \pi^{-1}(\Lambda^+) \cap M\}$ is a basis for $\mathbf{C}[M]^G$ with $\hat{}X(\lambda) = \lambda$. Notice that if $\xi \in M^G$, then $X(\xi) = \xi^*$.

LEMMA 6. *Let* E *be a set of peaked elements in* $\mathbf{C}[M]^G$ *such that*

$$\pi(\hat{}E) = \pi M \cap \Lambda^+.$$

Then $\mathbf{C}[M]^G$ *is generated by* E *as a* $\mathbf{C}[M^G]$*-module.*

PROOF. Let S be the $\mathbf{C}[M^G]$-submodule of $\mathbf{C}[M]^G$ spanned by E. According to the remarks above, we need only prove that $X(\lambda) \in S$, for all $\lambda \in \pi^{-1}(\Lambda^+) \cap M$.

Suppose not. Using the minimum condition (Theorem 4(i)), we can find $\lambda \in \pi^{-1}(\Lambda^+) \cap M$ such that $\pi(\lambda)$ is minimal subject to $X(\lambda) \notin S$. Choose $y \in E$ with $\pi(\hat{\ }y) = \pi(\lambda)$. That is, $\hat{\ }y = \lambda - \xi$, for some $\xi \in M^G$. Then the support of $X(\lambda) - \xi^* y$ consists of elements μ with $\pi(\mu) < \pi(\lambda)$, whence $X(\lambda) - \xi^* y \in S$. But then $X(\lambda) \in S$.

The next theorem is a broadening of the main result in [**1**, VI. 3.4]. Having found it dangerous to credit the proposition to any one individual, real or contrived, I will only say that its essence goes back to E. Cartan.

THEOREM 7. *Suppose $Y_1, \ldots, Y_n$ are peaked elements of* $\mathbf{C}[M]^G$. If *the function from the set of formally distinct monomials in $Y_1, \ldots, Y_n$ to $\pi M \cap \Lambda^+$, which sends Y to $\pi(\hat{\ }Y)$, is a surjection, then*

$$\mathbf{C}[M]^G = \mathbf{C}[M^G]\,[Y_1, \ldots, Y_n].$$

Furthermore, if this function is a bijection, then $Y_1, \ldots, Y_n$ are algebraically independent over $\mathbf{C}[M^G]$.

PROOF. Let E be the monoid of monomials in $Y_1, \ldots, Y_n$. By the lemma, $\mathbf{C}[M]^G$ is spanned by E as a module over $\mathbf{C}[M^G]$. This proves the first half of the theorem. We use the additional hypothesis to show that $\mathbf{C}[M]^G$ is a free $\mathbf{C}[M^G]$-module with basis E.

Suppose $\sum_{\text{finite}} r_Y Y = 0$, where $Y \in E$ and $r_Y \in \mathbf{C}[M^G]$. Assuming not all coefficients are zero, choose a monomial Z with $\pi\,(\hat{\ }Z)$ maximal among those Y with $r_Y \neq 0$. If b lies in the support of r_Z, then $b(\hat{\ }Z)$ must appear a second time as some cx, where c and x lie in the supports of r_T and the monomial T respectively. Thus, $\pi(\hat{\ }Z) = \pi(x) \leqq \pi(\hat{\ }T)$. By maximality and the bijection hypothesis, $Z = T$; by peakedness, $x = \hat{\ }Z$. We have found that $b(\hat{\ }Z)$ arises in only one way, a contradiction.

3. Applications. Given the base Δ, let $\omega_1, \ldots, \omega_m$ be fundamental dominant weights. We have called [**2**, **3**] a submodule of Λ which contains Φ a stretched weight lattice if it has a $\mathbf{Z}$-basis of the form $k_1\omega_1, \ldots, k_m\omega_m$, for some positive integers $k_1, \ldots, k_m$. The next statement is an immediate consequence of Theorem 7.

COROLLARY 8. *Suppose that πM is a stretched weight lattice with basis $k_1\omega_1, \ldots, k_m\omega_m$. For each $j = 1, \ldots, m$, choose a peaked element $Y_j \in$ $\mathbf{C}[M]^G$ such that $\pi(\hat{\ }Y_j) = k_j\omega_j$. (For instance, if $\pi(\lambda_j) = k_j\omega_j$ one might pick $Y_j = X(\lambda_j)$.) Then*

$$\mathbf{C}[M]^G = \mathbf{C}[M^G]\,[Y_1, \ldots, Y_m]$$

and $Y_1, \dots, Y_m$ *are algebraically independent over* $\mathbf{C}[M^G]$.

In order to further apply Theorem 7, we need to generate interesting collections of peaked elements in $\mathbf{C}[M]^G$.

LEMMA 9. *Suppose* $\lambda_1, \dots, \lambda_k \in M + \Lambda$ *are such that* $\lambda_1 + \dots + \lambda_k \in M$. *Then* $X(\lambda_1)X(\lambda_2) \dots X(\lambda_k) \in \mathbf{C}[M]^G$. (*Here, we regard* $X(\lambda_j) \in \mathbf{C}[M + \Lambda]$.)

PROOF. Details can be found in Lemmas 14 and 15 of **[2]**. A typical element in the support of $X(\lambda_1) \dots X(\lambda_k)$ has the form $g_1(\lambda_1) + \dots + g_k(\lambda_k)$ for some choice of $g_1, \dots, g_k \in G$. To show that this sum lies in M, it suffices to check that

$$\sigma_\alpha(\lambda_j) - \lambda_j \in M, \qquad \text{for } j = 1, \dots, k \text{ and } \alpha \in \Phi.$$

But $\sigma_\alpha(\lambda_j) - \lambda_j = - < \lambda_j, \alpha > \alpha$ lies in the root lattice. The result follows from RS 5.

Let $\mathbf{C}(M)$ denote the field of fractions of $\mathbf{C}[M]$. It is not difficult to prove [**5**, Lemma 2.5.12] that the field of fractions of $\mathbf{C}[M]^G$ coincides with the fixed field $\mathbf{C}(M)^G$.

THEOREM 10. $\mathbf{C}(M)^G$ *is always a rational function field.*

PROOF. First, imbed M in $N = M + \Lambda$. According to Corollary 8, $\mathbf{C}[N]^G = \mathbf{C}[N^G]\,[X(\omega_1), \dots, X(\omega_m)]$, where $X(\omega_1), \dots, X(\omega_m)$ are algebraically independent over the group algebra $\mathbf{C}[N^G]$. It follows that the multiplicative subgroup of $\mathbf{C}(N)$ generated by N^G and $X(\omega_1), \dots, X(\omega_m)$ is a free abelian group A of finite rank whose members are linearly independent over $\mathbf{C}$.

Each $\lambda \in \pi^{-1}(\Lambda^+) \cap M$ can be written in the form

$$\lambda = \xi + \sum_{j=1}^{m} c_j w_j,$$

where $\xi \in N^G$ and each c_j is a non-negative integer. Set

$$Y_\lambda = X(\xi) \prod_{j=1}^{m} X(\omega_j)^{c_j}.$$

According to Lemma 9, $Y_\lambda \in \mathbf{C}[M]^G$. It is peaked; indeed $\hat{}Y_\lambda = \lambda$. Since $M^G \subset \pi^{-1}(\Lambda^+) \cap M$, Theorem 7 yields

$$\mathbf{C}[M]^G = \mathbf{C}[Y_\lambda],$$

where λ runs over $\pi^{-1}(\Lambda^+) \cap M$. Moreover, each Y_λ lies in A. If B denotes the subgroup of A generated by $\{Y_\lambda | \lambda \in \pi^{-1}(\Lambda^+) \cap M\}$ then obviously B must be a free abelian group of finite rank as well. What is more, $\mathbf{C}[M]^G$ localized at B is isomorphic to the group algebra $\mathbf{C}[B]$.

It follows that the field of fractions of $\mathbf{C}[M]^G$ is isomorphic to the rational function field $\mathbf{C}(B)$.

LEMMA 11. *$M + \Lambda/M$ is finite.*

PROOF. $M + \Lambda/M \simeq \Lambda/\Lambda \cap M$ and $\Lambda \cap M$ lies between the root lattice and Λ. But the root lattice has finite index in the weight lattice.

THEOREM 12. *If $\mathbf{C}[M]^G$ is a unique factorization domain, then πM is a stretched weight lattice.*

PROOF. We closely follow the proof of Theorem 16 in **[2]**. Set $N = M + \Lambda$. Then $\mathbf{C}[N]^G$ is the polynomial ring $\mathbf{C}[N^G]\,[X(\omega_1), \ldots, X(\omega_m)]$; we will think of $\mathbf{C}[M]^G$ as a subring.

Call a nonzero element of $\pi M \cap \Lambda^+$ indecomposable if it cannot be written as a sum of two nonzero elements of $\pi M \cap \Lambda^+$. Suppose $\lambda \in M$ is such that $\pi(\lambda)$ is indecomposable. We may write

$$\lambda = \xi + (a_1\,\omega_1 + \ldots + a_m\omega_m)$$

where $\xi \in N^G$ and each a_j is a non-negative integer. As a consequence of Lemma 9,

$$Y = X(\xi)X(\omega_1)^{a_1}\ldots X(\omega_m)^{a_m}$$

lies in $\mathbf{C}[M]^G$. Any factoring of Y in $\mathbf{C}[M]^G$ is also a factoring of Y in the UFD $\mathbf{C}[N]^G$. As observed in **[2]**, the indecomposability of $\pi(\lambda)$ forces Y to be irreducible in $\mathbf{C}[M]^G$.

Lemma 11 implies that there is a positive integer d such that $d\cdot N \subset M$. Lemma 9 again tells us that $X(\xi)^d$ and all of the $X(\omega_j)^d$ are in $\mathbf{C}[M]^G$. Now,

$$Y^d = X(\xi)^d[X(\omega_1)^d]^{a_1}\ldots[X(\omega_m)^d]^{a_m}.$$

Since 0 cannot be indecomposable, at least one of the a_i is nonzero; this precludes Y from being a unit. Hence, under the assumption that $\mathbf{C}[M]^G$ is a UFD, Y divides $X(\omega_j)^d$, for some j. Now, unique factorization in $\mathbf{C}[N^G]\,[X(\omega_1), \ldots, X(\omega_m)]$ implies that $Y = X(\xi)X(\omega_j)^{a_j}$. In other words, $\pi(\lambda) = a_j\omega_j$.

As proved in **[2]**, πM is a stretched weight lattice if and only if its indecomposables have the form $a_j\omega_j$, for positive integers a_j.

COROLLARY 13. *Assume M is a lattice and $G \subset GL(M)$ is a finite reflection group. Then the following statements are equivalent.*

(a) *There is a root system which realizes M/M^G as its weight lattice and whose Weyl group is the group induced by G.*

(b) *$\mathbf{C}[M]^G$ is a polynomial ring over the group algebra $\mathbf{C}[M^G]$.*

(c) *$\mathbf{C}[M]^G$ is a* UFD.

PROOF. (a) $\rightarrow$ (b). Let V be the real span of M. By Maschke's Theorem, there is a G-submodule U such that $V = V^G \oplus U$. We can identify M/M^G with the projection of M into U. The root system contained in M/M^G thereby becomes a root system for U. We have produced a rooting section for M. Since the image of M in U is a weight lattice, (b) follows from Corollary 8.

(b) $\rightarrow$ (c). This is classical.

(c) $\rightarrow$ (a). Let (π, Φ) be the maximal rooting section for M constructed in §1. Theorem 12 asserts that πM is a stretched weight lattice for the Weyl group G induced by G. As $\mathbf{Z}[G]$-modules, M/M^G and πM are isomorphic. We observed in [**3**] (note added in proof) that every stretched weight lattice is isomorphic as a G-module to some ordinary weight lattice, at the possible expense of replacing the original root system by another one with the same Weyl group.

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