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A Q-ANALOGUE OF APPELL'S F_1 FUNCTION AND SOME QUADRATIC TRANSFORMATION FORMULAS FOR NON-TERMINATING BASIC HYPERGEOMETRIC SERIES

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ABSTRACT. A q-analogue of the integral representation of Appell's F_1 function is given as an extension of Askey and Wilson's q-beta integral and is evaluated as a sum of three very well-poised ${}_{10}\phi_9$ series. The formula is then applied to find two different types of quadratic transformation formulas between very well-poised ${}_{10}\phi_9$ series and balanced ${}_5\phi_4$ series. Special cases of balanced and very well-poised ${}_{10}\phi_9$ series are also examined.

1. Introduction. The Appell function F_1 is defined by the double infinite series [7, p. 224]

$$F_1(\alpha, \beta, \beta', \gamma; x, y) = \sum_m \sum_n \frac{(\alpha)_{m+n}(\beta)_m(\beta')_n}{m!n!(\gamma)_{m+n}} x^m y^n \tag{1.1}$$

subject to usual convergence restrictions, where the shifted factorials are defined by $(a)_0 = 1$, $(a)_m = a(a+1) \cdots (a+m-1)$, $m = 1,2,\ldots$. Of all the Appell functions this is the only one that has a representation in terms of a single integral [7, p. 231]

$$\begin{aligned} &F_1(\alpha, \beta, \beta', \gamma; x, y) \\ &= \frac{\Gamma(\gamma)}{\Gamma(\alpha)\Gamma(\gamma-\alpha)} \int_0^1 t^{\alpha-1}(1-t)^{\gamma-\alpha-1}(1-xt)^{-\beta}(1-yt)^{-\beta'}dt, \end{aligned} \tag{1.2}$$

where $0 < \operatorname{Re} \alpha < \operatorname{Re} \gamma$. Using the q-beta type integral of Askey and Wilson [3] the authors [8] recently found the following q-analogue of Euler's integral representation of Gauss' hypergeometric series ${}_2F_1$

$$\begin{aligned} &{}_8\phi_7\left[\begin{matrix}\lambda abcq^{-1}, q\sqrt{}, -q\sqrt{}, bc, ac, ab, \lambda d^{-1}, \lambda f^{-1} \\ \sqrt{}, -\sqrt{}, \lambda a, \lambda b, \lambda c, abcd, abcf\end{matrix}; q, df\right] \\ &= \frac{(q, ab, ac, ad, af, bc, bd, bf, cd, cf, \lambda abc; q)_\infty}{2\pi(\lambda a, \lambda b, \lambda c, abcd, abcf; q)_\infty} \\ &\quad \cdot \int_{-1}^1 w(z; a, b, c, d) \frac{h(z; \lambda)}{h(z; f)} dz, \end{aligned} \tag{1.3}$$

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where

$$(1.4)\qquad \begin{aligned} h(z;a) &= \prod_{n=0}^{\infty}(1-2azq^n+a^2q^{2n}) \\ &= (ae^{i\phi};q)_\infty(ae^{-i\phi};q)_\infty,\ z=\cos\phi, \end{aligned}$$

$$(1.5)\qquad w(z;a,b,c,d) = (1-z^2)^{-1/2}\frac{h(z;1)h(z;-1)\,h(z;\sqrt{q})h(z;-\sqrt{q})}{h(z;a)h(z;b)h(z;c)h(z;d)},$$

$$(1.6)\qquad (a;q)_\infty = \prod_{n=0}^{\infty}(1-aq^n),\ \text{whenever it converges,}$$

$$(1.7)\qquad (a_1,a_2,\ldots,a_k;q)_\infty = (a_1;q)_\infty\,(a_2;q)_\infty\cdots(a_k;q)_\infty$$

and the symbol on the left hand side of (1.3) represents a basic hypergeometric series defined by

$$(1.8)\qquad \begin{aligned} {}_{r+1}\phi_r\left[\begin{matrix} a_1, a_2, \ldots, a_{r+1} \\ b_1, \ldots, b_r \end{matrix}; q, z\right] \\ = \sum_{n=0}^{\infty}\frac{(a_1;q)_n\,(a_2;q)_n\cdots(a_{r+1};q)_n}{(q;q)_n(b_1;q)_n\ldots(b_r;q)_n}z^n \end{aligned}$$

provided $|z| < 1$ when the series does not terminate, with

$$(1.9)\qquad (a;q)_n = \frac{(a;q)_\infty}{(aq^n;q)_\infty}.$$

The series that occurs in (1.3) is very well-poised, and the open square root is over the top left-hand element, namely, $\lambda abcq^{-1}$. The convergence of the integral on the right of (1.3) requires that

$$(1.10)\qquad \max(|q|, |a|, |b|, |c|, |d|, |f|) < 1$$

under which condition the ${}_8\phi_7$ series is convergent. If we set

$$(1.11)\qquad \begin{aligned} a = q^{\alpha/2-1/4},\ b = q^{\alpha/2+1/4},\ c &= -q^{\gamma/2-\alpha/2-1/4}, \\ d &= -q^{\gamma/2-\alpha/2+1/4},\ f = \lambda q^{-\beta}, \end{aligned}$$

then use the q-gamma function [2]

$$(1.12)\qquad \Gamma_q(x) = \frac{(q;q)_\infty}{(q^x;q)_\infty}(1-q)^{1-x},\ 0<q<1,\ \lim_{q\to 1}\Gamma_q(x) = \Gamma(x),$$

and take the limit $q \to 1$, it can be easily shown that (1.3) approaches Euler's formula

$$(1.13)\qquad {}_2F_1(\alpha,\beta;\gamma;x) = \frac{\Gamma(\gamma)}{\Gamma(\alpha)\Gamma(\gamma-\alpha)}\int_0^1 t^{\alpha-1}(1-t)^{\gamma-\alpha-1}(1-xt)^{-\beta}dt,$$

where $x = -4\lambda(1-\lambda)^{-2}$, such that $|x| < 1$.

The analogy between (1.3) and (1.13) naturally leads us to consider the integral

$$S(a, b, c, d, f, g; \lambda, \mu) \equiv \int_{-1}^{1} w(z; a, b, c, d)\frac{h(z; \lambda)h(z; \mu)}{h(z; f)h(z; g)}\,dz \tag{1.14}$$

as a possible q-analogue of the integral in (1.2). Indeed, if we take $0 < q < 1$, identify the parameters a, b, c, d, f as in (1.11), set $g = \mu q^{-\beta'}$ and let $q \to 1$, we can show that the integral in (1.14) approaches that in (1.2) with $x = -4\lambda(1-\lambda)^{-2}$ and $y = -4\mu(1-\mu)^{-2}$.

Such an integral has recently been computed by Rahman **[10]**, but in the special case

$$\lambda\mu = abcdfg. \tag{1.15}$$

Rahman **[10]** found that

$$\begin{aligned}
&S(a, b, c, d, f, g; \lambda, \mu)\\
&= \frac{2\pi(\lambda\mu/ag, \lambda\mu/bg, \lambda\mu/cg, \lambda\mu/dg, \lambda f, \lambda/f, \mu f, \mu/f; q)_\infty}{(q, ab, ac, ad, af, bc, bd, bf, cd, cf, df, fg, g/f, \lambda\mu f/g; q)_\infty}\\
&\quad\cdot {}_{10}\phi_9\left[\begin{matrix}\lambda\mu f/gq, q\sqrt{\ }, -q\sqrt{\ }, af, bf, cf, fd, \lambda/g, \mu/g, \lambda\mu/q\\ \sqrt{\ }, -\sqrt{\ }, \lambda\mu/ag, \lambda\mu/bg, \lambda\mu/cg, \lambda\mu/dg, \mu f, \lambda f, fq/g\end{matrix}; q, q\right]\\
&+ \frac{2\pi(\lambda\mu/af, \lambda\mu/bf, \lambda\mu/cf, \lambda\mu/df, \lambda g, \lambda/g, \mu g, \mu/g; q)_\infty}{(q, ab, ac, ad, ag, bc, bd, bg, cd, cg, dg, gf, f/g, \lambda\mu g/f; q)_\infty}\\
&\quad\cdot {}_{10}\phi_9\left[\begin{matrix}\lambda\mu g/fq, q\sqrt{\ }, -q\sqrt{\ }, ag, bg, cg, dg, \lambda/f, \mu/f, \lambda\mu/q\\ \sqrt{\ }, -\sqrt{\ }, \lambda\mu/af, \lambda\mu/bf, \lambda\mu/cf, \lambda\mu/df, \mu g, \lambda g, gq/f\end{matrix}; q, q\right],
\end{aligned} \tag{1.16}$$

where the parameters are related by the balancing condition (1.15) as a result of which the ${}_{10}\phi_9$ series on the right are balanced. Because of this restriction, however, (1.16) does not provide a q-analogue of the general double series in (1.1). Specializing the parameters as in (1.11) with $g = \mu q^{-\beta'}$, one can see that (1.15) amounts to setting $\gamma = \beta + \beta'$ in (1.1) and, as is well-known, the F_1 series in this case reduces to a multiple of a ${}_2F_1$ series [7, p. 238].

One of the main objectives of this paper is to compute the general integral in (1.14) subject to the restriction

$$\max(|q|, |a|, |b|, |c|, |d|, |f|, |g|) < 1. \tag{1.17}$$

We shall, in fact, prove in §2 that

$$
\begin{aligned}
&S(a, b, c, d, f, g; \lambda, \mu) \\
&= \kappa(a, b, c, d) \frac{(q/abcd, aq/b, aq/c, aq/d, \lambda a, \lambda/a, \mu a, \mu/a; q)_\infty}{(qa^2, q/bc, q/bd, q/cd, af, f/a, ag, g/a; q)_\infty} \\
&\cdot {}_{10}\phi_9 \left[\begin{matrix} a^2, aq, -aq, ab, ac, ad, af, ag, aq/\lambda, aq/\mu \\ a, -a, aq/b, aq/c, aq/d, aq/f, aq/g, \lambda a, \mu a \end{matrix} ; q, \frac{\lambda\mu q}{abcdfg} \right] \\
&+ \text{idem}(a; f, g) \\
&\equiv S,
\end{aligned} \tag{1.18}
$$

where

$$
\kappa(a, b, c, d) = \frac{2\pi(abcd; q)_\infty}{(q, ab, ac, ad, bc, bd, cd; q)_\infty}, \tag{1.19}
$$

and idem$(a; f, g)$ means two similar expressions, one with $a \leftrightarrow f$ and the other with $a \leftrightarrow g$. Since the three series in (1.18) are, in general, non-terminating, their convergence requirement forces us to assume

$$
\left| \frac{\lambda\mu q}{abcdfg} \right| < 1. \tag{1.20}
$$

It is clear that under the balancing condition (1.15) all three series in (1.18) become balanced, as expected, but the right hand side does not resemble that of (1.16). However, using Bailey's four-term transformation formula **[4]** for balanced and non-terminating ${}_{10}\phi_9$'s one can show after a good deal of computations that (1.18) does indeed reduce to (1.16) in this special case.

It is somewhat curious that the right hand side of (1.18) is a sum of three single series while the left hand side is a q-analogue of the integral representation of the F_1 function and, therefore, in the limit $q \to 1$ leads to a double series, in general. One might begin to hope that the series on the right of (1.1) can perhaps be expressed as a single series. Unfortunately this is not the case. Using the parameters according to (1.11) and setting $g = \mu q^{-\beta'}$ on the right of (1.18) one does not get a single series limit. In fact, formal term-by-term limiting process leads to all sorts of divergence difficulties. The correct way to take the limit is to start with (2.15), which is a sum of three double series, and to show that one of them gives (1.1) while the other two go to zero. The exercise is by no means trivial and it might be useful for some readers to try to work it out. However, our main objective is not to say that (1.18) is a q-analogue of (1.1) but to prove that (1.18) is true for a wide range of values of the parameters subject only to the restrictions (1.17) and (1.20), and to use it to derive some quadratic transformation formulas between non-terminating basic hypergeometric series. In fact, there are two classes of formulas that we shall be able to handle by means of (1.18) and (1.16). One that connects very well-poised ${}_{10}\phi_9$ series in base q with balanced ${}_5\phi_4$ series in base q^2—we shall call it the transformation of type *I*—will be considered in §3. In §4 we shall deal with what we call the quadratic transformations of type II, where ${}_{10}\phi_9$ series in base q are transformed to balanced ${}_5\phi_4$ series in base $q^{1/2}$.

There is another interesting limit of (1.18) that we would like to point out. Replace the parameters a, b, c, d, f, g, λ, μ by q^a, q^b, q^c, q^d, q^f, q^g, q^λ and q^μ, respectively, replace $e^{i\psi}$ in the integral (1.14) by q^{ix} and then take the limit $q \to 1$. Use of (1.12) and the reflection formula for the gamma function leads to the following:

$$\begin{aligned}
&\int_{-\infty}^{\infty} \left| \frac{\Gamma(a+ix)\Gamma(b+ix)\Gamma(c+ix)\Gamma(d+ix)\Gamma(f+ix)\Gamma(g+ix)}{\Gamma(2ix)\Gamma(\lambda+ix)\Gamma(\mu+ix)} \right|^2 dx \\
&= \frac{4\pi^3 \sin(a+b+c+d)\pi\ \Gamma(a+b)\Gamma(a+c)\Gamma(a+d)\Gamma(1+2a)}{\sin(b+c)\pi\ \sin(b+d)\pi\ \sin(c+d)\pi\ \Gamma(1+a-b)} \\
&\quad \cdot \frac{\Gamma(a+f)\Gamma(f-a)\Gamma(g+a)\Gamma(g-a)}{\Gamma(1+a-c)\Gamma(1+a-d)\Gamma(\lambda+a)\Gamma(\lambda-a)\Gamma(\mu+a)\Gamma(\mu-a)} \\
&\quad \cdot {}_9F_8\left[\begin{matrix} 2a, 1+a, a+b, a+c, a+d, a+f, a+g, 1+a-\lambda, 1+a-\mu \\ a, 1+a-b, 1+a-c, 1+a-d, 1+a-f, 1+a-g, a+\lambda, a+\mu \end{matrix}; 1\right] \\
&\quad + \text{idem}(a; f, g),
\end{aligned} \tag{1.21}$$

provided

$$\begin{aligned}
&a, b, c, d, f, g > 0, \\
&a+b+c+d+f+g < \lambda+\mu+1, \\
&b+c+d < \min(1-a, 1-f, 1-g), \\
&\text{and } |a-g|, |a-f|, |f-g| \neq k, k = 0, 1, 2, \ldots.
\end{aligned} \tag{1.22}$$

2. Evaluation of $S(a, b, c, d, f, g; \lambda, \mu)$. One of the principal tools in this computation is Sears' formula for the sum of two balanced and non-terminating ${}_3\phi_2$'s [**12**, (5.2)], which *Al*-Salam and Verma [**1**, (1.3)] showed how to express in the compact notation of a q-integral

$$\begin{aligned}
&\int_a^b \frac{(qu/a, qu/b, cu; q)_\infty}{(du, eu, fu; q)_\infty} d_q u \\
&= \frac{b(1-q)(q, bq/a, a/b, c/d, c/e, c/f; q)_\infty}{(ad, ae, af, bd, be, bf; q)_\infty}
\end{aligned} \tag{2.1}$$

where $c = abdef$. The q-integral is defined by

$$\int_0^a f(x)d_qx = a(1-q)\sum_{n=0}^{\infty} f(aq^n)q^n,$$
$$\int_a^b f(x)d_qx = \int_0^b f(x)d_qx - \int_0^a f(x)d_qx. \tag{2.2}$$

Using (2.1) one can easily verify that

$$\frac{h(z;\mu)}{h(z;f)h(z;g)} = \frac{(\mu/f,\ \mu/g;\ q)_\infty}{g(1-q)\,(q,\ qg/f,\ f/g,\ fg;\ q)_\infty} \int_f^g \frac{(qu/f,\ qu/g,\ \mu u;\ q)_\infty}{(\mu u/fg;\ q)_\infty\, h(z;u)}\, d_qu. \tag{2.3}$$

Hence

$$S = \frac{(\mu/f,\ \mu/g;\ q)_\infty}{g(1-q)\,(q,\ qg/f,\ f/g,\ fg;\ q)_\infty} \int_f^g \frac{(qu/f,\ qu/g,\ \mu u;\ q)_\infty}{(\mu u/fg;\ q)_\infty}\, d_qu \cdot \int_{-1}^1 w(z;a,b,c,d)\,\frac{h(z;\lambda)}{h(z;u)}\, dz. \tag{2.4}$$

By (1.17) one can justify the interchange of integrals. The Riemann integral over z is similar to the one in (1.3) and so we find that

$$S = \kappa(a,b,c,d)\,\frac{(\lambda a,\ \lambda b,\ \lambda c,\ \mu/f,\ \mu/g;\ q)_\infty}{g(1-q)\,(q,\ qg/f,\ f/g,\ fg,\ \lambda abc;\ q)_\infty} \cdot \int_f^g \frac{(qu/f,\ qu/g,\ \mu u,\ abcu;\ q)_\infty}{(au,\ bu,\ cu,\ \mu u/fg;\ q)_\infty} \cdot {}_8\phi_7\left[\begin{matrix}\lambda abcq^{-1},\ q\sqrt{},\ -q\sqrt{},\ ab,\ ac,\ bc,\ \lambda d^{-1},\ \lambda u^{-1} \\ \sqrt{},\ -\sqrt{},\ \lambda c,\ \lambda b,\ \lambda a,\ abcd,\ abcu\end{matrix};\ q,\ du\right] d_qu. \tag{2.5}$$

It would be nice if we could find a transformation that would lead directly to (1.18). Such a short cut does not seem to exist at the moment, so we are forced into a rather long and tedious computation. First, we use Bailey's formula [5, (3), p. 69] to express the ${}_8\phi_7$ above as a sum of two balanced ${}_4\phi_3$ series:

$${}_8\phi_7\left[\begin{matrix}\lambda abcq^{-1},\ q\sqrt{},\ -q\sqrt{},\ ab,\ ac,\ bc,\ \lambda d^{-1},\ \lambda u^{-1} \\ \sqrt{},\ -\sqrt{},\ \lambda c,\ \lambda b,\ \lambda a,\ abcd,\ abcu\end{matrix};\ q,\ du\right]$$
$$= \frac{(\lambda abc,\ \lambda/a,\ cu,\ bu;\ q)_\infty}{(\lambda c,\ \lambda b,\ abcu,\ u/a;\ q)_\infty}\, {}_4\phi_3\left[\begin{matrix}ab,\ ac,\ ad,\ \lambda u^{-1} \\ \lambda a,\ abcd,\ aqu^{-1}\end{matrix};\ q,\ q\right]$$
$$+ \frac{(\lambda abc,\ ab,\ ac,\ ad,\ \lambda u^{-1},\ bcdu,\ \lambda u;\ q)_\infty}{(\lambda a,\ \lambda b,\ \lambda c,\ abcd,\ abcu,\ du,\ au^{-1};\ q)_\infty}\, {}_4\phi_3\left[\begin{matrix}bu,\ cu,\ du,\ \lambda a^{-1} \\ u,\ bcdu,\ qua^{-1}\end{matrix};\ q,\ q\right]. \tag{2.6}$$

Use of this in (2.5) gives

$$S = S_1 + S_2, \tag{2.7}$$

where

$$\begin{aligned} S_1 &= \kappa(a, b, c, d) \frac{(\lambda a, \lambda/a, \mu/f, \mu/g; q)_\infty}{g(1-q)(q, qg/f, f/g, fg; q)_\infty} \\ &\cdot \int_f^g \frac{(qu/f, qu/g, \mu u; q)_\infty}{(au, u/a, \mu u/fg; q)_\infty} {}_4\phi_3 \left[\begin{matrix} ab, ac, ad, \lambda/u \\ \lambda a, abcd, aq/u \end{matrix} ; q, q \right] d_q u, \end{aligned} \tag{2.8}$$

and

$$\begin{aligned} S_2 &= \kappa(a, b, c, d) \frac{(ab, ac, ad, \mu/f, \mu/g; q)_\infty}{g(1-q)(q, abcd, qg/f, f/g, fg; q)_\infty} \\ &\cdot \int_f^g \frac{(qu/f, qu/g, \mu u, bcdu, \lambda u, \lambda/u; q)_\infty}{(au, a/u, bu, cu, du, \mu u/fg; q)_\infty} \\ &\quad {}_4\phi_3 \left[\begin{matrix} bu, cu, du, \lambda/a \\ \lambda u, bcdu, qu/a \end{matrix} ; q, q \right] d_q u. \end{aligned} \tag{2.9}$$

Using (2.2) and [5, (3), p. 69] one can easily see that

$$\begin{aligned} S_1 &= \kappa(a, b, c, d) \frac{(\lambda a, \lambda/a, \mu a, \mu/a; q)_\infty}{(fa, f/a, ga, g/a; q)_\infty} \\ &\sum_k \frac{(ab, ac, ad, ag/\mu, q/fg, \lambda/g; q)_k}{(q, \lambda a, abcd, aq/f, aq/g, q/\mu g; q)_k} q^k \\ &\cdot {}_8\phi_7 \left[\begin{matrix} \mu g q^{-k-1}, q\sqrt{}, -q\sqrt{}, ag, \mu/f, \lambda\mu/q, gq^{-k}/a, q^{-k} \\ \sqrt{}, -\sqrt{}, \mu q^{-k}/a, fgq^{-k}, gq^{1-k}/\lambda, a\mu, \mu g \end{matrix} ; q, qf/\lambda \right] \end{aligned} \tag{2.10}$$

where, consistent with the notation (1.7), we have used $(a_1, a_2, \ldots, a_r; q)_k$ to mean $(a_1; q)_k (a_2; q)_k \ldots (a_r; q)_k$.

Use of Watson's formula [5, (2), p. 69]

$$\begin{aligned} &{}_8\phi_7 \left[\begin{matrix} a, q\sqrt{a}, -q\sqrt{a}, b, c, d, e, q^{-k} \\ \sqrt{a}, -\sqrt{a}, aq/b, aq/c, aq/d, aq/e, aq^{k+1} \end{matrix} ; q, \frac{a^2 q^{k+2}}{bcde} \right] \\ &= \frac{(aq, aq/de; q)_k}{(aq/d, aq/e; q)_k} {}_4\phi_3 \left[\begin{matrix} q^{-k}, d, e, aq/bc \\ aq/b, aq/c, deq^{-k}/a \end{matrix} ; q, q \right] \end{aligned} \tag{2.11}$$

in (2.10) gives the series

$${}_4\phi_3 \left[\begin{matrix} \mu/f, ag, aq/\lambda, q^{-k} \\ a\mu, aq/f, gq^{1-k}/\lambda \end{matrix} ; q, q \right]$$

which, when expressed back in terms of an ${}_8\phi_7$ series, leads to the following expression for S_1

$$S_1 = \kappa(a, b, c, d) \frac{(\lambda a, \lambda/a, \mu a, \mu/a; q)_\infty}{(fa, f/a, ga, g/a; q)_\infty} \sum_k \frac{(ab, ac, ad; q)_k}{(q, abcd, qa^2; q)_k} q^k$$

$$\cdot {}_8\phi_7\left[\begin{matrix} a^2, aq, -aq, af, ag, aq/\lambda, aq/\mu\, q^{-k} \\ a, -a, aq/f, aq/g, a\lambda, a\mu, a^2 q^{k+1} \end{matrix}; q, \frac{\lambda\mu q^k}{fg}\right].$$

We now turn to S_2. Note that, by (2.2),

$$\int_f^g \frac{(qu/f, qu/g, \mu u, bcdu, \lambda u, \lambda/u; q)_\infty}{(au, a/u, bu, cu, du, \mu u/fg; q)_\infty} {}_4\phi_3\left[\begin{matrix} bu, cu, du, \lambda/a \\ \lambda u, bcdu, qu/a \end{matrix}; q, q\right] d_q u$$

$$= g(1-q) \frac{(q, qg/f, \mu g, bcdg, \lambda g, \lambda/g; q)_\infty}{(ag, bg, cg, dg, \mu/f, a/g; q)_\infty}$$

$$\sum_k \sum_\ell \frac{(bg, cg, dg; q)_{k+\ell} (\lambda/a; q)_k (ag, \mu/f, qg/\lambda; q)_\ell}{(\lambda g, bcdg, gq/a; q)_{k+\ell} (q; q)_k (q, \mu g, qg/f; q)_\ell} \left(\frac{\lambda}{a}\right)^\ell q^{k+\ell} \tag{2.13}$$

$$- f(1-q) \frac{(q, qf/g, \mu f, bcdf, \lambda f, \lambda/f; q)_\infty}{(af, bf, cf, df, \mu/g, a/f; q)_\infty}$$

$$\sum_k \sum_\ell \frac{(bf, cf, df; q)_{k+\ell} (\lambda/a; q)_k (af, \mu/g, qf/\lambda; q)_\ell}{(\lambda f, bcdf, fq/a; q)_{k+\ell} (q; q)_k (q, \mu f, qf/g; q)_\ell} \left(\frac{\lambda}{a}\right)^\ell q^{k+\ell}.$$

Using this in (2.9) and simplifying we get

$$S_2 = \kappa(a, b, c, d) \frac{(ab, ac, ad, \lambda g, \lambda/g, \mu g, \mu/g; q)_\infty}{(ag, bg, cg, dg, fg, a/g, f/g; q)_\infty}$$

$$\cdot \sum_m \frac{(bg, cg, dg, \lambda/a; q)_m}{(q, \lambda g, qg/a, bcdg; q)_m} b^m\, {}_4\phi_3\left[\begin{matrix} q^{-m}, ag, \mu/f, qg/\lambda \\ \mu g, qg/f, aq^{1-m}/\lambda \end{matrix}; q, q\right]$$

$$+ \kappa(a, b, c, d) \frac{(ab, ac, ad, \lambda f, \lambda/f, \mu f, \mu/f; q)_\infty}{(af, bf, cf, df, gf, a/f, g/f; q)_\infty} \tag{2.14}$$

$$\cdot \sum_m \frac{(bf, cf, df, \lambda/a; q)_m}{(q, \lambda f, qf/a, bcdf; q)_m} q^m\, {}_4\phi_3\left[\begin{matrix} q^{-m}, af, \mu/g, qf/\lambda \\ \mu f, qf/g, aq^{1-m}/\lambda \end{matrix}; q, q\right].$$

Note that we may express the two ${}_4\phi_3$'s as ${}_8\phi_7$'s by use of (2.11) that are similar to the ${}_8\phi_7$ in (2.12). Once this is done we combine (2.14) with (2.12), simplify the coefficients and obtain

$$
\begin{aligned}
&S(a, b, c, d, f, g; \lambda, \mu) \\
&= \kappa(a, b, c, d) \frac{(\lambda a, \lambda/a, \mu a, \mu/a; q)_\infty}{(fa, f/a, ga, g/a; q)_\infty} \sum_k \frac{(ab, ac, ad; q)_k}{(q, abcd, qa^2; q)_k} q^k \\
&\cdot {}_8\phi_7\left[\begin{matrix} a^2, aq, -aq, af, ag, aq/\lambda, aq/\mu, q^{-k} \\ a, -a, aq/f, aq/g, \lambda a, a\mu, a^2 q^{k+1} \end{matrix}; q, \frac{\lambda\mu q^k}{fg}\right] \\
&+ \text{idem}(a; f, g).
\end{aligned} \tag{2.15}
$$

The high degree of symmetry in this formula will be found very useful in proving our main result, namely, (1.18). But first, let us point out an interesting special case. Let $\lambda\mu = q$. Then each of the ${}_8\phi_7$ series in (2.15) becomes a ${}_6\phi_5$ which is summable by Jackson's formula [**14**, (iv. 9), p. 247]. The ${}_8\phi_7$ that is displayed in (2.15), for example, has the sum $(qa^2, q/fg; q)_k/(aq/f, aq/g; q)_k$. Thus we find that

$$
\begin{aligned}
&S(a, b, c, d, f, g; \lambda, q/\lambda) \\
&= \kappa(a, b, c, d) \frac{(\lambda a, \lambda/a, aq/\lambda, q/a\lambda; q)_\infty}{(fa, f/a, ga, g/a; q)_\infty} {}_4\phi_3\left[\begin{matrix} ab, ac, ad, q/fg \\ aq/f, aq/g, abcd \end{matrix}; q, q\right] \\
&+ \text{idem}(a; f, g).
\end{aligned} \tag{2.16}
$$

In particular, if $f = \lambda q^n$, $g^m = q^{+1}/\lambda$, where m, n are non-negative integers and $\max(|\lambda|, |q/\lambda|) < 1$, then, since $(\lambda/f; q)_\infty = (q^{-n}; q)_\infty = 0$ and $(q/g\lambda; q)_\infty = (q^{-m}; q)_\infty = 0$, we get

$$
\begin{aligned}
&S(a, b, c, d, \lambda q^n, q^{m+1}/\lambda; \lambda, q/\lambda) \\
&= \kappa(a, b, c, d)\, (\lambda a, \lambda/a; q)_n\, (aq/\lambda, q/a\lambda; q)_m \\
&\cdot {}_4\phi_3\left[\begin{matrix} q^{-m-n}, ab, ac, ad \\ aq^{1-n}/\lambda, a\lambda q^{-m}, abcd \end{matrix}; q, q\right].
\end{aligned} \tag{2.17}
$$

A special case of this formula has already been used by Rahman [**11**] in computing a 2-dimensional q-analogue of Selberg's integral [**13**].

Returning to (2.15), we note that Sears' formula [**12**, (5.2)], to which (2.1) is equivalent, can be spelled out as

$$
\begin{aligned}
&{}_3\phi_2\left[\begin{matrix} a, b, c \\ e, f \end{matrix}; q, q\right] \\
&= \frac{(q/e, f/a, f/b, f/c; q)_\infty}{(aq/e, bq/e, cq/e, f; q)_\infty} - \frac{(a, b, c, q/e, fq/e; q)_\infty}{(aq/e, bq/e, cq/e, e/q, f; q)_\infty} \\
&\quad {}_3\phi_2\left[\begin{matrix} aq/e, bq/e, cq/e \\ q^2/e, qf/e \end{matrix}; q, q\right].
\end{aligned} \tag{2.18}
$$

where $ef = abcq$.

Hence, for $\ell = 0, 1, 2, \ldots$,

$$
\begin{aligned}
&{}_3\phi_2\left[\begin{matrix} abq^\ell, acq^\ell, adq^\ell \\ abcdq^\ell, a^2q^{2\ell+1} \end{matrix}, q, q\right] \\
&= \frac{(aq^{\ell+1}/b, aq^{\ell+1}/c, aq^{\ell+1}/d, q^{1-\ell}/abcd; q)_\infty}{(q/bc, q/bd, q/cd, a^2q^{2\ell+1}; q)_\infty} \\
&\quad - \frac{(abq^\ell, acq^\ell, adq^\ell, aq^{\ell+2}/bcd, q^{1-\ell}/abcd; q)_\infty}{(q/bc, q/bd, q/cd, abcdq^{\ell-1}, a^2q^{2\ell+1}; q)_\infty} \\
&\quad \cdot {}_3\phi_2\left[\begin{matrix} q/bc, q/bd, q/cd \\ aq^{\ell+2}/bcd, q^{2-\ell}/abcd \end{matrix}; q, q\right].
\end{aligned} \tag{2.19}
$$

Using this we get

$$
\begin{aligned}
&\sum_k \frac{(ab, ac, ad; q)_k}{(q, abcd, qa^2; q)_k} q^k \\
&\quad \cdot {}_8\phi_7\left[\begin{matrix} a^2, aq, -aq, af, ag, aq/\lambda, aq/\mu, q^{-k} \\ a, -a, aq/f, aq/g, a\lambda, a\mu, a^2q^{k+1} \end{matrix}; q, \frac{\lambda\mu q^k}{fg}\right] \\
&= \sum_\ell \frac{(a^2, aq, -aq, af, ag, aq/\lambda, aq/\mu; q)_\ell}{(q, a, -a, aq/f, aq/g, a\lambda, a\mu, q)_\ell}\left(-\frac{\lambda\mu}{fg}\right)^\ell \\
&\quad \cdot q^{\ell(\ell+1)/2} \frac{(ab, ac, ad; q)^\ell}{(abcd; q)_\ell (qa^2; q)_{2\ell}} {}_3\phi_2\left[\begin{matrix} abq^\ell, acq^\ell, adq^\ell \\ abcdq^\ell, a^2q^{2\ell+1} \end{matrix}; q, q\right] \\
&= \frac{(q/abcd, aq/b, aq/c, aq/d; q)_\infty}{(qa^2, q/bc, q/bd, q/cd; q)_\infty} \\
&\quad \cdot {}_{10}\phi_9\left[\begin{matrix} a^2, aq, -aq, ab, ac, ad, af, ag, aq/\lambda, aq/\mu \\ a, -a, aq/b, aq/c, aq/d, aq/f, aq/g, a\lambda, a\mu \end{matrix}; q, \frac{\lambda\mu q}{abcdfg}\right] \\
&\quad - \frac{(q/abcd, aq^2/bcd, ab, ac, ad; q)_\infty}{(abcd/q, qa^2, q/bc, q/bd, q/dc; q)_\infty} \sum_{k=0}^\infty \frac{(q/bc, q/bd, q/cd; q)_k}{(q, q^2/abcd, aq^2/bcd; q)_k} q^k \\
&\quad \cdot {}_8\phi_7\left[\begin{matrix} a^2, aq, -aq, af, ag, aq/\lambda, aq/\mu, abcdq^{-k-1} \\ a, -a, aq/f, aq/g, a\lambda, a\mu, aq^{k+2}/bcd \end{matrix}; q, \frac{\lambda\mu q^k}{abcdfg}\right].
\end{aligned} \tag{2.20}
$$

From (2.15) and (2.20) it then follows that

$$
\begin{aligned}
&S(a, b, c, d, f, g; \lambda, \mu) \\
&= \kappa(a, b, c, d) \frac{(q/abcd, aq/b, aq/c, aq/d, \lambda a, \lambda/a, \mu a, \mu/a; q)_\infty}{(qa^2, q/bc, q/bd, q/cd, fa, f/a, ga, g/a; q)_\infty}
\end{aligned} \tag{2.21}
$$

$$\cdot {}_{10}\phi_9\left[\begin{matrix} a^2,\ aq,\ -aq,\ ab,\ ac,\ ad,\ af,\ ag,\ aq/\lambda,\ aq/\mu \\ a,\ -a,\ aq/b,\ aq/c,\ aq/d,\ aq/f,\ aq/g,\ a\lambda,\ a\mu \end{matrix};q,\frac{\lambda\mu q}{abcdfg}\right]$$

$$+ \text{idem}(a; f, g)$$

$$- T, \text{ say,}$$

where

$$T = \rho(b, c, d)\left\{\frac{(\lambda a, \lambda/a, \mu a, \mu/a, q^2/abcd, aq^2/bcd; q)_\infty}{(qa^2, af, f/a, ag, g/a; q)_\infty} a^{-1}\right.$$

$$\cdot\sum_{k=0}^{\infty}\frac{(q/bc, q/bd, q/cd; q)_k}{(q, q^2/abcd, aq^2/bcd; q)_k} q^k \tag{2.22}$$

$$\cdot {}_8\phi_7\left[\begin{matrix} a^2,\ aq,\ -aq,\ af,\ ag,\ aq/\lambda,\ aq/\mu,\ abcdq^{-k-1} \\ a,\ -a,\ aq/f,\ aq/g,\ a\lambda,\ a\mu,\ aq^{k+2}/bcd \end{matrix};q,\frac{\lambda\mu q^k}{abcdfg}\right]$$

$$\left. + \text{idem}(a; f, g)\right\},$$

with

$$\rho(b, c, d) = \frac{2\pi}{bcd(q, bc, bd, cd, q/bc, q/bd, q/cd; q)_\infty}, \tag{2.23}$$

provided $0 \neq bc, bd, cd \neq q^{\pm k}$, $k = 0, 1, 2, \ldots$ . .

All we need to prove now is that T actually vanishes. Fortunately we have an identity due to Bailey [**6**, (4.6)], see also [**14**, (7.1.1.5)],

$$a^{-1}\frac{(aq/d, aq/e, aq/f, q/ad, q/ae, q/af; q)_\infty}{(qa^2, ab, ac, b/a, c/a; q)_\infty}$$

$$\cdot {}_8\phi_7\left[\begin{matrix} a^2,\ aq,\ -aq,\ ab,\ ac,\ ad,\ ae,\ af \\ a,\ -a,\ aq/b,\ aq/c,\ aq/d,\ aq/e,\ aq/f \end{matrix};q,q^2/abcdef\right] \tag{2.24}$$

$$+ \text{idem}(a; b, c) = 0.$$

Replacing b, c, d, e, f by $f, g, q/\lambda, q/\mu$ and $bcdq^{-k-1}$, respectively, immediately proves that $T = 0$. Hence the proof of (1.18) is complete.

3. Quadratic transformation formulas of type I. In (1.18) let us set

$$\mu = -\lambda, g = -f, b = -a \tag{3.1}$$

and assume that

$$\left|\frac{\lambda^2 q}{a^2 cdf^2}\right| < 1. \tag{3.2}$$

Denoting $p = q^2$ and using the identity $(a;q)_\infty(-a;q)_\infty = (a^2;q^2)_\infty$ we get

$$S(a,\ -a,\ c,\ d,f,\ -f;\ \lambda,\ -\lambda)$$

$$(3.3)\quad = \frac{2\pi\begin{pmatrix}-cda^2, & -q/cda^2, \\ aq/c, & aq/d, & -q;\ q\end{pmatrix}_\infty (\lambda^2a^2,\ \lambda^2/a^2;\ p)_\infty}{\begin{pmatrix}q, & qa^2, & -a^2, & cd, & -q/ac, \\ -q/ad, & q/cd;\ q\end{pmatrix}_\infty (a^2c^2,\ a^2d^2,\ a^2f^2,\ f^2/a^2;\ q)_\infty}$$
$$\cdot {}_{10}\phi_9\left[\begin{matrix}a^2, qa, -qa, -a^2, ac, ad, -af, af, aq/\lambda, -aq/\lambda \\ a, -a, -q, aq/c, aq/d, aq/f, -aq/f, a\lambda, -a\lambda\end{matrix};\ q,\ -\frac{\lambda^2q}{cda^2f^2}\right]$$
$$+\frac{2\pi\begin{pmatrix}-acdf, & -q/acdf, & -fq/a, \\ fq/c, & fq/d;\ q)\end{pmatrix}_\infty (\lambda^2f^2,\ \lambda^2/f^2;\ q)_\infty}{\begin{pmatrix}q, & qf^2, & -f^2, & -1, & cd, & -q/ac, & -q/ad, \\ q/cd, & -ac, & -ad, & fc, & fd, & a/f;\ q\end{pmatrix}_\infty (a^2f^2;\ p)_\infty}$$
$$\cdot {}_{10}\phi_9\left[\begin{matrix}f^2, qf, -qf, -f^2, fa, -fa, fc, fd, fq/\lambda, -fq/\lambda \\ f, -f, -q, fq/a, -fq/a, fq/c, fq/d, f\lambda, -f\lambda\end{matrix};\ q,\ -\frac{\lambda^2q}{cda^2f^2}\right]$$
$$+\frac{2\pi\begin{pmatrix}acdf, & q/acdf, & fq/a, \\ -fq/c, & -fq/d;\ q\end{pmatrix}_\infty (\lambda^2f^2,\ \lambda^2/f^2;\ p)_\infty}{\begin{pmatrix}q, & qf^2, & -f^2, & -1, & cd, & -q/ac, & -q/ad, \\ q/cd, & -ac, & -ad, & -fc, & -fd, & -a/f;\ q\end{pmatrix}_\infty (a^2f^2, p)_\infty}$$
$$\cdot {}_{10}\phi_9\left[\begin{matrix}f^2, qf, -qf, -f^2, fa, -fa, -fc, -fd, fq/\lambda, -fq/\lambda \\ f, -f, -q, fq/a, -fq/a, -fq/c, -fq/d, f\lambda, -f\lambda\end{matrix};q,-\frac{\lambda^2q}{cda^2f^2}\right].$$

In the special case when the series are balanced, that is, when $a^2cdf^2 = -\lambda^2$, we use (1.16) instead of (1.18) since it reflects the corresponding simplifications explicitly. Thus for the balanced case

$$S(a,\ -a,\ c,\ d,f,\ -f;\ \lambda,\ -\lambda)$$

$$(3.4)\quad = \frac{2\pi(\lambda^2/cf,\ \lambda^2/df;\ q)_\infty(\lambda^4/a^2f^2,\ \lambda^2f^2,\ \lambda^2/f^2;\ p)_\infty}{(q,\ -1,\ -a^2,\ -f^2,\ \lambda^2,\ cd,\ cf,\ df;\ q)_\infty\ (a^2c^2,\ a^2d^2,\ a^2f^2;\ p)_\infty}$$
$$\cdot {}_{10}\phi_9\left[\begin{matrix}\lambda^2q^{-1}, \lambda\sqrt{q}, -\lambda\sqrt{q}, -\lambda^2q^{-1}, af, -af, \lambda/f, -\lambda/f, cf, df \\ \lambda/\sqrt{q}, -\lambda/\sqrt{q}, -q, \lambda^2/af, -\lambda^2/af, \lambda f, -\lambda f, \lambda^2/cf, \lambda^2/df\end{matrix};q,\ q\right]$$
$$+\frac{2\pi(-\lambda^2/cf,\ -\lambda^2/df;\ q)_\infty\ (\lambda^4/a^2f^2,\ \lambda^2f^2,\ \lambda^2/f^2;\ p)_\infty}{(q,\ -1,\ -a^2,\ -f^2,\lambda^2,\ cd,\ -cf,\ -df;q)_\infty(a^2c^2,\ a^2d^2,\ a^2f^2;p)_\infty}$$
$$\cdot {}_{10}\phi_9\left[\begin{matrix}\lambda^2q^{-1}, \lambda\sqrt{q}, -\lambda\sqrt{q}, -\lambda^2q^{-1}, -af, af, -\lambda/f, \lambda/f, -cf, -df \\ \lambda/\sqrt{q}, -\lambda/\sqrt{q}, -q, -\lambda^2/af, \lambda^2/af, -\lambda f, \lambda f, -\lambda^2/cf, -\lambda^2/df\end{matrix};q,\ q\right].$$

We now turn to the left hand sides of (3.3) and (3.4). Note that

$$(3.5)\quad h(x;\ a)h(x;\ -a) = |(ae^{i\theta};\ q)_\infty\ (-ae^{i\theta};\ q)_\infty|^2 = |(a^2e^{2i\theta};\ p)_\infty|^2 \equiv h_p(\xi;\ a^2),$$

where $x = \cos\theta$, $\xi = \cos 2\theta = 2x^2 - 1$. So

(3.6)
$$\begin{aligned}
&\frac{h(x;\lambda)h(x;-\lambda)}{h(x;a)h(x;-a)h(x;f)h(x;-f)}\\
&=\frac{h_p(\xi;\lambda^2)}{h_p(\xi;a^2)h_p(\xi;f^2)}\\
&=\frac{(\lambda^2/a^2,\lambda^2/f^2;p)_\infty}{f^2(1-p)\,(p,pf^2/a^2,a^2/f^2,a^2f^2;p)_\infty}\\
&\quad\int_{a^2}^{f^2} d_pu\,\frac{(pu/a^2,pu/f^2,\lambda^2u;p)_\infty}{(\lambda^2u/a^2f^2;p)_\infty h_p(\xi;u)},\ \text{by (2.1).}
\end{aligned}$$

Hence

(3.7)
$$\begin{aligned}
&S(a,-a,c,d,f,-f;\lambda,-\lambda)\\
&=\frac{(\lambda^2/a^2,\lambda^2/f^2;p)_\infty}{f^2(1-p)(p,pf^2/a^2,a^2/f^2,a^2f^2;p)_\infty}\int_{a^2}^{f^2} d_pu\,\frac{(pu/a^2,pu/f^2,\lambda^2u;p)_\infty}{(\lambda^2u/a^2f^2;p)_\infty}\\
&\quad\cdot\int_{-1}^{1} w(x;c,d,\sqrt{u},-\sqrt{u})dx.
\end{aligned}$$

Since

(3.8)
$$\begin{aligned}
&\int_{-1}^{1} w(x;c,d,\sqrt{u},-\sqrt{u})dx\\
&=\frac{2\pi(-cdu;q)_\infty}{(q,cd,c\sqrt{u},-c\sqrt{u},d\sqrt{u},-d\sqrt{u},-u;q)_\infty}\\
&=\frac{2\pi(-cdu;p)_\infty(-cdu\sqrt{p};p)_\infty}{(q,cd;q)_\infty(-u,c^2u,d^2u,-u\sqrt{p};p)_\infty},
\end{aligned}$$

by virtue of the identity

(3.9)
$$(a;q)_\infty=(a;q^2)_\infty(aq;q^2)_\infty,$$

we get

(3.10)
$$\begin{aligned}
&S(a,-a,c,d,f,-f;\lambda,-\lambda)\\
&=\frac{2\pi}{(q,cd;q)_\infty}\,\frac{(\lambda^2/a^2,\lambda^2/f^2;p)_\infty}{f^2(1-p)(p,pf^2/a^2,a^2/f^2,a^2f^2;p)_\infty}\\
&\quad\cdot\int_{a^2}^{f^2}\frac{(pu/a^2,pu/f^2,\lambda^2u;p)_\infty}{(\lambda^2u/a^2f^2,c^2u,d^2u;p)_\infty}\cdot\frac{(-cdu,-cdu\sqrt{p};p)_\infty}{(-u,-u\sqrt{p};p)_\infty}\\
&=\frac{2\pi(\lambda^2/f^2,\lambda^2f^2;p)_\infty(-cdf^2;q)_\infty}{(a^2/f^2,a^2f^2,c^2f^2,d^2f^2;p)_\infty(q,cd,-f^2;q)_\infty}\\
&\quad\cdot{}_5\phi_4\left[\begin{matrix}c^2f^2,d^2f^2,\lambda^2/a^2,-f^2,-f^2\sqrt{p}\\ pf^2/a^2,\lambda^2f^2,-cdf^2,-cdf^2\sqrt{p}\end{matrix};p,p\right]\\
&\quad+\frac{2\pi(\lambda^2/a^2,\lambda^2a^2;p)_\infty(-cda^2;q)_\infty}{(f^2/a^2,a^2f^2,c^2a^2,d^2a^2;p)_\infty(q,cd,-a^2;q)_\infty}\\
&\quad\cdot{}_5\phi_4\left[\begin{matrix}c^2a^2,d^2a^2,\lambda^2/f^2,-a^2,-a^2\sqrt{p}\\ pa^2/f^2,\lambda^2a^2,-cda^2,-cda^2\sqrt{p}\end{matrix};p,p\right],
\end{aligned}$$

by (2.2) and some simplifications. If $f^2 = -\lambda^2/a^2\,cd$ then (3.4) gives

$$
\begin{aligned}
&\frac{(\lambda^4/a^2 f^2, \lambda^2 f^2, \lambda^2/f^2; p)_\infty (\lambda^2/cf, \lambda^2/df; q)_\infty}{(a^2c^2, a^2d^2; p)_\infty (-1, -a^2, -f^2, \lambda^2, cf, df; q)_\infty} \\
&\cdot {}_{10}\phi_9\left[\begin{matrix}\lambda^2q^{-1}, \lambda\sqrt{q}, -\lambda\sqrt{q}, -\lambda^2q^{-1}, af, -af, \lambda/f, -\lambda/f, cf, df \\ \lambda/\sqrt{q}, -\lambda/\sqrt{q}, -q, \lambda^2/af, -\lambda^2/af, \lambda f, -\lambda f, \lambda^2/cf, \lambda^2/df\end{matrix}; q, q\right] \\
&+ \frac{(\lambda^4/a^2f^2, \lambda^2f^2, \lambda^2/f^2; p)_\infty (-\lambda^2/cf, -\lambda^2/df; q)_\infty}{(a^2c^2, a^2d^2; p)_\infty (-1, -a^2, -f^2, \lambda^2, -cf, -df; q)_\infty} \\
&\cdot {}_{10}\phi_9\left[\begin{matrix}\lambda^2q^{-1}, \lambda\sqrt{q}, -\lambda\sqrt{q}, -\lambda^2q^{-1}, af, -af, \lambda/f, -\lambda/f, -cf, -df \\ \lambda/\sqrt{q}, -\lambda/\sqrt{q}, -q, \lambda^2/af, -\lambda^2/af, \lambda f, -\lambda f, -\lambda^2/cf, -\lambda^2/df\end{matrix}; q, q\right] \\
(3.11)\quad &= \frac{(-a^2cd, -\lambda^4/a^2cd; p)_\infty (\lambda^2/a^2; q)_\infty}{(-a^4cd/\lambda^2, -c\lambda^2/a^2d, -d\lambda^2/a^2c; p)_\infty (-\lambda^2/a^2cd; q)_\infty} \\
&\cdot {}_4\phi_3\left[\begin{matrix}-d\lambda^2/a^2c, -c\lambda^2/a^2d, \lambda^2/a^2cd, \lambda^2\sqrt{p}/a^2cd \\ -\lambda^2p/a^4cd, -\lambda^4/a^2cd, \lambda^2\sqrt{p}/a^2\end{matrix}; p, p\right] \\
&+ \frac{(\lambda^2/a^2, \lambda^2a^2; p)_\infty (-cda^2; q)_\infty}{(-\lambda^2/a^4cd, a^2c^2, a^2d^2; p)_\infty (-a^2; q)_\infty} \\
&\cdot {}_4\phi_3\left[\begin{matrix}a^2c^2, a^2d^2, -a^2, -a^2\sqrt{p} \\ -a^4cdp/\lambda^2, \lambda^2a^2, -cda^2\sqrt{p}\end{matrix}; p, p\right]
\end{aligned}
$$

Use of [5, (3), p. 69] and some simplifications yield the following transformation formula

$$
\begin{aligned}
&\frac{(\lambda^2/cf, \lambda^2/df; q)_\infty}{(cf, df; q)_\infty} \\
&\cdot {}_{10}\phi_9\left[\begin{matrix}\lambda^2q^{-1}, \lambda\sqrt{q}, -\lambda\sqrt{q}, -\lambda^2q^{-1}, af, -af, \lambda/f, -\lambda/f, cf, df \\ \lambda/\sqrt{q}, -\lambda/\sqrt{q}, -q, \lambda^2/af, -\lambda^2/af, \lambda f, -\lambda f, \lambda^2/cf, \lambda^2/df,\end{matrix}; q, q\right] \\
&+ \frac{(-\lambda^2/cf, -\lambda^2/df; q)_\infty}{(-cf, -df; q)_\infty} \\
(3.12)\quad &\cdot {}_{10}\phi_9\left[\begin{matrix}\lambda^2q^{-1}, \lambda\sqrt{q}, -\lambda\sqrt{q}, -\lambda^2q^{-1}, af, -af, \lambda/f, -\lambda/f, -cf, -df \\ \lambda/\sqrt{q}, -\lambda/\sqrt{q}, -q, \lambda^2/af, -\lambda^2/af, \lambda f, -\lambda f, -\lambda^2/cf, -\lambda^2/df\end{matrix}; q, q\right] \\
&= \frac{(\lambda^2a^2, \lambda^2/a^2, \lambda^2d/c, \lambda^2\sqrt{p}/a^2cd; p)_\infty (-1, \lambda^2, \lambda^2c/d, -cda^2; q)_\infty}{(-cda^2, -\lambda^4/a^2cd, -\lambda^2d/a^2c, -\lambda^2c/a^2d, a^2\lambda^2cd, \lambda^2c\sqrt{p}/d; p)_\infty} \\
&\cdot {}_8\phi_7\left[\begin{matrix}a^2\lambda^2cd/p, p\sqrt{\ }, -p\sqrt{\ }, a^2c^2, a^2d^2, cd, -a^2, -\lambda^2/\sqrt{p} \\ \sqrt{\ }, -\sqrt{\ }, \lambda^2d/c, \lambda^2c/d, a^2\lambda^2, -\lambda^2cd, -cda^2\sqrt{p}\end{matrix}; p, -\frac{\lambda^2\sqrt{p}}{a^2cd}\right],
\end{aligned}
$$

provided $|\lambda^2\sqrt{p}/a^2cd| < 1$, unless the series on the right terminates.

In the general case, that is, when $f^2 \neq -\lambda^2/a^2cd$ no further simplification of (3.10) seems possible, so after some rearrangement of the coefficients (3.3) and (3.10) give the following formula

$$
\begin{aligned}
&\frac{(-cda^2,\ -q/cda^2,\ aq/c,\ aq/d,\ -q;\ q)_\infty\,(\lambda^2a^2,\ \lambda^2/a^2;\ p)_\infty}{(qa^2,\ -a^2,\ -q/ac.\ -q/ad,\ q/cd;\ q)_\infty\,(a^2c^2,\ a^2d^2;\ p)_\infty} \\
&\cdot {}_{10}\phi_9\left[\begin{matrix} a^2, qa, -qa, -a^2, ac, ad, af, af, -af, aq/\lambda, -aq/\lambda \\ a, -a, -q, aq/c, aq/d, aq/f, -aq/f, a\lambda, -a\lambda \end{matrix}\ ;q,\ -\frac{\lambda^2q}{cda^2f^2}\right] \\
&+\frac{(-acdf.\ -q/acdf, fq/c, fq/d, -fq/a; q)_\infty(\lambda^2f^2, \lambda^2/f^2, f^2, f^2/a^2; p)_\infty}{(qf^2, -f^2, -q/ac, -q/ad, q/cd, -ac, -ad, -1, fc, fd, a/f; q)_\infty} \\
&\cdot {}_{10}\phi_9\left[\begin{matrix} f^2, qf, -qf, -f^2, fa, -fa, fc, fd, fq/\lambda, -fq/\lambda \\ f, -f, -q, fq/a, -fq/a, fq/c, fq/d, f\lambda, -f\lambda, \end{matrix}\ ;q,\ -\frac{\lambda^2q}{cda^2f^2}\right] \\
&+\frac{(acdf,\ q/acdf,\ -fq/c,\ -fq/d, fq/a;\ q)_\infty\,(\lambda^2f^2,\ \lambda^2/f^2, f^2/a^2;\ p)_\infty}{(qf^2, -f^2, -q/ac, -q/ad, q/cd, -ac, -ad, -1, -fc, -fd, -a/f;\ q)_\infty} \\
(3.13)\qquad &\cdot {}_{10}\phi_9\left[\begin{matrix} f^2, qf, -qf, -f^2, fa, -fa, -fc, -fd, fq/\lambda, -fq/\lambda \\ f, -f, -q, fq/a, -fq/a, -fq/c, -fq/d, f\lambda, -f\lambda \end{matrix}\ ;q,\ -\frac{\lambda^2q}{cda^2f^2}\right] \\
&= \frac{(\lambda^2/f^2,\ \lambda^2f^2,\ f^2/a^2;\ p)_\infty\,(-cdf^2;\ q)_\infty}{(a^2/f^2,\ c^2f^2,\ d^2f^2;\ p)_\infty\,(-f^2;\ q)_\infty} \\
&{}_5\phi_4\left[\begin{matrix} c^2f^2,\ d^2f^2,\ \lambda^2/a^2,\ -f^2,\ -f^2\sqrt{p} \\ pf^2/a^2,\ \lambda^2f^2,\ -cdf^2,\ -cdf^2\sqrt{p} \end{matrix}\ ;p,\ p\right] \\
&+ \frac{(\lambda^2/a^2,\ \lambda^2a^2;\ p)_\infty\,(-cda^2;\ q)_\infty}{(a^2c^2,\ a^2d^2;\ p)_\infty\,(-a^2;\ q)_\infty} \\
&{}_5\phi_4\left[\begin{matrix} a^2c^2,\ a^2d^2,\ \lambda^2/f^2,\ -a^2,\ -a^2\sqrt{p} \\ pa^2/f^2,\ \lambda^2a^2,\ -cda^2,\ -cda^2\sqrt{p} \end{matrix}\ ;p,\ p\right].
\end{aligned}
$$

Since the ordinary hypergeometric limit of this formula does not seem to exist in the literature it should be of interest to see what it is. To this end, let us choose the parameters in the following way.

$$a^2 = -q^\alpha,\ c^2 = -q^{\alpha+2-2w},\ d^2 = -q^{2\delta-\alpha}, f^2 = -q^{2\gamma-\alpha},\ \lambda^2 = -q^{\alpha+2-2\beta}$$

and

$$ac = -q^{1+\alpha-w},\ ad = q^\delta,\ ab = q^{1+\alpha-\beta},\ cf = -q^{1+\gamma-w},\ df = q^{\gamma+\delta-\alpha}.$$

Then, using (1.9) and (1.12) we can rewrite (3.13) in the form

$$\frac{\Gamma_q(\alpha)\Gamma_q(w-\alpha)\Gamma_q(w-\delta)\Gamma_p(\delta)\Gamma_p(\alpha-w+1)}{\Gamma_q(w)\Gamma_q(w-\alpha-\delta)\Gamma_q(1+\alpha+\delta-w)\Gamma_p(1-\beta)} \cdot \frac{(-q^{1+\alpha-\delta};q)_\delta}{(-q^{1-\delta};q)_\delta}$$
$$\Gamma_p(\alpha-\beta+1)$$
$$\cdot {}_{10}\phi_9\left[\begin{matrix}-q^{\alpha}, q\sqrt{\ }, -q\sqrt{\ }, q^{\alpha}, -q^{1+\alpha-w}, q^{\delta}, q^{\gamma}, -q^{\gamma}, -q^{\beta}, q^{\beta} \\ \sqrt{\ }, -\sqrt{\ }, -q, q^{w}, -q^{1+\alpha-\delta}, -q^{1+\alpha-\gamma}, q^{1+\alpha-\gamma}, q^{1+\alpha-\beta}, -q^{1+\alpha-\beta}\end{matrix}; q, q^{\alpha+w-\delta-2\beta-2\gamma+1}\right]$$
$$+\frac{\Gamma_q(2\gamma-\alpha)\Gamma_q(w-\alpha)\Gamma_q(w-\delta)\Gamma_q(1+\alpha-w)\Gamma_q(\gamma+\delta-\alpha)}{\Gamma_q(1+\gamma+\delta-w)\Gamma_q(w-\gamma-\delta)\Gamma_q(\gamma-\alpha+w)\Gamma_q(1+\gamma-\beta)}$$
$$\frac{\Gamma_q(\alpha-\gamma)(-q^{\gamma-\alpha+1};q)_\gamma(-q^{\gamma-\alpha};q)_{1+\alpha-w}}{\Gamma_q(1+\alpha-\beta-\gamma)\Gamma_q(\gamma-\alpha)(-1;q)_{\gamma-\beta+1}(-q^{1-\delta};q)_\gamma}$$
$$\cdot\frac{(1+q)^{w+1-\delta-2\beta}}{(-q^{\delta};q)_{1+\alpha-\beta-\gamma-\delta}}\,{}_{10}\phi_9\left[\begin{matrix}-q^{2\gamma-\alpha}, q\sqrt{\ }, -q\sqrt{\ }, q^{2\gamma-\alpha}, q^{\gamma}, -q^{\gamma}, q^{\gamma+\beta-\alpha}, -q^{\gamma+\beta-\alpha}, -q^{1+\gamma-w}, q^{\gamma+\delta-\alpha} \\ \sqrt{\ }, -\sqrt{\ }, -q, -q^{1+\gamma-\alpha}, q^{1+\gamma-\alpha}, -q^{1+\gamma-\beta}, q^{1+\gamma-\beta}, q^{\gamma-\alpha+w}, -q^{1+\gamma-\delta}\end{matrix}; q, q^{\alpha+w-\delta-2\beta-2\gamma+1}\right]$$

(3.15)
$$+\frac{\Gamma_q(2\gamma-\alpha)\Gamma_q(w-\alpha)\Gamma_q(w-\delta)\Gamma_q(1+\alpha-w)\Gamma_q(1+\gamma-w)}{\Gamma_q(1+\gamma-\delta)\Gamma_q(1+\gamma-\alpha)\Gamma_q(1+\gamma-\beta)\Gamma_q(\gamma-\alpha)\Gamma_q(1+\alpha-\beta-\gamma)}$$
$$\cdot\frac{(-q^{\gamma-\alpha};q)_{\gamma+1}(-q^{\gamma-\alpha+w};q)_{\delta-w}(-q^{1+\gamma+\delta-w};q)_{w+\alpha-2\gamma-\delta-1}\,(1+q)^{w+1-\delta-2\beta}}{(-q^{\delta};q)_{\alpha-\beta-\gamma-\delta+1}(-1;q)_{\delta-\beta+1}(-q^{1-\delta};q)_{w-\gamma-1}}$$
$$\cdot{}_{10}\phi_9\left[\begin{matrix}-q^{2\gamma-\alpha}, q\sqrt{\ }, -q\sqrt{\ }, q^{2\gamma-\alpha}, q^{\gamma}, -q^{\gamma}, q^{\gamma+\beta-\alpha}, -q^{\gamma+\beta-\alpha}, q^{1+\gamma-w}, -q^{\gamma+\delta-\alpha} \\ \sqrt{\ }, -\sqrt{\ }, -q, -q^{1+\gamma-\alpha}, q^{1+\gamma-\alpha}, -q^{1+\gamma-\beta}, q^{1+\gamma-\beta}, -q^{\gamma-\alpha+w}, q^{1+\gamma-\delta}\end{matrix}; q, q^{\alpha+w-\delta-2\beta-2\gamma+1}\right]$$
$$=\frac{\Gamma_p(\alpha-\gamma)\Gamma_p(1+\gamma-w)\Gamma_p(\gamma-\alpha+\delta)\Gamma_q(2\gamma-\alpha)}{\Gamma_p(\gamma-\alpha)\Gamma_p(1+\gamma-\beta)\Gamma_p(1+\alpha-\beta-\gamma)\Gamma_q(1+2\gamma-\alpha+\delta-w)}$$
$$\cdot{}_5\phi_4\left[\begin{matrix}p^{\gamma-\alpha/2}, p^{\gamma-\alpha/2+\frac{1}{2}}, p^{1-\beta}, p^{1+\gamma-w}, p^{\gamma+\delta-\alpha} \\ p^{1+\gamma-\alpha}, p^{1+\gamma-\beta}, p^{\gamma+(1+\delta-\alpha-w)/2}, p^{\gamma+(2+\gamma-\alpha-w)/2}\end{matrix}; p, p\right]$$
$$+\frac{\Gamma_p(1+\alpha-w)\Gamma_p(\delta)\Gamma_q(\alpha)}{\Gamma_p(1-\beta)\Gamma_p(1+\alpha-\beta)\Gamma_q(1+\alpha+\delta-w)}$$
$$\cdot{}_5\phi_4\left[\begin{matrix}p^{\alpha/2}, p^{\alpha+1/2}, p^{1+\alpha-\beta-\gamma}, p^{\delta}, p^{1+\alpha-w} \\ p^{1+\alpha-\gamma}, p^{1+\alpha-\beta}, p^{(1+\alpha+\delta-w)/2}, p^{(2+\alpha+\delta-w)/2}\end{matrix}; p, p\right],$$

provided $0 < q < 1$ and

(3.16) $$\alpha + w + 1 > 2\beta + 2\gamma + \delta,$$

when the series are non-terminating. If we now take the limit $q \to 1$ we obtain

$$\frac{\Gamma(\alpha)\Gamma(w-\alpha)\Gamma(w-\delta)\Gamma(\delta)\Gamma(1+\alpha-w)}{\Gamma(w)\Gamma(w-\alpha-\delta)\Gamma(1+\alpha+\delta-w)\Gamma(1-\beta)\Gamma(\alpha-\beta+1)} \cdot {}_4F_3\left[\begin{matrix}\alpha, \beta, \gamma, \delta, \\ 1+\alpha-\beta, 1+\alpha-\gamma, w\end{matrix}; 1\right]$$

$$+ \frac{\Gamma(2\gamma-\alpha)\Gamma(w-\alpha)\Gamma(w-\delta)\Gamma(1+\alpha-w)\Gamma(\gamma+\delta-\alpha)\Gamma(\alpha-\gamma)}{\Gamma(1+\gamma+\delta-w)\Gamma(w-\gamma-\delta)\Gamma(\gamma-\alpha+w)\Gamma(1+\gamma-\beta)\,\Gamma(1+\alpha-\beta-\gamma)\Gamma(\gamma-\alpha)} \cdot {}_4F_3\left[\begin{matrix}2\gamma-\alpha, \gamma, \gamma+\beta-\alpha, \gamma+\delta-\alpha, \\ 1+\gamma-\alpha, 1+\gamma-\beta, \gamma-\alpha+w\end{matrix}; 1\right]$$

$$+ \frac{\Gamma(2\gamma-\alpha)\Gamma(w-\alpha)\Gamma(w-\delta)\Gamma(1+\alpha-w)\Gamma(1+\gamma-w)}{\Gamma(1+\gamma-\delta)\Gamma(1+\gamma-\alpha)\Gamma(1+\gamma-\beta)\Gamma(\gamma-\alpha)\Gamma(1+\alpha-\beta-\gamma)}$$

(3.17) $$\cdot {}_4F_3\left[\begin{matrix}2\gamma-\alpha, \gamma, \gamma+\beta-\alpha, 1+\gamma-w \\ 1+\gamma-\alpha, 1+\gamma-\beta, 1+\gamma-\delta\end{matrix}; 1\right]$$

$$= \frac{\Gamma(\alpha-\gamma)\Gamma(1+\gamma-w)\Gamma(\gamma-\alpha+\delta)\Gamma(2\gamma-\alpha)}{\Gamma(\gamma-\alpha)\Gamma(1+\gamma-\beta)\Gamma(1+\alpha-\beta-\gamma)\Gamma(1+2\gamma-\alpha+\delta-w)} \cdot {}_5F_4\left[\begin{matrix}\gamma-\frac{\alpha}{2}, \gamma-\frac{\alpha-1}{2}, 1-\beta, 1+\gamma-w, \gamma+\delta-\alpha \\ 1+\gamma-\alpha, 1+\gamma-\beta, \gamma+\frac{1+\delta-\alpha-w}{2}, \gamma+\frac{2+\delta-\alpha-w}{2}\end{matrix}; 1\right]$$

$$+ \frac{\Gamma(1+\alpha-w)\Gamma(\alpha)\Gamma(\delta)}{\Gamma(1-\beta)\Gamma(1+\alpha-\beta)\Gamma(1+\alpha+\delta-w)} \cdot {}_5F_4\left[\begin{matrix}\frac{\alpha}{2}, \frac{\alpha+1}{2}, 1+\alpha-\beta-\gamma, \delta, 1+\alpha-w \\ 1+\alpha-\gamma, 1+\alpha-\beta, \frac{1+\alpha+\delta-w}{2}, \frac{2+\alpha+\delta-w}{2}\end{matrix}; 1\right].$$

If we divide by $\Gamma(\delta)$ and let $\delta \to -m$, $m = 0, 1, 2 \ldots$ then (3.17) reduces to Whipple's transformation [5, 4.5 (1)]. On the other hand if we divide by $\Gamma(\alpha)$ and then let $\alpha \to -n$, $n = 0, 1, 2, \ldots$ we obtain another formula of Whipple [5, 4.7 (1)]. So (3.17) can be regarded as a non-terminating extension of Whipple's formulas while (3.15) is a non-terminating generalization of [9, (1.8)]. One should keep in mind that (3.16) alone is not enough for the validity of (3.17). Further conditions that we choose to leave unstated must be satisfied so that the ratios of gamma functions on both sides remain meaningful.

§4. *Quadratic transformation formulas of type II.* We now consider transformations that relate series with base q to those with base $\sqrt{q}$. By virtue of the identity (3.9) we have

$$h(x;a)h(x;a\sqrt{q}) = |(ae^{i\theta};q)(a\sqrt{q}e^{i\theta};q)|^2 = |(ae^{i\theta};\sqrt{q})|^2 = h_{\sqrt{q}}(x;a), \text{ say.} \tag{4.1}$$

So we set

$$\mu = \lambda\sqrt{q},\ b = c\sqrt{q},\ d = f\sqrt{q} \tag{4.2}$$

in (1.18). Since

$$\frac{h_{\sqrt{q}}(x;\lambda)}{h_{\sqrt{q}}(x;c)h_{\sqrt{q}}(x;f)} = \frac{(\lambda/c,\lambda/f;\sqrt{q})_\infty}{c(1-\sqrt{q})(\sqrt{q},c\sqrt{q}/f,f/c,fc;\sqrt{q})_\infty} \cdot\int_f^c d_{\sqrt{q}}u\,\frac{(u\sqrt{q}/f,u\sqrt{q}/c,\lambda u;\sqrt{q})_\infty}{(u\lambda/fc;\sqrt{q})_\infty\,h_{\sqrt{q}}(x;u)} \tag{4.3}$$

and $h_{\sqrt{q}}(x;u) = h(x;u)h(x;u\sqrt{q})$, we get

$$\begin{aligned}
&S(a,c,c\sqrt{q},f,f\sqrt{q},g;\lambda,\lambda\sqrt{q})\\
&= \frac{(\lambda/c,\lambda/f;\sqrt{q})_\infty}{c(1-\sqrt{q})(\sqrt{q},c\sqrt{q}/f,f/c,fc;\sqrt{q})_\infty}\cdot\int_f^c d_{\sqrt{q}}u\,\frac{(u\sqrt{q}/f,u\sqrt{q}/c,\lambda u;\sqrt{q})_\infty}{(\lambda u/fc;\sqrt{q})_\infty}\int_{-1}^1 w(x;a,g,u,u\sqrt{q})\,dx\\
&= \frac{2\pi(\lambda/c,\lambda/f;\sqrt{q})_\infty}{c(1-\sqrt{q})(\sqrt{q},c\sqrt{q}/f,f/c\,fc;\sqrt{q})_\infty(q,ag;q)_\infty}\\
&\quad\cdot\int_f^c d_{\sqrt{q}}u\,\frac{(u\sqrt{q}/f,u\sqrt{q}/c,\lambda u,\sqrt{aguq^{1/4}},-\sqrt{aguq^{1/4}};\sqrt{q})_\infty}{(au,gu,\lambda u/fc,uq^{1/4},-uq^{1/4};\sqrt{q})_\infty}\\
&= \frac{2\pi(\lambda/c,\lambda c,c\sqrt{ag}\,q^{1/4},-c\sqrt{ag}\,q^{1/4};\sqrt{q})_\infty}{(q,ag;q)_\infty(ac,fc,f/c,gc,cq^{1/4},-cq^{1/4};\sqrt{q})_\infty}\\
&\quad\cdot{}_5\phi_4\left[\begin{matrix}ac,cg,\lambda/f,cq^{1/4},-cq^{1/4}\\ \lambda c,c\sqrt{q}/f,c\sqrt{ag}\,q^{1/4},-c\sqrt{ag}\,q^{1/4}\end{matrix};\sqrt{q},\sqrt{q}\right]\\
&\quad+\frac{2\pi(\lambda/f,\lambda f,f\sqrt{ag}\,q^{1/4},-f\sqrt{ag}\,q^{1/4};\sqrt{q})_\infty}{(q,ag;q)_\infty(af,cf\,c/f,gf,fq^{1/4},-fq^{1/4};\sqrt{q})_\infty}\\
&\quad\cdot{}_5\phi_4\left[\begin{matrix}af,fg,\lambda/c,fq^{1/4},-fq^{1/4}\\ \lambda f,f\sqrt{q}/c,f\sqrt{ag}\,q^{1/4},-f\sqrt{ag}\,q^{1/4}\end{matrix};\sqrt{q},\sqrt{q}\right].
\end{aligned}\tag{4.4}$$

Thus we have the transformation formula

$$
\begin{aligned}
&\frac{(afc^2q,\ 1/afc^2,\ a\sqrt{q}/f;\ q)_\infty\,(\lambda a,\ \lambda/a,\ a\sqrt{q}/c;\ \sqrt{q})_\infty}{(qa^2,\ c^2\sqrt{q},\ \sqrt{q}/c^2, f/a,\ g/a,;\ q)_\infty\,(ac,\ af,\ 1/cf,\ cf\sqrt{q};\ \sqrt{q})_\infty} \\
&\cdot{}_{10}\phi_9\left[\begin{matrix} a^2,\ qa,\ -qa,\ ac,\ ac\sqrt{q},\ af,\ af\sqrt{q},\ ag,\ aq/\lambda,\ a\sqrt{q}/\lambda \\ a,\ -a,\ aq/c,\ a\sqrt{q}/c,\ ag/f,\ a\sqrt{q}/f,\ aq/g,\ a\lambda,\ a\lambda\sqrt{q}\end{matrix};\ q,\ \frac{\lambda^2\sqrt{q}}{agc^2f^2}\right] \\
&+\frac{(c^2f^2q,\ 1/c^2f^2,\ ag\sqrt{q};\ q)_\infty\,(\lambda f, \lambda/f,\ f\sqrt{q}/c;\ \sqrt{q})_\infty}{(c^2\sqrt{q},\ \sqrt{q}/c^2,\ af,\ a/f,\ gf,\ g/f;\ q)_\infty\,(cf,\ 1/cf,\ cf\sqrt{q},\ f^2\sqrt{q};\ \sqrt{q})_\infty} \\
&\cdot{}_{10}\phi_9\left[\begin{matrix} f^2,\ qf,\ -qf,\ fa,\ fc,\ fc\sqrt{q},\ f^2\sqrt{q},\ fg,\ fq/\lambda,\ f\sqrt{q}/\lambda \\ f,\ -f,\ fq/a,\ fq/c,\ f\sqrt{q}/c,\ \sqrt{q},\ fq/g,\ \lambda f,\ \lambda f\sqrt{q}\end{matrix};\ q,\ \frac{\lambda^2\sqrt{q}}{agc^2f^2}\right] \\
&+\frac{(fgc^2q,\ 1/fgc^2,\ g\sqrt{q}/f;\ q)_\infty\,(\lambda g,\ \lambda/g,\ g\sqrt{q}/c;\ \sqrt{q})_\infty}{(c^2\sqrt{q},\ \sqrt{q}/c^2,\ qg^2,\ a/g,\ f/g;\ q)_\infty\,(cg,\ fg,\ cf\sqrt{q},\ 1/cf;\ \sqrt{q})_\infty} \qquad (4.5)\\
&\cdot{}_{10}\phi_9\left[\begin{matrix} g^2,\ qg,\ -qg,\ ga,\ gc,\ gc\sqrt{q},\ gf,\ gf\sqrt{q}, gq/\lambda,\ g\sqrt{q}/\lambda \\ g,\ -g,\ gq/a,\ gq/c,\ g\sqrt{q}/c,\ qg/f,\ g\sqrt{q}/f,\ g\lambda,\ g\lambda\sqrt{q}\end{matrix};\ q,\ \frac{\lambda^2\sqrt{q}}{agc^2f^2}\right] \\
&=\frac{(agc^2\sqrt{q};\ q)_\infty\,(\lambda c,\ \lambda/c;\ \sqrt{q})_\infty}{(c^2\sqrt{q};\ q)_\infty\,(ac,\ fc,\ f/c,\ gc;\ \sqrt{q})_\infty} \\
&\cdot{}_5\phi_4\left[\begin{matrix} ac,\ cg,\ \lambda/f,\ cq^{1/4},\ -cq^{1/4} \\ \lambda c,\ c\sqrt{q}/f,\ c\sqrt{ag}\,q^{1/4},\ -c\sqrt{ag}\,q^{1/4}\end{matrix};\ \sqrt{q},\ \sqrt{q}\right] \\
&+\frac{(agf^2\sqrt{q};\ q)_\infty\,(\lambda f,\ \lambda/f;\ \sqrt{q})_\infty}{(f^2\sqrt{q};\ q)_\infty\,(af,\ cf,\ c/f,\ gf;\ \sqrt{q})_\infty} \\
&\cdot{}_5\phi_4\left[\begin{matrix} af, fg,\ \lambda/c,\ fq^{1/4},\ -fq^{1/4} \\ \lambda f, f\sqrt{q}/c, f\sqrt{ag}\,q^{1/4},\ -f\sqrt{ag}\,q^{1/4}\end{matrix};\ \sqrt{q},\ \sqrt{q}\right]
\end{aligned}
$$

provided

$$\left|\frac{\lambda^2\sqrt{q}}{agc^2f^2}\right| < 1. \qquad (4.6)$$

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